Günçavdi, Öner; Akhiev, Seyidali S.; McKay, Andrew
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Instantaneous optimal investment decisions with costly and costless reversibility

Öner Günçavdi
Seyidali S. Akhiev
Andrew McKay[1]

- **Abstract**: While many different theories have been put forward to explain investment behaviour, they are all generally based on dynamic optimization, and there are a number of different methods available to solve such problems. In such models, the optimal values of the control and state variables, namely investment and the capital stock respectively, become *forward-looking*, dependent on the *future* values of prices of both output and the factors of production, and on the (unknown) end period value of the capital stock. In this paper we suggest a new method to obtain optimal investment levels without requiring information on the future, or end period conditions. Thus the optimal paths of control and state variables are obtained without needing to know future values of variables. Instead of maximizing the discounted value of the cash flows from unit capital accumulation over an unobservable future time interval as a performance index, the firm is assumed to maximize the current value of the cash flow of a unit capital accumulation at each time *t*.

- **Resumen**: En general, las diferentes teorías que se han planteado para explicar la inversión se basan en problemas de optimización dinámica cuyas soluciones cuentan con diferentes métodos. En dichos modelos, los valores óptimos de las variables de control y de estado, etiquetadas como inversión y capital respectivamente, dependen de los precios futuros del producto y de los factores de la producción y del nivel de capital al final del periodo. En este artículo sugerimos un nuevo método para obtener los niveles óptimos de inversión sin requerir información sobre los precios futuros o las condiciones de transversalidad. Así, en lugar de maximizar el valor presente de los flujos de caja por unidad de capital, como un índice de desempeño, la empresa maximiza el valor actual del flujo de caja por unidad de capital.

1   Öner Günçavdi, Seyidali S. Akhiev Istanbul Technical University, Maçka 80680, Istanbul, Turkey. Corresponding author. Tel: +90-212-293 13 00 (60 lines)/2008; Fax: +90-212-240 72 60; e-mail: guncavdi@itu.edu.tr Andrew McKay, School of Economics, University of Nottingham,University Park, Nottingham, NG7 2RD, UK

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- *Introduction*

While many different theories have been put forward to explain invest-
ment behaviour, they are all generally based on dynamic optimization,
and a number of different methods are available to solve such problems.
In such models, the optimal values of the control and state variables,
namely investment and the capital stock respectively, become forward-
looking, dependent on the future values of prices of both output and the
factors of production (e. g. see Abel, 1980; Kort, 1989 and 1990). This
is mainly because firms are represented as maximizing the discounted
value of the expected cash flows from an incremental addition to the
capital stock over a time interval in which the future state of economy is
not observable, and subject to an end-period terminal condition for the
capital stock. Difficulties arise both in specifying the terminal condition,
and in representing expectations of future values.

Concerning the terminal condition, investment decisions are typi-
cally taken over a finite horizon, so that in dynamic optimization a ter-
minal condition is needed giving the value of the capital stock at the end
of the horizon. In practice, though, this value is unknown in advance,
and indeed the length of the horizon need not be known with certainty.
This issue is often avoided by representing the investment decision as
an infinite horizon problem, but this is not very realistic. Firms are often
concerned about the returns they are able to make on their investments
over a relatively short time horizon (perhaps due to costs of financing or
even uncertainty about their continued survival). This is especially likely
to be the case in a high technology industry where technologies change
rapidly; firms then may have very short time horizons over which they
must earn returns to their investments.

The second major issue concerns the representation of unknown fu-
ture values. The evolution of market conditions in future will matter to
the firm, which therefore must form expectations of these; of course
these expectations are unobserved. For modelling purposes then, it is
necessary to make assumptions regarding the future paths of the vari-
ables, such as rational expectations (or simply assuming that the vari-
ables concerned remain unchanged). However, in either case, we are
unable to determine the correct paths of the optimal controls, but only

able to find their approximate values because of the inevitable errors in predicting the future paths of the relevant variables.

Given these two issues, our aim in this paper is to suggest a new method to obtain optimal investment levels which avoids both of these problems. The novelty in this paper is twofold. First, the optimal paths of control and state variables are obtained without needing to know future values of variables. Second, instead of maximizing the discounted value of the cash flows from unit capital accumulation over an unobservable future time interval [0,T] as a performance index, the firm is assumed to maximize the current value of the cash flow of a unit capital accumulation at every time t for all $0 \le t \le T$ recursively. In this optimization method at time t the firm considers only the market conditions for time t, thus avoiding the need for a terminal condition. At the end of time *t*, firms maximize the objective function for time *t+1*, and so on. This representation may be particularly plausible in an economy where the economic conditions that influence investment decisions change very rapidly. In such circumstances, firms will mostly be concerned with the current returns of unit investment. The optimal levels of control and state variables with this method are therefore derived as being dependent only on the current and past levels of prices of output and the factors of production, but not on their future values, on which information is then not required.

