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A CHARACTERIZATION OF HEIGHT-BASED EXTENSIONS OF PRINCIPAL FILTRAL OPPORTUNITY RANKINGS

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A parameterized characterization of height-based total extensions of principal filtral opportunity rankings is provided and shown to include, as a special case, a version of the well-known Pattanaik-Xu characterization of the cardinality-based ranking.

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INTRODUCTION

In the last two decades, a considerable amount of work has been devoted to the analysis of opportunity inequality and related issues concerning rankings of opportunity sets (see e.g. Alcalde-Unzu & Ballester, 2005; Alcalde-Unzu & Ballester, 2010; Arlegi & Nieto, 1999; Barberà, Bossert & Pattanaik, 2004; Dutta & Sen, 1996; Herrero, 1997; Herrero, Iturbe-Ormaetxe & Nieto, 1998; Kolm, 2010; Kranich, 1996; Kranich, 1997; Ok, 1997; Ok & Kranich, 1998; Pattanaik & Xu, 1990; Pattanaik & Xu, 2000; Savaglio & Vannucci, 2007; Savaglio & Vannucci, 2009; Sen, 1991; Xu, 2004, among many others). In particular, Savaglio and Vannucci (2007) suggest the introduction of minimal opportunity thresholds, modelling them by *set-inclusion filtral preorders*.

A *set-inclusion filtral preorder* on a finite set X of basic opportunities amounts to the standard set-inclusion partial order as augmented with a *minimal opportunity threshold* which is induced by an *order-filter* (to be defined below). Under the threshold, opportunity sets are indifferent to each other and to the null opportunity set, whereas, over the threshold *the set-inclusion partial order is simply replicated*. Therefore, the behaviour of a set-inclusion filtral preorder (henceforth SIFP) over the threshold is arguably *non-controversial,* at least in the following sense: if over the threshold, plausible preferences on opportunity sets are taken to be monotonic with respect to set-inclusion, SIFPs include their common core. On the other hand, since the threshold itself can be chosen in many different ways, SIFPs—unlike e.g. the cardinality preorder—also accommodate a non-negligible diversity of judgments concerning the most appropriate ranking of opportunity sets. Thus, SIFPs can be regarded as a format for opportunity rankings that, building upon a common and essentially `objective' basis, gives some scope to a modicum of diversity in judgments, and therefore strikes a balance between any such judgment and the former common basis. Distinct choices of the relevant threshold enable an assessment of the extent of opportunity deprivation as seen from different perspectives. Furthermore, SIFPs are amenable to nice *and strategy-proof* aggregation methods including majority voting (see Savaglio & Vannucci, 2012; Vannucci, 1999). Therefore, even the threshold of any given SIFP may be possibly regarded as the outcome of a fair amalgamation of reliable information on *private judgments of experts and/or stakeholders concerning the most appropriate choice of minimal living standards*. Indeed, under a judicious selection of the set of basic opportunities, SIFPs arguably embody valuable information that is both reliable and comparatively easy to collect. That information can be deployed to assess several aspects of extant inequality and its social perception e.g. by majorization criteria. The implied assessments of the relevant opportunity distributions in terms of inequality and deprivation may in turn *help diagnose those critical situations of severe inequality and polarization that could strain and erode several key social networks, and undermine social cohesion*. All in all, carefully constructed SIFPs may arguably provide a key source of reliable *and* comparatively affordable information to several public and private agencies.

However, when it comes to using SIFPs as a key input for the assessment of opportunity inequality and polarization, one has to reckon with the central role of *majorization rankings* in the analysis of inequality. Hence, if inequality of opportunity profiles is to be assessed relying on a suitable *majorization preorder*, the very fact that a SIFP is, in general, a *non-total* preorder is undoubtedly a rather fastidious inconvenience. Two basic strategies may be devised to escape the foregoing difficulty while sticking to the notion of a majorization preorder, namely 1) reformulating (and generalizing) the majorization construct in order to adapt it to the general case of arbitrary non-total preorders, or 2) extending SIFPs to *total* preorders in a suitably 'natural' manner.

Strategy (α) is quite radical a move. Indeed, the main problem here is that majorization—as it is usually conceived—relies on comparisons between suitable sequences of pairs of partial *numerical sums* running over pairs of (α) *equally sized subpopulations,* each of which (β) forms a *maximal chain* with respect to the ranking of population units as induced by their respective individual endowments. Now, if the underlying preorder in endowment space is *partial* a typical joint effect of requirements (α) and (β) will consist in singling out *distinct numbers of relevant partial sums at different opportunity profiles*. But then, which partial sum of one opportunity profile should be compared to which partial sum of another one? It can be shown that, essentially, such a situation demands some special principled tactics to cope with pairs of non-isomorphic lattices of order filters as defined below. Thus, such an approach runs deep to the very foundations of the majorization construct.

