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Roig, Ana-Isabel; Llinares, Salvador; Penalva, Maria del Carmen
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Different Moments in the Participatory Stage of the Secondary Students' Abstraction of Mathematical Conceptions

Diferentes Momentos na Abstração de Conceitos Matemáticos de Estudantes Secundários

Ana-Isabel Roig[*]
Salvador Llinares[**]
Maria del Carmen Penalva[***]

Abstract

This study provides support to the characteristics of participatory and anticipatory stages in secondary school pupils' abstraction of mathematical conceptions. We carried out clinical task-based interviews with 71 secondary-school pupils to obtain evidence of the different constructed mathematical conceptions (Participatory Stage) and how they were used (Anticipatory Stage). We distinguish two moments in the Participatory Stage based on the coordination of information from particular cases by activity-effect reflection which, in some cases, lead to a change of focus enabling secondary-school

[*] Doctora em Didactica de la Matemática por la Universidad de Alicante, UA. Profesora del Departamento de Innovación y Formación Didáctica, Facultad de Educación, Universidad de Alicante – UA, Campus de San Vicente del Raspeig, Alicante, España. Endereço para correspondência: Departamento de Innovación y Formación Didáctica, Campus de San Vicente del Raspeig, 03080, Alicante, España. *E-mail*: aroigal@hotmail.com

[**] Doctor en Ciencias de la Educacion por la Universidad de Sevilla, US. Profesor del Departamento de Innovación y Formación Didáctica, Facultad de Educación. Universidad de Alicante – UA, Campus de San Vicente del Raspeig, Alicante, España. Endereço para correspondência: Departamento de Innovación y Formación Didáctica, Campus de San Vicente del Raspeig, 03080, Alicante, España. *E-mail*: sllinares@ua.es

[***] Doctora en Matemática por la Universidad de Valencia, UV. Profesora del Departamento de Innovación y Formación Didáctica, Facultad de Educación, Universidad de Alicante – UA, Campus de San Vicente del Raspeig, Alicante, España. Endereço para correspondência: Departamento de Innovación y Formación Didáctica, Campus de San Vicente del Raspeig, 03080, Alicante, España. *E-mail*: carmina.penalva@ua.es

pupils to achieve a reorganization of their knowledge. We argue that (a) the capacity of perceiving regularities in sets of particular cases is a characteristic of activity-effect reflection in the abstraction of mathematical conceptions in secondary school, and (b) the coordination of information by pupils provides opportunities for changing the attention-focus from the particular results to the structure of properties.

Key words: Mathematical Abstraction. Activity-effect Reflection. Reflective Abstraction. Formation of Mathematical Objects. Participatory and Anticipatory Stages.

Resumo

Este estudo discute características das etapas participativa e antecipatória da abstração de conceitos matemáticos em alunos do ensino secundário. Foram realizadas 71 entrevistas clínicas baseadas em tarefas com alunos desse nível de ensino para obter evidências sobre a construção de diferentes conceitos matemáticos (Etapa Participativa) e sobre como eles foram mobilizados (Etapa Antecipatória). Na etapa participativa distinguem-se dois momentos. Argumenta-se que (a) a capacidade de perceber regularidades em um conjunto de casos particulares é uma característica da reflexão atividade-efeito na abstração de conceitos matemáticos no ensino secundário, e (b) a coordenação das informações por parte dos alunos cria oportunidades para alterar o foco de atenção do caso particular para a estrutura das propriedades.

Palavras-chave: Abstração Matemática. Reflexão Atividade-Efeito. Abstração Reflexionante. Formação de Objetos Matemáticos. Etapas Participativa e Antecipatória.