This paper is structured as follows. The next section summarizes the conventional neoclassical investment model and its well-known results. In section 3, we introduce our proposed method of optimization by using an instantaneous performance index, this being considered both for the cases where investment is reversible and where it is irreversible. Section 4 summarizes main conclusions of the paper.

■   *The model*

In the standard neoclassical investment theory, the problem that the firm encounters is to find piecewise continuous control variables, *I(t)*, and *L(t)*, and an associated continuous and piecewise differentiable state variable, *K(t)*, defined on the fixed time interval [0, T] to maximize the value of the following performance index (e.g. see Jorgenson, 1963, and Takayama, 1996):

(1) $\quad \underset{I(t),L(t)}{Max} \quad J = \int_0^T e^{-rt} \left[p(t) f\left(K(t), L(t)\right) - w(t)L(t) - v(t)I(t) - c\left(I(t)\right) \right] dt$

subject to:

(2) $$K(t) = I(t) - aK(t)$$

(3) $$K(0) = K_0, \ K_0 > 0$$

(4) and an end period condition for $K(T)$;

where $p(t)$ is the price of output, $w(t)$ the wage of unit labour, $L(t)$: labour used in production, $K(t)$: the capital stock the firm owns, $v(t)$: the price of capital goods, $I(t)$: investment, all at time t, r : the discount factor (assumed to be fixed for simplicity), a: constant depreciation rate of capital, the function $f\ (K, L)$ is the production function, assumed to exhibit decreasing returns to scale[2] with $f_K{>}0, f_L{>}0, f_{KL}{>}0$, and $f_{KK}{<}0, f_{LL}{<}0$, for $K,\ L \in (0, + \infty)$, and $c(I)$ is the non-negative convex cost of adjustment function associated with changing the rate of investment, with $c'(I) > 0$ $(I > 0)$, $c'(I) \le 0$ $(I \le 0)$ and $c''(I) > 0$ on $I \in R = (- \infty, + \infty)$ (Lucas, 1967; Mussa, 1977; Abel, 1979; Kort and Jørgensen, 1993). Constraint (2) represents the capital accumulation rule implying that net changes in the capital stock equal gross investment minus replacement investment. The functions of $p(t)$, $v(t)$ and $w(t)$ are assumed to be continuous and positive over the interval $[0, T]$.

The following necessary conditions for optimality can be derived by applying Pontryagin's Maximum Principle:

(5) $$p(t) f_L\big(K(t), L(t)\big) = w(t)$$

(6) $$\lambda(t) = v(t) + c'\big(I(t)\big)$$

(7) $$\lambda(t) = (r + a)\lambda(t) - p(t) f_K\big(K(t), L(t)\big)$$

(8) $$\lambda(T) = 0$$

where $\lambda(t)$ is the Lagrange multiplier associated with constraint (2). Condition (5)-(8) together with (3) and (4) are the necessary conditions of optimality. The sufficient condition for a maximum can be derived by

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2   The assumption of decreasing returns to scale is the necessary and sufficient condition for profit maximization. In general, the positivity of the determinant of the Hessian matrix of second-order derivatives of the cash flow function (1) with respect to $K(t)$ and $L(t)$ ensures the existence of a unique solution for a profit maximizing firm. This assumption can mathematically be introduced into the model by requiring that $f_{LL} f_{KK} - f_{KL}^2 > 0$ (see Chirinko, 1993; Brechling, 1975).

showing that the Hessian matrix of second-order derivatives of Hamiltonian function of the problem (1)-(4) is negative definite. This is so given the assumption of decreasing returns to scale (that is $f_{LL} f_{KK} - f^2{}_{KL} > 0$) and the convexity of the adjustment cost function (that is $c''(I) > 0$).

In the determination of the optimal investment behaviour of the firm, $\lambda(t)$ is a critical variable representing the willingness of the firm to sacrifice current revenue for capital accumulation. Given the transversality condition (8), the discounted value of a unit investment can be obtained from condition (7) as follows

$$(9) \qquad \lambda(t) = \int_t^T e^{-(r+a)(s-t)} p(s) f_K\left(K(s), L(s)\right) ds$$

The right hand side of (9) represents the *expected* discounted stream of marginal profits that a unit increase in the capital stock generates from the present to the end of planning horizon [*t, T*] which is currently *not* observable. As easily seen in (9), the value of $\lambda(s)$ $\forall s \in (t, T)$ is a function of the price of output, *p(s)*, wage *w(s)* (through condition (5)), and the value of *K(s)* for the time interval [*t, T*].