By comparison, strategy (β) namely extending SIFPs to *total* preorders is much more conservative. One way to pursue strategy (β) is implicitly proposed and explored in Savaglio and Vannucci (2007). It consists in relying on the height function of a SIFP in order to extend the latter to a total preorder, resulting in the 'higher than' relation. Now, the height of an element x counts the size of the longest strictly ascending chain having x as its maximum. Does this notion qualify as a 'natural' extension of the underlying SIFP?

In general, the answer is admittedly bound to be disputable. In fact, the main problem here is that, generally speaking, SIFPs admit maximal strictly ascending chains of different sizes having the same minimum and the same maximum, i.e. SIFPs do *not* satisfy the so called Jordan-Dedekind chain condition, hence are *not* graded i.e do not have any rank function for their elements (see e.g. Barbut & Monjardet, 1970, for a general review of the foregoing notions as defined below in the text). Thus, reliance on height functions, which provide an instance of rank functions in the graded case, but are well-defined anyway, is a second-best choice of sorts. However, it turns out that in order to remedy that inconvenience, one may select a suitably defined well-behaved class of SIFPs. In the present paper, I focus on *principal* SIFPs, namely on those SIPFs whose threshold consists of precisely *one minimum living standard* (as opposed to a set of several mutually non-comparable minimal living standards). The reasons for doing so are the following. First, it is shown below that principal SIFPs are indeed graded and their height functions do amount to rank functions. This implies that in the principal case, heights provide a much more reliable numerical scale to rank the elements of a SIFP, than they do in the general case. Therefore, height-based extensions are arguably more 'natural' and strongly grounded for principal SIFPs than they are for general SIFPs. Second, Savaglio and Vannucci (2007) prove that principal SIFPs do support an opportunity-profile counterpart to the classic characterization theorems of the majorization preorder on real sequences due to Hardy, Littlewood and Pólya (1952).

Starting from the foregoing background and motivation, a simple characterization of height-based extensions of principal SIFPs is provided below. Our characterization relies on conditions that use the relevant filter as a fixed parameter. Indeed, it is quite clear that over the filtral threshold height-based extensions of principal SIFPs behave, essentially, as the cardinality-based preorder. Therefore, one should expect that a suitable reformulation of standard characterizations of the cardinality-based preorder would also work for height-based extensions of SIFPs. As a matter of fact, we show that SIFPs can indeed be characterized by a suitably adapted version of the axiom set employed by Pattanaik and Xu (1990) to obtain their well-known characterization of the cardinality-based preorder. We also show that in our setting a version of the Pattanaik-Xu characterization mentioned above is essentially recaptured as a special case which arises from a particular choice of the relevant filtral parameter. Arguably, that result highlights the significance of the cardinality-ranking as a benchmark and a limiting case within the larger family of SIFP-rankings, while confirming, at the same time, the remarkable scope and flexibility of SIFPs themselves.

MODEL AND RESULTS

Let $(X,\succcurlyeq)$ be a *preset* (i.e. $\succcurlyeq$ is a preorder, namely a reflexive and transitive binary relation on set X). We shall denote by $([X]_\sim,[\succcurlyeq]_\sim)$ its quotient partially ordered set or *quotient poset* w.r.t. the symmetric component $\sim$ of $\succcurlyeq$, namely the *antisymmetric* preset on the set $[X]_\sim$ of $\sim$-equivalence classes as defined by the rule $[x]_\sim[\succcurlyeq]_\sim[y]_\sim$ if and only if $x \succcurlyeq y$. If in particular $\succcurlyeq$ is *antisymmetric* then preset $(X,\succcurlyeq)$ itself is a *partially ordered set* or *poset.* A (non-empty) *antichain* of $(X,\succcurlyeq)$ is a (non-empty) set $Z \subseteq X$ such that for any $z_1, z_2 \in Z$ if $z_1 \neq z_2$ then z_1 and z_2 are *not* $\succcurlyeq$-comparable. For any (non-empty) antichain Z of a *finite* non-empty preset $(X,\succcurlyeq)$ an *order filter* of $(Z,\succcurlyeq)$ with *basis* Z is the minimal set $F = F(Z) \subseteq X$ such that $X \supseteq Z$ and for any $y,z \in X$ if $y \in F$ and $z \succcurlyeq y$ then $z \in F$.

Thus, whenever X is *finite*, an order filter F of preset $(X,\succcurlyeq)$ is uniquely defined by a finite set $Z = Z(F) = \{z_1,...,z_l\}$ such tha $F = \{x \in X : x \succcurlyeq z_i \text{ for some } i, i = 1,...l\}$: Z is also denoted as the *basis* of F.