1 Introduction

Understanding how mathematical conceptions are constructed can help in thinking about teaching with the aim of encouraging learning. In this sense it is essential to have accurate fine descriptions of the processes by which mathematical knowledge is developed. This situation generates issues about what it means to know something about mathematical objects, and how the learner develops or constructs that knowledge (DÖRFLER, 2002; DUBINSKY, 1991; HERSHKOWITZ; SCHWARTZ; DREYFUS, 2001). Cognitive theories based on Piagetian stances assume that mathematical conceptions reflect regularities from human actions and mental operations. Piaget (2001) postulated reflective abstraction as the process by which new, more advanced conceptions develop from existing ones. Reflective abstraction is understood as the process by which higher-level mental structures could develop from lower-level structures through two phases: a Projection Phase in which the actions at one level become the objects of reflection at the next level, and a Reflection Phase in which reorganization takes place. These two phases as postulated by Piaget are described from a general viewpoint and provide too little information to make them operatively useful in the analysis of the processes of construction of mathematical conceptions. These two phases in the construction process have been treated differently by different researchers (TALL; THOMAS; DAVIS; GRAY; SIMPSON, 1999). Dörfler (2002), for instance, postulates that the construction process must be supplemented by a deliberate decision to view, treat, use and investigate a structure or a collection of items as a unified object, in which the formation of mathematical objects is conceived as a point of view taken by the learner. Simon and his colleagues (SIMON et al., 2004) postulate the existence of a mechanism that they call Reflection on Activity-Effect Relationship to explain this construction process. To make this process clear Simon et al. (2004) suggest that the records of experience from which this abstraction derives are records of learners' activity associated with the effects of that activity. This is a way of explaining the Projection Phase and Reflection Phase postulated by Piaget in the process of constructing new mathematical conceptions. Taking into account the two phases of reflective abstraction (Projection Phase and Reflection Phase) described by Piaget (2001), Tzur and Simon (2004) point out that in the Projection Phase where the actions at one level become the objects of the reflection at the next level, learners sort the activity-effect records in terms of the established goal, distinguishing between the records that get closer to their goal and those that do not. During the solution of a problem, the student may call up a mathematical conception already constructed (anticipatory stage), but in cases in which this conception doesn't exist, learners trigger certain actions whose goal is to obtain information in order to solve the problem.

In this case the process of solving the problem produces information from the activity-effect reflection, thereby enabling the learner to anticipate the effects of new activities (LLINARES; ROIG, 2008). In the Participatory Stage of the abstraction process of a mathematical conception, a learner who is asked to solve a problem in which this mathematical conception is relevant sorts the activity-effect records, compares, relates, searches and coordinates information to identify a regularity that is the emergence of the mathematical conception. Initially, the conception is only available in the context of the activity through which it has been developed. That is to say, the abstracted conception (the regularity) is provisional and unstable; since, in other contexts (problem situations) learners may not be able to recognise on their own the involvement of the conception and come to depend on external prompts. In this process, one key notion is the learners' *awareness* of the new structure (DÖRFLER, 2002;

GARCÍA; LLINARES; SÁNCHEZ-MATAMOROS, 2010). The extent to which a learner is aware of a constructed conception is considered an indicator of the construction process. That is to say, a learner at some point decides to consider something (e.g. a regularity in the relation between an activity and its effect) as an object in itself and then treats and uses it as an integrated whole.

Simon and his colleagues (SIMON et al., 2004; TZUR, 2007) have developed case studies to explain the mechanism of Reflection on Activity-Effect Relationship and the learners' construction process, and have provided empirical evidence to support theoretical stage-distinction in the process of constructing a new mathematical conception with primary school children in the context of teaching experiments. Here, our focus is on the cognitive characteristics of the construction process in different mathematical conceptions. As a consequence, from a methodological perspective we need to consider several mathematical conceptions and one broad sample of students. The goal of this research is to provide empirical support to (i) a hypothetical distinction between a Participatory Stage and an Anticipatory Stage in the abstraction of mathematical conceptions with secondary school pupils and (ii) a finer description of how the participatory stage proceeds.

2 Method

2.1 Participants and instruments

511 secondary school pupils in the last year of compulsory education (15-16 years old) answered a questionnaire containing five mathematical problems in the domains of variability, divisibility, and generalization as a part of a previous study about the use of mathematical knowledge as tool in problem solving. The analysis of the replies to the problems displayed the secondary students' diverse behaviours. To obtain further information we carried out 40-minute task-based clinical interviews with 71 of these secondary students. The interviews focused on how the mathematical conceptions were used during problem solving as a manifestation of the conception constructed. There were 38 boys and 33 girls. The clinical interview provided us with a context in which it was possible to identify how some students generated an activity leading to the possibility of recording and obtaining information about the activity-effect relation. The data come from audio-recordings of secondary students' justifications and their written answers to the five problems. Findings presented in this paper come from the analysis of the 71 task-based clinical interviews.