Using (9) and (6) together, the following equality can be written.

$$(10) \qquad v(t) + c'\left(I(t)\right) = \int_t^T e^{-(r+a)(s-t)} p(s) f_K\left(K(s), L(s)\right) ds$$

From (10), we can obtain the forward-looking optimal path of investment. In the economic literature, it is often convenient to extend the planning horizon indefinitely into the future, and change the interval of integration in the objective functional from [0,T] to [0,∞] (see Léonard and Van Long, 1992). One of the reasons for doing so is to avoid the problem of specifying the end-point-horizon stock value function. Also this representation often leads to simplified formulae. In such cases, the forward-looking nature of the dynamic problem (1)-(4) is examined with saddle-point behaviour, which is described by a pair of first-order differential equations for *K(t)* and *I(t)*. Commonly this system of first-order differential equations yields an equilibrium point (*K**, *L**) with the saddle-point property.[3] Considering the steady-state values of *K(t)* and *I(t),* such

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[3]   The dynamic behaviour of the optimal solution *(K(t), I(t))* in the *KOI* phase space is determined by the eigenvalues (roots) of this dynamic system of differential equations. Such saddle-point dynamics involves both unstable and stable roots. In problem (2)-(4) with

analysis in the infinite horizon yields only *qualitative* information about the direction of the movements in the optimal paths of *K(t)* and *I(t)*.

Although this type of analysis in an infinite horizon provides a general insight into the dynamic behaviour of investment, it cannot allow us to determine the optimal values of *(K(t), I(t))* numerically in a finite horizon. In particular, the transversality condition does not disappear in a finite horizon problem and the value of K_T is unknown (so that $\lambda(t)$ is unknown) in practice.

■ *Instantaneous performance indices*
  and optimal investment decisions

In this section we introduce a new solution algorithm to find the optimal trajectories of the control and state variables. The method employed here is recursive in nature, and the optimal solution defined for each time *t* is obtained so as to maximize a particular objective function. Instead of describing the performance index (objective function) in the form of an integral (such as (1)) over a time interval [0,T], the following time-dependent function is chosen.

$$(11) \quad \underset{I(t),L(t)}{Max} \quad J(t) = e^{-rt} h \left[f\left(K(t), L(t)\right) - w(t)L(t) - v(t)I(t) - c\left(I(t)\right) \right].$$

where *h>0* is a given number. The optimal control conditions are derived by maximizing *J(t)* at every time *t* for all $0 \le t \le T$. Hence, the control conditions are referred to as instantaneous control algorithms, and function (11) can be regarded as an instantaneous performance index. We find the optimal values of *(K(t),I(t),L(t))* at time *t* so as to maximize *J(t)* in (11), assuming that the values of *(K(t-h),I(t-h),L(t-h))* at time *(t-h)* are known. Also in the case where *h* is sufficiently small, function (11) can be considered as the total cash flows of the firm over the interval *[t,t+h]*. Since *h* and e^{-rt} are constant for a given time *t*, their values have no effect on the optimization of (11).

The capital accumulation rule (equation 2 above) continues to apply. In general this is a first order differential equation, which for a given initial value *K(t₀)* can be written as follows:

$$(12) \qquad K\left(t\right) = K\left(t_0\right) e^{-a\left(t-t_0\right)} + \int_{t_0}^{t} I\left(s\right) e^{-a\left(t-s\right)} ds$$

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a positive discount rate, if a steady-state *(K*,L*)* exists, then it cannot be locally stable in the *KOI* space due to the positive roots of the system, so that we have only conditional stability.

In this specific case where $t_0=t-h$, equation (12) reduces to

(13) $$K(t)=K(t-h)e^{-ah}+\int_{t-h}^{t}I(s)e^{-a(t-s)}ds$$

The integral on the right-hand side of (13) can be calculated by the Trapezium rule (Ketter and Prawel, 1969, Collatz, 1966) to give:

(14) $$K(t)=K(t-h)e^{-ah}+\frac{h}{2}\left[I(t)+e^{-ah}I(t-h)\right]+O(h^3)$$

where $O(h^3)$ is the remainder. Ignoring the term $O(h^3)$, equation (14) can be written as

(15) $$K(t)=\frac{h}{2}I(t)+F(t-h)$$

where

(16) $$F(t-h)=K(t-h)e^{-ah}+\frac{h}{2}e^{-ah}I(t-h)$$

Equation (15) is a discrete approximation of equation (2) yielding the value of the capital stock $K(t)$ at time t depending on both the value of investment $I(t)$ and the value of $F(t-h)$. Now, the problem becomes one of maximizing (11) subject to discrete equation (15) as follows.