In particular, if Z is a singleton i.e. $l = 1$ then F is said to be a *principal* order filter of $(X, \succcurlyeq)$. It should be remarked that if $(X, \succcurlyeq)$ is a *lattice* (namely, if $\succcurlyeq$ is antisymmetric and for any $x, y \in Y$ the pair $\{x, y\}$ has both a *least upper bound* w.r.t. $\succcurlyeq$, and a *greatest lower bound* w.r.t. $\succcurlyeq$, denoted by $x \vee y$ and $x \wedge y$, respectively) then a principal order fillter of $(X, \succcurlyeq)$ is also $\wedge$*-closed* or equivalently a *latticial* filter, namely $x \wedge y \in F$ whenever both $x \in F$ and $y \in F$ (it can also be shown that the converse also holds for any finite lattice).

A *chain* of $(X, \succcurlyeq)$ is a subset $Y \subseteq X$ which is *totally* (pre)ordered by $\succcurlyeq$ and such that for any $x, y \in Y$ $x \sim y$ only if $x = y$: by definition, the *length* of chain Y is $l(Y) = |Y| - 1$. A chain Y of $(X, \succcurlyeq)$ is *maximal* if there is no chain U of $(X, \succcurlyeq)$ such that $X' \subset U$. If $(X, \succcurlyeq)$ has a *minimum* or bottom element $\perp$, one may define its *height function* $h(\succcurlyeq)$: $X \to \mathbb{Z}_+ \cup \{\infty\}$ by declaring the *height* $h(\succcurlyeq)(x)$ of any $x \in X$ to be the lowest upper bound of the set of the lenghts of all (maximal) chains Y of $(X, \succcurlyeq)$ having x as their maximum.

A preset $(X, \succcurlyeq)$ is said to satisfy the *Jordan-Dedekind chain condition* if for any $x, y \in X$, and any pair of maximal chains Y, Z of $(X, \succcurlyeq)$ having x as their common minimum and y as their common maximum, $l(Y) = l(Z)$ i.e. equivalently $|Y| = |Z|$. Furthermore, a preset $(X, \succcurlyeq)$ is *graded* if it admits a *rank function* i.e. an integer-valued function $r : X \to \mathbb{Z}$ such that for any $x, y \in X$:

i) if $x \succ y$ then $r(x) > r(y)$ and ii) $r(x) = r(y) + 1$ whenever x *covers* y i.e. $x \succ y$ and $\{z \in X : x \succ z \succ y\} = \varnothing$.

We are now ready to turn to set-inclusion filtral preorders. We shall confine ourselves to a *finite* set X, and its power set $\mathcal{P} \boxplus X)$. For any order filter F of poset $(\mathcal{P}(X), \supseteq)$ the F-generated *set-inclusion filtral preorder* (SIFP) is the binary relation $\succcurlyeq_F$ on $\mathcal{P}(X)$ defined as follows: for all $A, B \in \mathcal{P}(X)$, $A \succcurlyeq_F B$ if and only if $A \supseteq B$ or $B \notin F$.

Let F be an order filter of $(\mathcal{P}(X), \supseteq)$ and $\succcurlyeq_F$ the (set-inclusion) filtral preorder induced by F.

Then, the $\succcurlyeq_F$*-induced height function* h_F : $\mathcal{P}(X) \to \mathbb{Z}_+$ is defined as follows: for any $A \subseteq X$,

$$h_F(A) = max \left\{ \begin{array}{c} |\mathbf{C}| : \mathbf{C} \text{ is a } \succcurlyeq_F - \text{chain such that} \\ A \in \mathbf{C} \text{ and } A \succ_F B \text{ for any } B \in \mathbf{C} \setminus \{A\} \end{array} \right\}.$$

The *height-based (total) extension* of $\succcurlyeq_F$ is the total preorder $\succcurlyeq_{h_F}$ defined as follows: for any $A, B \subseteq X$, $A \succcurlyeq_{h_F} B$ if and only if $h_F(A) \geq h_F(B)$.

As mentioned in the Introduction, the main aim of the present paper is to provide a characterization of the height-based total preorder $\succcurlyeq_{h_F}$ when the relevant order filter F is *principal*. Indeed, it turns out that in the latter case the SIFP $(\mathcal{P}(X), \succcurlyeq_F)$ is graded, hence the height function h_F is a well-defined rank function which provides an unambiguous criterion to assess the comparative 'status' of opportunity sets according to $\succcurlyeq_F$. This claim is made precise by the following

Proposition 1. *Let F be a principal order filter of $(\mathcal{P}(X),\supseteq)$. Then, the F-generated SIFP $(\mathcal{P}(X),\succcurlyeq_F)$ is a graded preset.*