There were two pattern-generalization problems and three word-problems (see Figure 1 for examples of the problems used). Each pattern-generalization problem was composed of three kinds of tasks identified by Tzur (1999) as tasks that can lead the learner to the construction of a new mathematical structure. The aim of these tasks is that the student should carry out actions that could later be used to facilitate the coordination of information by comparing, relating, and searching in order to obtain the abstraction of a pattern.

P1: THE JOB OFFER
Job offers for pizza delivery workers have appeared in a local newspaper.
Pizza takeaway A pays each delivery worker 0.6 euros for each pizza delivered and a fixed sum of 60 euros a month. Pizza takeaway B pays 0.9 euros for each pizza delivered and a fixed sum of 24 euros a month.
Which do you think is the better-paid job?
Make a decision and explain why your choice is the better one.

P4: THE DANCE FLOOR
A floor-tile manufacturer has donated a quantity of floor tiles to the festival committee. Each tile is 33 centimetres long and 30 centimetres wide. The committee has decided to lay a square dance floor within the festival enclosure, but you have to tell them:
a) the length of each side of the smallest square that can be made with this size of tile without cutting any of them
b) what other sizes of square dance-floors could be laid using only uncut tiles of this size, and why?
In your reply to the committee, explain what you have done.

P5: THE SEQUENCE

The following representation was found in an ancient document:

```
* *        * * *       * * * *
 *          * *         * * *
```

Continue the graphic sequence up to the 5^{th} *term.*
Express with numbers the first five terms of the sequence.
What is the number at the 10^{th} *position?*
What is the number at the 25^{th} *position? What is the one at the* 50^{th} *position?*
What is the number at the n position?
Solve the task and explain how you have done it.

Figure 1 – Examples of problems used

To solve the three word-problems students were asked to make a decision regarding the situation and to express the reasons why they thought their decision was the right one. In contrast to the pattern-generalization problems, these problems do not include prompts to guide the student. The mathematical conceptions in each problem were the comparison of linear functions (P1), the identification of the minimum value of a rational function (P2), the conception of an odd number as an even number minus/plus 1 (P3), the idea of the lowest common multiple (lcm) (P4) and the arithmetical progression ($a_n = 2n + 1$) (P5). The mathematical conceptions were different in each problem, and provided the means to supply the necessary information in order to infer general characteristics of the activity-effect relationship reflection. The set of tasks taking into account different mathematical conceptions and level of prompts was designed to trigger the coordination process of information and to facilitate the identification of changes in the students' awareness of the structure of the situation.

The interviews were carried out after the pupils had completed the questionnaire and the researchers undertook a first analysis of their replies. The aim of the clinical interview was to get the pupils to verbalise their thought-processes used in solving the problems (GOLDIN, 2000) in order to obtain evidence of cognitive characteristics of the use of the mathematical conception. During the interviews the pupils had their written answers with them. The interviewer had a prior interview script, constructed according to the characteristics of each problem and the type of answer given by the pupils. In any case, the interviewer could modify her questions in view of the pupil's behaviour, in order to clarify or investigate more deeply the reasoning processes followed. To undertake the analysis, the interviews were transcribed and the text was integrated with the answers given in the questionnaire. In order to simplify the analysis of the interviews, each intervention was numbered, with E representing the interviewer and A the interviewee.