(i) $$\underset{I(t),L(t)}{\text{Max}}\quad J(t)=e^{-rt}h\left[p(t)f(K(t),L(t))-w(t)L(t)-v(t)I(t)-c(I(t))\right]$$

(17)   *Subject to*

(ii) $$K(t)=\frac{h}{2}I(t)+F(t-h)$$

where $t\in\left[t-h,T\right]$ is given. We must note that problem (17i) can be referred to as an independent maximization problem of (2)-(4) for the discrete system subject to (17ii). Problem (17) is a convex programming problem, and the conditions for optimality can be obtained by applying the Kuhn-Tucker theorem (Takayama, 1985). However, these optimality conditions can also be derived as follows. Upon substituting (17ii) into (17i), we are able to transform problem (17) into one of maximizing the following equation in the open region $R\times(0,\infty)$ with respect to $(I(t),L(t))$:

$$(17') \quad \underset{I(t),\,L(t)}{Max}\ J(t) = e^{-rt}h\left[p(t)f\left(\frac{h}{2}I(t)+F(t-h),\,L(t)\right)\right.$$

$$\left. -\,w(t)L(t)-v(t)I(t)-c\big(I(t)\big)\right]$$

The optimality conditions for *(K(t), I(t), L(t))* from *(17')* are then given by the following system of equations

$$(18) \qquad \frac{h}{2}p(t)f_K\big(K(t),L(t)\big)=v(t)+c'\big(I(t)\big)$$

$$(19) \qquad K(t)=\frac{h}{2}I(t)+F(t-h)$$

$$(20) \qquad p(t)f_L\big(K(t),L(t)\big)=w(t)$$

Due to the convexity of functional *J(t)* with respect to its arguments *(K(t), I(t), L(t))*, conditions (18)-(20) are also the sufficient conditions for optimality. Consequently, the optimal solution *(K(t), I(t), L(t))* at time *t* can be obtained from (18)-(20), if the value *F(t-h)* is known. The next task is to prove the existence and uniqueness of the optimal solution (18)-(20).

■ *The existence and the uniqueness of the solution*

In this respect, we analyze two different cases under different assumptions regarding the cost of adjustment function. First, we examine the optimal investment decision in the case where the adjustment of capital stock involves adjustment costs for both purchasing and re-selling unit capital; in other words, capital is assumed to be subject to costly reversibility. In the second case, the investment expenditure is assumed to be perfectly reversible without any adjustment cost when used capital goods are sold in the secondary capital goods market.

Case I: Optimal Investment Decisions with Costly Reversibility

This is specified by introducing a convex cost of adjustment function, which postulates that revenue from selling a unit used capital good in the secondary capital goods market is lower than its purchasing price due to sunk costs. Now, let the following assumptions hold.

Assumption 1. *As before, the production technology is assumed to possess decreasing returns to scale. The production function f(K,L) and its first- and second-order partial derivatives are assumed to be continuous in the region* $(K, L) \in (0, + \infty) \times (0, + \infty)$, *where the partial derivatives satisfy the same properties as in section 2 above.*

Assumption 2. *Suppose that* $f_L(K, L)$ *and* $f_K(K, \varphi(K, Z))$ *satisfy the conditions*

$$(21) \qquad \lim_{L \to +0} f_L(K, L) = + \infty \quad and \quad \lim_{L \to +\infty} f_L(K, L) = + 0,$$

$$for \ given \ K \in (0, + \infty)$$

and

$$(22) \quad \lim_{K \to +0} f_K(K, \varphi(K, Z)) = + \infty \quad and \quad \lim_{K \to +\infty} f_K(K, \varphi(K, Z)) = + 0$$

$$for \ given \ Z \in (0, + \infty)$$

where $L = \phi(K, Z)$ *is the inverse function of* $Z = f_L(K, L)$ *with respect to* $L \in (0, + \infty)$.[4]

Assumption 3. *c (I) is a convex continuous non-negative function with continuous first and second order derivatives c′ (I) and c′ (I) for* $I \in R$, *and satisfying the following conditions:*

$$c'(I) < 0 \ (I < 0), \quad c'(I) > 0 \ (I > 0), \quad c(0) = c'(0) = 0,$$

$$c'(-\infty) = -\infty, \quad c'(+\infty) = +\infty \quad and \quad c''(I) > 0 \ (I \in R).$$

Proposition 1: Given assumptions 1 – 3 above, the system of equations (18)-(20) have a unique solution *(K(t), I(t), L(t))* for a given *F(t-h)* and arbitrary *t*, where $h \leq t \leq T$.