Proof. It is a well-known fact that a finite poset—hence indeed any finite preset, by definition—is graded if and only if it satisfies the Jordan-Dedekind condition as defined above (see e.g. Barbut & Monjardet, 1970, chpt. 1, p.39). Thus, it suffices to show that $(\mathcal{P}(X),\succcurlyeq_F)$ does satisfy the latter condition. Indeed, suppose it does not. Then, by definition there exist $A,B\subseteq X$ and maximal chains $\mathbf{C}=\{[C_0]_{\sim F},..,[C_k]_{\sim F}\},\mathbf{C}'=\{[C_0']_{\sim F},..,[C_{k'}']_{\sim F}\}$ of $([\mathcal{P}(X)]_{\sim F},[\succcurlyeq]_{\sim F})$ with $C_i\succ_F C_{i+1}$, $i=0,.,k-1$, $C_i'\succ_F C_{i+1}'$, $i=0,.,k'-1$, $[C_0]_{\sim F}=[C_0']_{\sim F}=[B]_{\sim F}$, $[C_k]_{\sim F}=[C_{k'}']=[A]_{\sim F}$, and $|\mathbf{C}|\neq|\mathbf{C}'|$. Now, if $A=B$ or $\{A,B\}\cap F=\varnothing$ then by construction $|\mathbf{C}|=|\mathbf{C}'|=1$, a contradiction. Hence, $A\neq B$ and either $\{A,B\}\cap F=\{A\}$ or $\{A,B\}\subseteq F$ holds. If in fact, $\{A,B\}\cap F=\{A\}$ then $B\notin F$ hence, by definition of $\succcurlyeq_F$ and maximality of chains $\mathbf{C},\mathbf{C}'$, it must be the case that both C_1 and C_1' belong to the basis of F and for any $i\in\{1,..k-1\}$, $j\in\{1,..,k'-1\}$ there exist $x_i\in X\setminus C_i, y_j\in X\setminus C'_j$ such that $C_i=C_i\cup\{x_i\}$, $C_{j+1}'=C_j'\cup\{x_j\}$. But then, since F is principal, $C_1=C_1'$. It follows that, by construction, $k-1=|\mathbf{C}\setminus\{C_0,C_1\}|=|\mathbf{C}'\setminus\{C_0',C_1'\}|='-1$ whence $k=k'$, a contradiction. Finally, if $B\in F$ as well then again, by definition of $\succcurlyeq_F$ and maximality of chains $\mathbf{C},\mathbf{C}'$ it must be the case that for any $i\in\{0,..k-1\}$, $j\in\{0,..,k'-1\}$ there exist $x_i\in X\setminus C_i, y_j\in X\setminus C_j'$ such that $C_i=C_i\cup\{x_i\}$, $C_{j+1}'=C_j'\cup\{x_j\}$ whence by the same argument presented above $k=k'$, a contradiction, and the thesis is established.

Remark. Of course, a general SIFP need not be graded. To check this fact, consider the following elementary example: let $X=\{x,y,z\}$, F the order filter of $(\mathcal{P}(X),\supseteq)$ having $\{\{x\},\{y,z\}\}$ as its basis, and $(\mathcal{P}(X),\succcurlyeq_F)$ the resulting SIFP. Then consider $\mathbf{C}=\{[X]_{\sim F},[\{x,y\}]_{\sim F}, [\{x\}]_{\sim F},[\varnothing]_{\sim F}\}$ and $\mathbf{C}'=\{[X]_{\sim F},[\{y,z\}]_{\sim F},[\varnothing]_{\sim F}\}$. Notice that $\mathbf{C}$ and $\mathbf{C}'$ are two maximal chains of $([\mathcal{P}(X)]_{\sim F},[\succcurlyeq_F]_{\sim F})$ of different size (and length), having $[X]_{\sim F}$ as their common maximum and $[\varnothing]_{\sim F}$ as their common minimum. Thus, $(\mathcal{P}(X),\succcurlyeq_F)$ does *not* satisfy the Jordan-Dedekind chain condition and as a consequence -being finite- is *not* graded.

Let us now proceed to the announced characterization of $\succcurlyeq_{h_F}$. In order to accomplish that task, a few more definitions are needed.

Let F be any (non-empty) principal order filter of the (finite) poset $(\mathcal{P}(X),\supseteq)$, i.e. equivalently a (non-empty) latticial filter of the (finite) lattice $(\mathcal{P}(X),\cup,\cap)$. Then, for an arbitrary binary relation $\succcurlyeq$ on $\mathcal{P}(X)$ (with asymmetric and symmetric components denoted as usual by $\sim$ and $\succ$, respectively) the following F-parameterized properties can be defined:

F-Restricted Indifference between Singletons (F-RIS):

$(\mathcal{P}(X),\succcurlyeq)$ satisfies F-RIS if for all $A\in F$ and $x,y\in X\setminus A$, $A\cup\{x\}\sim A\cup\{y\}$.

F-Restricted Strict Monotonicity (F-RSM):

$(\mathcal{P}(X),\succcurlyeq)$ satisfies F-RSM if for all $A \in F$ and $x,y \in X$ such that $x \neq y$, $y \notin A$ entails $A \cup \{x,y\} \succ A \cup \{x\}$.

F-Restricted Independence (F-RIND):

$(\mathcal{P}(X),\succcurlyeq)$ satisfies F-RIND if for all $A,B \in F$ and $x \in X$, $x \notin A \cup B$ and $A \succcurlyeq B$ if and only if $A \cup \{x\} \succcurlyeq B \cup \{x\}$.

F-Threshold Effect (F-TE):

$(\mathcal{P}(X),\succcurlyeq)$ satisfies F-TE if $A \succ B \sim \varnothing$ for all $A,B \subseteq X$ such that $\varnothing \neq A \in F$ and $B \in \mathcal{P}(X) \setminus F$.

It turns out that, in general, the foregoing properties are *not* mutually independent. Indeed, we have the following:

Proposition 2. *Let F be a principal filter of $(\mathcal{P}(X),\supseteq)$ and $(\mathcal{P}(X),\succcurlyeq)$ a preset which satisfies both F-RIS and F-RSM. Then $(\mathcal{P}(X),\succcurlyeq)$ satisfies F-RIND as well.*

Proof. Let us assume that $A,B \in F, x \in X \setminus (A \cup B)$. Since, by definition of F, there exists $Y \subseteq X$ such that $F = \{C \subseteq X : C \supseteq Y\}$, it follows that there also exist non-negative integers h,k and $\{a_1,..,a_h\} \subseteq X \setminus Y$, $\{b_1,..,b_k\} \subseteq X \setminus Y$ such that $A = Y \cup \{a_1,..,a_h\}$, $B = Y \cup \{b_1,..,b_k\}$.

Now, suppose $A \succcurlyeq B$. If $h < k$ then $A \sim Y \cup \{b_1,..,b_h\}$ by a repeated application of F-RIS. Therefore, by a repeated application of F-RSM, $B \succ Y \cup \{b_1,..,b_h\} \sim A$ whence, by transitivity of $\succcurlyeq$, $B \succ A$, a contradiction. Let us then assume without loss of generality that $h \geq k$. Thus, by a repeated application of F-RIS to $Y \cup \{x\}$, $Y \cup \{a_1,..,a_k,x\} \sim Y \cup \{b_1,..,b_k,x\}$. If $h = k$, then $A \cup \{x\} \sim B \cup \{x\}$ follows immediately. Otherwise, $A \cup \{x\} = Y \cup \{a_1,..,a_h,x\} \succ Y \cup \{b_1,..,b_k\}$ follows by a repeated application of F-RSM, and by transitivity of $\succcurlyeq$.

Conversely, let us assume that $A \cup \{x\} \succcurlyeq B \cup \{x\}$. If $A = Y \cup \{a_1,..,a_h\} \succcurlyeq B = Y \cup \{b_1,..,b_k\}$ does *not* hold, then it must be the case that $h < k$. But then, it follows by a repeated application of F-RIS to $Y \cup \{x\} \in F$ that $Y \cup \{b_1,..,b_h,x\} \sim Y \cup \{a_1,..,a_h,x\} = A \cup \{x\}$. Thus, by a repeated application of F-RSM and by transitivity of $\succcurlyeq$, it also follows that $B \cup \{x\} \succ A \cup \{x\}$, a contradiction. Hence $A \succcurlyeq B$, and F-RIND holds.

We are now in a position to state and prove the main characterization result of the present paper.

Theorem 3. *Let F be a principal filter of $(\mathcal{P}(X),\supseteq)$ and $(\mathcal{P}(X),\succcurlyeq)$ a preset. Then, $\succcurlyeq$ is the height-based (total) extension $\succcurlyeq_{h_F}$ of the set-inclusion principal filtral preorder $\succcurlyeq_F$ if and only if $(\mathcal{P}(X),\succcurlyeq)$ satisfies F-RIS, F-RSM and F-TE.*

Proof. It is straightforward to check that $(\mathcal{P}(X),\succcurlyeq_{h_F})$ is in fact a totally preordered set that satisfy F -RIS, F -SM, and F -TE. Indeed, let $F = \{A \subseteq X : A \supseteq Y\}$ where $Y \subseteq X$. If $A \in F$, and $x,y \in X \setminus F$ then by definition $h_F(A \cup \{x\}) = h_F(A \cup \{y\}) = |A \setminus Y| + 1$ whence $A \cup \{x\} \sim_F A \cup \{y\}$. Moreover, $A \in F$, $x,y \in X$ and $y \notin A$ clearly entail $h_F(A \cup \{x,y\}) = |A \setminus Y| + 3$ and

$h_F(A \cup \{x\}) = |A \setminus Y| + 2$ if $x \notin A$, while $h_F(A \cup \{x,y\}) = |A \setminus Y| + 2$ and $h_F(A \cup \{x\}) = |A \setminus Y| + 1$ if $x \in A$: in any case, by definition, $A \cup \{x,y\} \succ_{h_F} A \cup \{x\}$. Finally, for all $A \in \mathcal{P}(X) \setminus F$ and $B \in F$, $h_F(A) = 0 = h_F(\varnothing)$ while $h_F(B) \geq 1$ i.e. $B \succ_{h_F} A \sim_{h_F} \varnothing$.