Proof: Since $f_{LL} < 0$, the function $Z = f_L(K, L)$ possesses the inverse function $L = \varphi(K, Z)$ with respect to $L \in (0, + \infty)$, and $\varphi_Z(K, Z) < 0$. Condition (21) indicates that the function $L = \varphi(K, Z)$ is defined for all $Z \in (0, + \infty)$. Therefore, we obtain *L(t)* from (20) as follows

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[4]   The argument *Z* can be considered to be the real wage from condition (20).

$$(23) \qquad L(t) = \varphi\left(K(t), \frac{w(t)}{p(t)}\right)$$

Then substituting (23) into (18) yields

$$(24) \qquad \frac{h}{2} p(t) f_K\left(K(t), \varphi\left(K(t), \frac{w(t)}{p(t)}\right)\right) = v(t) + c'\left(I(t)\right)$$

Let $A(K, Z) = f_K\left(K, \varphi(K, Z)\right)$ for all $K \in (0, +\infty)$ and $Z \in (0, +\infty)$. Also it can easily be shown that $A_K(K, Z) < 0$. Hence, the function $S = A(K, Z)$ has the inverse function $K = \Phi(S, Z)$ with respect to $K \in (0, +\infty)$, and $\Phi_S(S, Z) < 0$. Additionally, condition (22) postulates that the function $K = \Phi(S, Z)$ is defined for all $S \in (0, +\infty)$.

We are now able to write (24) as follows

$$(25) \qquad K(t) = \Phi\left(\frac{2}{hp(t)}\left[v(t) + c'\left(I(t)\right)\right], \frac{w(t)}{p(t)}\right)$$

In order to obtain the optimal $I(t)$, we can derive the following using (19) and (25).

$$(26) \qquad \frac{h}{2} I(t) + F(t - h) = \Phi\left(\frac{2}{hp(t)}\left[v(t) + c'\left(I(t)\right)\right], \frac{w(t)}{p(t)}\right)$$

The right-hand side of (26) is defined for all $I(t) \in \left(I_0(t), +\infty\right)$, where $I_0(t)$ is the unique solution of equation $c'\left(I(t)\right) + v(t) = 0$ and $I_0(t) < 0$, and satisfies the following conditions

$$(27) \qquad \lim_{I(t) \to I_0(t) + 0} \Phi\left(\frac{2}{hp(t)}\left[v(t) + c'\left(I(t)\right)\right], \frac{w(t)}{p(t)}\right) = +\infty$$

and

$$(28) \qquad \lim_{I(t) \to +\infty} \Phi\left(\frac{2}{hp(t)}\left[v(t) + c'\left(I(t)\right)\right], \frac{w(t)}{p(t)}\right) = +0$$

The left-hand-side of (26), $g\big(I(t)\big) = (h/2)I(t) + F(t-h)$, is a linear function of $I(t)$ with a positive slope *(h/2)*, and $g\big(I_0(t)\big) < +\infty$ and $g(+\infty) = +\infty$. As a result, equation (26) yields a unique solution $I(t) \in \big(I_0(t), +\infty\big)$ by using (27) and (28). The optimal values of *K(t)* and *L(t)*, in turn, can be obtained from (25) and (23) respectively (see Figure 1).

We must note that the optimal value of investment can also be obtained from (24) in terms of *K(t)* as shown below.

$$(29) \qquad I(t) = \Psi^{-1}\left[\frac{h}{2}p(t)A\left(K(t),\frac{w(t)}{p(t)}\right) - v(t)\right]$$

where $I = \Psi^{-1}(Z)$ is the inverse of the function $Z = c'(I)$. The value of *I(t)* given by (29) can be referred to as an optimal regulator. Since the right-hand sides of (23) and (29) are considered as a function defined on the state function *K(t)*, these formulas yield a closed-loop solution for the optimal control variables *I(t)* and *L(t)*. Additionally, they can be considered as an open-loop with respect to exogenous variables *p(t)*, *w(t)* and *v(t)*. Substituting (29) into (19) renders the following equation for the optimal value of the capital stock.

$$(30) \qquad K(t) = \frac{h}{2}\Psi^{-1}\left[\frac{h}{2}p(t)A\left(K(t),\frac{w(t)}{p(t)}\right) - v(t)\right] + F(t-h)$$

The existence of the solution $K(t) \in (0, +\infty)$ of (30) is equivalent to the existence of the solution $I(t) \in \big(I_0(t), +\infty\big)$ of the equation (26). Hence, equation (30) has a unique solution $K(t) \in (0, +\infty)$.