Conversely, let $(\mathcal{P}(X), \succcurlyeq)$ be a preset that satisfies F-RIS, F-RSM and F-TE (hence in particular F-RIND by Proposition 2. To begin with, we define an auxiliary function $l_F : \mathcal{P}(X) \to \mathbb{N}$ as follows: for any $A \subseteq X$, $l_F(A) = max\{|A \setminus Z| : Z \in F, Z \subseteq A\} = |A \setminus Y|$, if $A \supseteq Y$ and $l_F(A) = -1$ otherwise (i.e. $l_F(A) = h_F(A) - 1$: l_F is the so-called *length function* of $(\mathcal{P}(X), \succcurlyeq_F)$).

Next, we show that since $(\mathcal{P}(X), \succcurlyeq)$ satisfies F-RIS and F-RIND it follows that for any $A, B \in F$:

$l_F(A) = l_F(B)$ entails $A \sim B$ (or equivalently $h_F(A) = h_F(B)$ entails $A \sim B$).

We proceed by induction on $l_F(A)$. The case $l_F(A) = l_F(B) = 0$ is trivial in that it entails - by definition - $A = Y = B$ whence $A \sim B$.

Let us now suppose by inductive hypothesis that for any nonnegative integer m not larger than n, $l_F(A) = l_F(B) = m$ entails $A \sim B$. Then, take a pair $C, D \subseteq X$ such that $l_F(C) = l_F(D) = n+1$. If $C = D$ there is nothing to prove. If $C \neq D$ then there exist $x, y \in X$ and $A, B \subseteq X$ such that $A \cap B \supseteq Y$, $|A \setminus Y| = |B \setminus Y| = n$, $x \notin B$, $y \notin A$ and $C = A \cup \{x\}$, $D = B \cup \{y\}$. It follows that $\{x,y\} \cap Y = \varnothing$ hence, by definition, $l_F(A) = l_F(C) - 1 = l_F(D) - 1 = l_F(B) = n$, which entails $A \sim B$, by the inductive hypothesis. Moreover, if $x \in A$ then $C = A$, a contradiction since $l_F(A) \neq l_F(C)$, thus indeed $x \notin A \cup B$, and $y \notin A \cup B$ by a similar argument. Therefore, $A \cup \{x\} \sim B \cup \{x\}$ (and $A \cup \{y\} \sim B \cup \{y\}$) by F-RIND. Furthermore, $A \cup \{x\} \sim A \cup \{y\}$ (and $B \cup \{y\} \sim B \cup \{x\}$) by F-RIS. As a result, $A \cup \{x\} \sim B \cup \{y\}$ i.e. $C \sim B$, by transitivity of $\sim$, and the inductive thesis follows.

Now, take any pair $A, B \subseteq X$ such that $h_F(A) > h_F(B)$ or equivalently $l_F(A) > l_F(B)$. Two cases should be distinguished, namely: i) $A \supseteq Y$ and $B \not\supseteq Y$; $ii) A \cap B \supseteq Y$. If case 1) obtains, then, by definition of F, $A \in F$ and $B \notin F$ hence $A \succ B$ by F-TE. Under case 2) both $A \in F$ and $B \in F$, and there exist $A' \subseteq X \setminus Y$, $B' \subseteq X \setminus Y$ such that $A = Y \cup A'$, $B = Y \cup B'$ and $|A'| > |B'|$. Then, there also exists $A'' \subset A'$ such that $|A''| = |B'|$. We also posit $|A' \setminus A''| = k$ and $A' = A'' \cup \{x_1, .., x_k\}$. Therefore, $l_F(Y \cup A'') = l_F(Y \cup B') = l_F(B)$ whence $(Y \cup A'') \sim B$ by the first part of this proof. Since $Y \cup A'' \in F$, it follows from F-RSM that $Y \cup A'' \cup \{x_1\} \succ Y \cup A''$. By a repeated application of a similar argument - and by transitivity of $\succ$ - we can eventually establish that $A \succ Y \cup A'' \sim B$ whence $A \succ B$.