The determination of the optimal and unique value of investment and the capital stock are graphically shown in Figure 1. Both sides of (26) are depicted in the figure separately. In Figure 1, the capital stock and investment are plotted on the vertical and horizontal axis respectively. It is seen from the left-hand side of (26), the function yielding *K(t)* for a given value of *F(t-h)* at time *t-h* is linear and increasing with respect to investment, with slope *(h/2)*. It is seen from (17') that *F(t-h)* is defined based on the levels of the capital stock and investment at time *(t-h)*; the higher the level of the capital stock is, the higher will be the level of *F(t-h)*. In the figure, two linear functions are drawn for two different values of *F(t-h)*; the solid line representing its higher value at point *D*. The case

of a lower value of *F(t-h)* indicates a lower value of *K(t)* and higher va-
lue of *I(t)* at time *t*, and *vice versa*.

In the case where *F(t-h)* is higher than $K_0(t)$ at point D,[5][5] it is clearly
seen from Figure 1 that the firm wishes to disinvest (i.e. *I(t)<0*), and the
following condition (from 17') will hold for time *t*

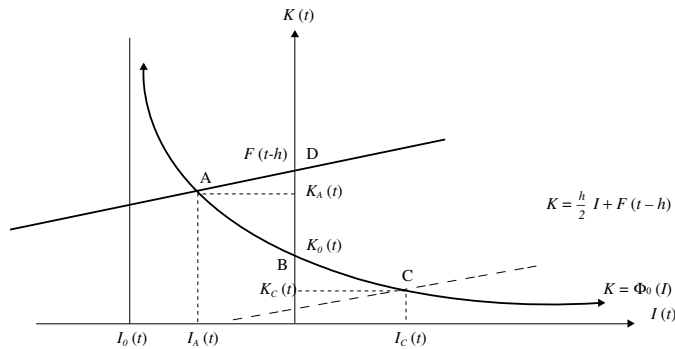$$F\left(t\right) = e^{-ah}\left[\frac{h}{2}I\left(t\right) + K\left(t\right)\right] < F\left(t - h\right)$$

Upon substituting *K(t)* from (19), we then get

$$e^{-ah}\left[hI\left(t\right) + F\left(t - h\right)\right] < F\left(t - h\right)$$

The firm continues to adjust its capital stock by disinvesting as long
as $K_0(t) < F(t+mh) < F(t-h)$, where *m* is the number of steps (the number
of times the optimization is repeated). At a particular step, the firm fina-
lly reaches to the point where $F(t+(m+1)h) < K_0(t)$. This then shows that
if the firm starts adjusting its capital stock by disinvesting, then it must
definitely wish to invest at some point in the stage of the adjustment. If
F(t-h) is low enough, it is seen from Figure 1 that the optimal *I(t)* and
K(t) become significantly high at the next stage.[6]

Figure 1
Function $K = \Phi_0 (t, I)$, $I \in (I_0 (t), + \infty)$, notes the function given by (25)

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5    $K_0(t)$ and $I_0(t)$ respectively indicate the values of the optimal capital stock and investment
     in the beginning of each time period *t*. However the initial capital stock and investment at
     t=0 are represented by *K(0)* and *I(0)* respectively.
6   Also we must note that since the time interval is considered as a finite horizon, then it
     would be misleading to consider that the optimal values of *K(t+nh)* and *I(t+nh)* move
     around a particular steady-state equilibrium values I^* and K^* as the number of steps, *n*,
     increases.

The right-hand side of (26) in fact represents function (25), and is unbounded when $I(t) \to I_0(t) + 0$ and bounded when $I(t) \to +\infty$, as seen in the figure, under condition (27) and (28). It is easily seen from (25) that the different locations of this function are determined by the different levels of price variables (namely *p(t), w(t)* and *v(t)*) and the marginal costs of capital adjustment. In Figure 1, the function is depicted for given levels of prices. The intersection of (25) with the vertical axes *OK* at point *B* can be called the zero-investment point for the initial optimal capital stock $K_0(t)$ for time t. At this point, the firm has no motivation to invest (or disinvest) unless prices vary. The intersection of the lines representing the two sides of (26) yields unique optimal values of *K(t)* and *I(t)*. This solution together with (23) then shows a recursive dependence of the optimal values *(K(t), I(t), L(t))* upon *F(t-h)*. In order to see this recursive relationship clearly, it is sufficient to derive *I(t)* from (26). Additionally, deriving the optimal solution values *(K(t), I(t), L(t))* at time t, the value of *F(t)* immediately becomes determined, and the optimal values of values *(K(t+h), I(t+h), L(t+h))* at time *(t+h)* can, similarly, be obtained, depending on *F(t)*. However, the optimal behaviour at the initial stage where *t=0*, must be considered separately with particular care. In this case, equation (19) is regarded as *K(0)=K₀*. From (23), we have

(31)
$$L(0) = \varphi\left(K_0, \frac{w(0)}{p(0)}\right) = L_0$$

Using (18), we have

(32)
$$\frac{h}{2} p(0) f_K(K_0, L_0) = v(0) + c'(I(0))$$

Then *I(0)* can be derived from (32) by substituting (31). Whether *I(0)* is positive or negative is dependent on the sign of the following scalar obtained from the left-hand side of (32).

(33)
$$\alpha_0 = \frac{h}{2} p(0) f_K(K_0, L_0) - v(0)$$

In the case where the initial capital stock, K_0, is small, it is likely that α_0 will positive. On the other hand, having a small demand for labour increases the possibility of negative value of α_0.

Additionally, the right side of equation (26) given by (25) is a function of $p(t)$, $v(t)$ and $w(t)$. It is easily shown that the signs of the partial derivatives of (25) with respect to $p(t)$, $v(t)$ and $w(t)$ are as follows: $\partial \Phi / \partial p(t) > 0$, $\partial \Phi / \partial v(t) < 0$ and $\partial \Phi / \partial w(t) < 0$. As everything remains constant, it is evident in a particular case where $p_1(t) < p_2(t)$, then $K_1(t) < K_2(t)$ and $I_1(t) < I_2(t)$ for the optimal solution $(K_1(t), I_1(t), L_1(t))$ and $(K_2(t), I_2(t), L_2(t))$ corresponding to $p_1(t)$ and $p_2(t)$ respectively. Also it can easily be obtained from (23) and (25) that since $\partial L(t) / \partial p(t) > 0$, we get $L_2(t) > L_1(t)$. The response of the optimal solution $(K(t), I(t), L(t))$ both to $v(t)$ for given $(p(t), w(t))$ and to $w(t)$ for given $(p(t), v(t))$ can similarly be obtained.

Case II-Optimal Investment Decisions with Costless Reversibility

We now assume that investment expenditure is reversible, and re-selling capital already in use requires no costs.

Proposition 2: Let $c(I) \equiv 0$ for $I \leq 0$; $c'(I) > 0$ and $c''(I) > 0$ for $I > 0$; and $c'(+\infty) = +\infty$, and let conditions (21) and (22) hold. Then the system of equations (18)-(20) has a unique solution $(K(t), I(t), L(t))$ for a given $F(t-h)$ and arbitrary t, where $h \leq t \leq T$.

Proof: Although functions $V(t)$ and $H(t)$ are not concave with respect to the triple arguments $(K(t), I(t), L(t))$, it is seen that function (17') is concave in a pair of arguments $(I(t), L(t))$. Hence, the optimality conditions are given by the system of (18)-(20). In such a case, the optimal solution of the system (18)-(20) is reduced to the solution of (26) with respect to $I(t)$. The following function must however be considered on the right-hand side of (26).

$$(34) \qquad K(t) = \begin{cases} K_0(t), & I(t) \leq 0 \\[2em] \Phi\left(\dfrac{2}{hp(t)}\left(v(t) + c'\left(I(t)\right)\right), \dfrac{w(t)}{p(t)}\right), & I(t) > 0 \end{cases}$$

where $K_0(t) > 0$ is the unique solution of the following equation

$$(35) \qquad f_K\left(K(t), \varphi\left(K(t), \frac{w(t)}{p(t)}\right)\right) = \frac{2}{hp(t)} v(t)$$

Having substituted (34) into the right-hand side of (26), the resulting equation has a unique solution $I(t) \in R$ in the case of costless reversibility (see Figure 2). It is obvious that the optimal value of $L(t)$ can also be obtained from (23).

According to the definition of an optimal solution, the values of $(K(0), I(0), L(0)) \in (0, +\infty) \times R \times (0, +\infty)$ for $t=0$ can be obtained so as to maximize the following function

$$(36) \quad J(0) = h\left[p(0) f\left(K(0), L(0)\right) - v(0) I(0) - w(0) L(0) - c\left(I(0)\right)\right]$$

However, the initial capital stock is given at time $t=0$; that is $K(0)=K_0$. Then the problem must be considered as maximizing the following function with respect to $(I(0), L(0))$.

$$(37) \quad \begin{aligned} Max\ J(0) &= h\left[p(0) f\left(K_0, L(0)\right) - v(0) I(0) - w(0) L(0) - c\left(I(0)\right)\right] \\ & \quad I(0) \in R, \quad L(0) \in (0, +\infty) \end{aligned}$$