Thus, we have just shown that for any $A, B \subseteq X : h_F(A) = h_F(B)$ entails $A \sim B$ and $h_F(A) > h_F(B)$ entails $A \succ B$, i.e. $(\succcurlyeq_{h_F}) \subseteq (\succcurlyeq)$. Hence, in particular, $\succcurlyeq$ is a *total* preorder. But notice that if there exist $A, B \subseteq X$ such that $A \succcurlyeq B$ and *not* $A \succcurlyeq_{h_F} B$, then -since $\succcurlyeq_{h_F}$ is also a *total* preorder by definition - it must be the case that $B \succcurlyeq_{h_F} A$ hence $h_F(B) \geq h_F(A)$ and therefore $B \succcurlyeq A$ since $(\succcurlyeq_{h_F}) \subseteq (\succcurlyeq)$. Moreover, *not* $A \succcurlyeq_{h_F} B$ entails $h_F(B) > h_F(A)$ and *not* $A \sim B$, whence $B \succ A$, a contradiction. It follows that $(\succcurlyeq) \subseteq (\succcurlyeq_{h_F})$ as well, so that $(\succcurlyeq) = (\succcurlyeq_{h_F})$ and our thesis is thus established.

The foregoing characterization is tight. To see this, consider the following list of examples.

Example 1. Take a principal order filter F of $(\mathcal{P}(X), \supseteq)$ and the corresponding set-inclusion filtral preorder $\succcurlyeq_F$ on $\mathcal{P}(X)$ defined as follows: for any $A, B \subseteq X$, $A \succcurlyeq_F B$ if and only if $[A \supseteq B \;\; B / \in F]$ (see Vannucci, 1999; Savaglio & Vannucci, 2007). It is easily checked that $\succcurlyeq_F$ is indeed a preorder, and satisfies F-RSM and F-TE. Moreover, let $A \in F, B \in F$ and $x \in X \setminus (A \cup B)$. Thus, $A \succcurlyeq_F B$ entails $A \supseteq B$ whence $A \cup \{x\} \supseteq B \cup \{x\}$ which in turn entails $A \cup \{x\} \succcurlyeq_F B \cup \{x\}$. Conversely, since obviously $\{A \cup \{x\}, B \cup \{x\}\} \subseteq F$, $A \cup \{x\} \succcurlyeq_F B \cup \{x\}$ entails $A \cup \{x\} \supseteq B \cup \{x\}$. Then $A \supseteq B$ as well, hence by definition $A \succcurlyeq_F B$. It follows that $\succcurlyeq_F$ also satisfies F-RIND. However, for any $A \in F$ and $x, y \in A$ such that $x \neq y$, $A \cup \{x\}$ and $A \cup \{y\}$ are *not* $\succcurlyeq_F$-comparable, hence F-RIS fails.

Example 2. Let us consider again a principal order filter F of $(\mathcal{P}(X), \supseteq)$, and the binary relation $\succcurlyeq_F^{\#\partial} \subseteq \mathcal{P}(X) \times \mathcal{P}(X)$ defined as follows: for any $A, B \subseteq X$, $A \succcurlyeq_F^{\#\partial} B$ if and only if [either $(B \in F, A \in F$ and $|B| \geq |A|)$ or $B \notin F$]. Notice that $\succcurlyeq_F^{\#\partial}$ is indeed a preorder: to check this, first observe that reflexivity of $\succcurlyeq_F^{\#\partial}$ follows trivially from the definition, and assume that $A \succcurlyeq_F^{\#\partial} B$ and $B \succcurlyeq_F^{\#\partial} C$. The following mutually exclusive and jointly exhaustive cases should be distinguished:

1) $|B| \geq |A|, |C| \geq |B|$ and $\{A, B, C\} \subseteq F$: in this case $|C| \geq |A|$ hence $A \succcurlyeq_F^{\#\partial} C$ by the first clause; 2) $|B| \geq |A|, \{A, B\} \subseteq F$ and $C \notin F$: in this case $A \succcurlyeq_F^{\#\partial} C$ by the second clause; 3) $B \notin F$ and $C \notin F$: here again $A \succcurlyeq_F^{\#\partial} C$ follows immediately from the second clause. Thus, $A \succcurlyeq_F^{\#\partial} C$ is transitive. Also, if $A \in F$, $x \notin A$ and $y \notin A$, then clearly $A \cup \{x\} \subseteq F$, $A \cup \{y\}\} \subseteq F$ and $|A \cup \{x\}| = |A \cup \{y\}|$ whence by definition $A \cup \{x\} \sim_F^{\#\partial} A \cup \{y\}$ i.e. F-RIS is satisfied. Similarly, if $A \in F$, $B \in F$, $x \in X \setminus (A \cup B)$ and $A \succcurlyeq_F^{\#\partial} B$ then $|B| \geq |A|$ and $\{A \cup \{x\}, B \cup \{x\}\} \subseteq F$. Thus, $|B \cup \{x\}| \geq |A \cup \{x\}|$ whence by definition $A \cup \{x\} \succcurlyeq_F^{\#\partial} B \cup \{x\}$. Conversely, if $A \in F$, $B \in F$, $x \in X \setminus (A \cup B)$ and $A \cup \{x\} \succcurlyeq_F^{\#\partial} B \cup \{x\}$ then, by definition $|B \cup \{x\}| \geq |A \cup \{x\}|$: it follows that $|B| \geq |A|$ as well hence by definition $A \succcurlyeq_F^{\#\partial} B$. Therefore, F-RIND is also satisfied by $(\mathcal{P}(X), \succcurlyeq_F^{\#\partial})$. Finally, F-TE follows immediately from the definition. However, F-RSM is definitely *not* satisfied by $(\mathcal{P}(X), \succcurlyeq_F^{\#\partial})$: indeed, if $A \in F$, $x \in X \setminus A$, $y \in X \setminus A$ and $x \neq y$ then, by definition, $A \cup \{x\} \succcurlyeq_F^{\#\partial} A \cup \{x, y\}$ hence F-RSM is violated.