It is clear that function $J(0)$ does not have a finite maximum value under the constraint imposed on the adjustment cost function $c(I)$. Nevertheless, the maximum value of the function $J(0)$ with respect to $L(0)$ can be obtained from the following condition

$$(38) \quad p(0) f_L\left(K_0, L(0)\right) = w(0)$$

Hence, we may derive the initial value of $L(0)$ as

$$(39) \quad L(0) = \varphi\left(K_0, \frac{w(0)}{p(0)}\right)$$

However, function $J(0)$ is not bounded above with respect to the argument $I(0) \in R$. In order to determine the initial value of $I(0)$, we must use economic intuition. For instance, the initial value of $I(0)$ can be derived from the condition that the capital stock at the first stage is equal to its initial value; that is $K(0)=K(h)$. Since the time interval h from the initial step to the first one is too small, the firm is to be unable to generate enough positive revenue to cover all the purchasing cost of a unit of capital (see equation (18)). The only way of producing income for the firm over such a small interval appears to sell the capital stock already in use. We, however, assume that at the initial state, the firm sells no capi-

tal stock to earn positive income, and aims only to maintain the existing capital stock by undertaking replacement investment at *t=0*. It is clear that using (16), the condition *K(0)=K(h)* can be written as follows.

(40)
$$\frac{h}{2}I\left(h\right)+F\left(0\right)=K_0$$

or

(40')
$$\frac{h}{2}I\left(h\right)+\frac{h}{2}I\left(0\right)e^{-ah}+K\left(0\right)e^{-ah}=K_0$$

From (40') we have

(41)
$$K_0\left(1-e^{-ah}\right)=\frac{h}{2}\left(I\left(h\right)+I\left(0\right)e^{-ah}\right)$$

Considering the coefficients with the *h*-order term in (41), we obtain $I(0)=aK_0$ for the initial value of *I(0)*. The other features of the optimal solution in the case of costless reversible investment, such as the dependence of the optimal solution on function *p(t)*, *w(t)* and *v(t)*, can be examined in a similar way to those in the case of costly reversibility.

Figure 2
Function $K = \Phi_0\left(t, I\right), I \in R$, indicates the function given by (34)
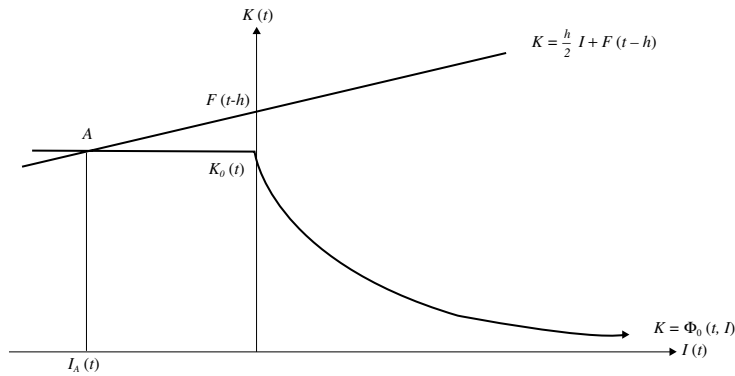
The determination of a unique optimal level of investment in the case of costless reversibility is graphically shown in Figure 2. The adjustment of the capital stock in this case differs from the previous case only when *F(t-h)* is high, and the firm wishes to sell the excess capital stock at the

given level of prices. It is seen from Figure 2 that this adjustment process takes place instantaneously (at one step) requiring no adjustment cost. However, having disinvested the excess capital stock at the initial stage, the firm faces the case where $F(t) < K_0(t)$ at the first stage, indicating both a shortage of capital stock and a motivation for the firm to invest.

■ *Conclusion*

The optimal control theory has commonly used in the neoclassical theory of investment. While the Calculus of Variation, Dynamic Programming and Pontryagin's Maximum Principle have been the most popular techniques to find the optimal paths of capital accumulation and investment of a hypothetical firm, the trajectory of the control variable becomes dependent on the future evaluation of other economic variables in the model, such as the prices of output and capital goods and wages, and on the end period condition in a finite horizon. Since the future state of the economy is unknown at current time for the firm, the optimal path of investment can only be derived under a particular case where either the future values of relevant variable are predicted and substituted in the function of the optimal investment, or they can simply be ignored. In both cases, we can only derive approximated optimal path of investment. However, in this paper we introduce a new method, known as Instantaneous Performance Index, to find the solution of the optimal investment behaviour without requiring any information on the future.

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