Example 3. Fix a principal order filter F of $(\mathcal{P}(X), \supseteq)$ and take the binary relation $\succcurlyeq_F^{\#} \subseteq \mathcal{P}(X) \times \mathcal{P}(X)$ defined as follows: for any $A, B \subseteq X$, $A \succcurlyeq_F^{\#} B$ if and only if either ($A \notin F$) or ($A \in F, B \in F$ and $|A| \geq |B|$). It can be readily checked that $\succcurlyeq_F^{\#}$ is a preorder: to see this, observe that reflexivity follows immediately from the definition. As for transitivity, if $A \succcurlyeq_F^{\#} B$ and $B \succcurlyeq_F^{\#} C$ then the following two mutually exclusive and jointly exhaustive cases are to be distinguished: i) $A \notin F$, and ii) $A \in F$, $B \in F$, $C \in F$, $|A| \geq |B|$ and $|B| \geq |C|$. In both cases, $A \succcurlyeq_F^{\#} C$ follows immediately from the definition. Furthermore, F-RIS and F-RSM of $\succcurlyeq_F^{\#}$ are also easily seen to follow trivially from the definition. On the other hand, $(\mathcal{P}(X), \succcurlyeq_F^{\#})$ obviously fails to satisfy F-TE since by definition *not* $A \succcurlyeq_F^{\#} B$ for any $A, B \subseteq X$ such that $\varnothing \neq A \in F$ and $B \notin F$.

It should be emphasized here that the axioms used by Pattanaik and Xu (1990) in their well-known, and seminal, characterization of the cardinality-based preorder —namely Indifference between Singletons, Strict Monotonicity, and Independence— are implied by the corresponding axioms in our list when the reference filter F is taken to be the trivial or maximum filter $\mathcal{P}(X)$. Moreover, it is immediately visible that for $F = \mathcal{P}(X)$ the fourth axiom of our list i.e. F-Threshold Effect, which has no counterpart in the Pattanaik-Xu list, is, in fact, trivially satisfied when restricted to the original Pattanaik-Xu domain, which only includes *non-empty* opportunity sets. The remarkable flexibility and scope of SIFPs is thereby confirmed.

CONCLUDING REMARKS

As mentioned in the Introduction, the characterization of height-based extensions of principal filtral opportunity preorders provided in the present paper does not extend to the general case of arbitrary filtral opportunity preorders. This is due to the fact that when an order filter is *not* principal, the height function of the corresponding SIFP may exhibit a highly irregular behaviour.

Therefore, the height-based extension of a SIFP does *not* mimic the behaviour of the cardinality-based preorder over the filtral threshold. A simple example may help clarify this point.

Example 4. Let $X = \{x_1, .., x_7\}$, $Z = \{\{x_1, x_2\}, \{x_3, x_4, x_5, x_6\}, \{x_7\}\}$, and $F = F(Z)$ (notice that Z is indeed an antichain of $(\mathcal{P}(X), \supseteq)$). Then, consider the height-based extension $(\mathcal{P}(X), \succcurlyeq_{h_F})$ of the F-induced SIFP $(\mathcal{P}(X), \succcurlyeq_F)$ and take $A = \{x_3, x_4, x_5, x_6\}$. Clearly, $A \in F$. However, $h_F(A \cup \{x_1\}) = 2$ while $h_F(A \cup \{x_7\}) = 5$, hence $A \cup \{x_7\} \succ_{h_F} A \cup \{x_1\}$ and F-RIS fails.

By contrast, our characterization is, in fact, amenable to a simple generalization in another direction. Indeed, a counterpart to Theorem 3 for arbitrary (finite) lattices of sets is readily available provided that the axioms are suitably reformulated by replacing join-irreducibles (elements that cover precisely one element) for singletons/atoms. The details of that extension, however, are best left as a topic for future research.

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