2.2 Data Analysis

There were two phases in the analysis process. Firstly, the pupils' responses to the problems and the interviews were analysed using a constant-comparative methodology (STRAUSS; CORBIN, 1994) taking into account the way in which each pupil set up and used elements of mathematics knowledge as tools in order to interpret the situation and then make a decision. We characterised the way in which the pupils' answers indicated the construction and/or use of a mathematical conception in each problem. We identified the way in which they considered the variability of the quantities, the conditions that had to be fulfilled by these quantities in the situations given, and the way in which they discerned generalities from the registers of particular data. Next, we interpreted these characteristics from the process involving the pupils' goal-directed activity and the reflection process. The discrepancies in the interpretations of the pupils' behaviour were discussed by the two researchers, which led to modifications in the initial interpretation of the answer or modifications in the characterisation of the stages in the abstraction process (CLEMENT, 2000). Secondly, we considered the characteristics and the interpretations generated and classified them according to the stage distinction. Here, we considered that the effect of reflection on the activity-effect relationship involves a coordination of the available conceptions. We identified different moments in the Participatory Stage with similar characteristics in the different mathematical conceptions taking into account how students created records of experience, sorted and compared the records, and identified patterns in those records. In this case, the processes that we were characterising in this way were constructive since we were able to identify how pupils reflected on a pattern in an activity-effect relationship that led to an anticipation (a mathematical conception necessary in the given situation) that in some cases could be provisional and unstable considering the student's capacity to access it.

3 Results

Table 1 shows the results obtained from the combined analysis of the interviews and the answers of the questionnaire. These results reveal different student performances in each problem. This variation in the students' performance supports the idea that the problems offered a wide range of opportunities to infer the characteristics of the use of mathematical conceptions and the abstraction process generated by the pupils and can be considered suitable contexts to inquiry about how mathematical conceptions are used. In addition to it, rather more than 10% of the total number of answers indicated that students had anticipated the mathematical conception in the situation (Anticipatory Stage). For example, in problem 1 some students recognized the comparison of linear functions for their solving process or in problem 4 some applied the idea of lowest common multiple. Problem 4 (*The dance floor*) provided the highest number of Anticipatory Stage answers, showing that the pupils recognised and used the idea of the lowest common multiple in this situation. In problem 5 (*The sequence*) nearly 17% of the pupils identified the conception of arithmetical progression (with d=2) and used it to solve the problem.

Table 1 – Percentages in Different Stages of the Abstraction Process

	Problem 1 *The job offer*	Problem 2 *The festival enclosure*	Problem 3 *Even number*	Problem 4 *The dance floor*	Problem 5 *The sequence*	Total of answers (71x5=355)
Participatory Stage						
Frequency	30	23	68	13	55	189
Percentage	42.2%	32.4%	95.8%	18.3%	77.4%	53.2%
Anticipatory Stage						
Frequency	7	1	3	15	12	38
Percentage	9.9%	1.4%	4.2%	21.1%	16.9%	10,7%
Others						
Frequency	34	47	0	43	4	128
Percentage	47.9%	66.2%	0%	60.6%	5.7%	36,1%
Total						
Frequency	71	71	71	71	71	355
Percentage	100%	100%	100%	100%	100%	100%

On the other hand, the answers considered to be at the Participatory Stage revealed two moments in the pupils' construction process. These two moments indicate how students generate registers, obtain some kind of information from them and coordinate them in an attempt to use a mathematical conception useful in solving the problem situation. In these cases we characterize how some pupils obtained information from the registers of information (e.g. particular cases, organized information by tables…) and coordinated it in order to generate an answer which reflected a certain degree of coordination which had not been present in their original written answers, thus indicating a change of focus. The student's activity-effect relationship reflection during the interview revealed the existence of two cognitive states (moments) and a pathway between them. We have labelled them Projection and Local Anticipation, and the transition from one to other has been characterized considering how learners change focus when identifying regularities in the set of registers (Table 2). In the following two sections we shall describe these two cognitive moments in the Participatory Stage.

Table 2 – Percentages in Different Moments of Participatory Stage

Participatory Stage	Problem 1 *The job offer*	Problem 2 *The festival enclosure*	Problem 3 *Even number*	Problem 4 *The dance floor*	Problem 5 *The sequence*	Total of answers (71x5=355)
Projection						
Frequency	4	11	13	3	38	69
Percentage	5.6%	15.5%	18.3%	4.2%	53.5%	19.4%
Local Anticipation						
Frequency	26	12	55	10	17	120
Percentage	36.6%	16.9%	77.5%	14.1%	23.9%	33.8%
Total						
Frequency	30	23	68	13	55	189
Percentage	42.2%	32.4%	95.8%	18.3%	77.4%	53.2%

3.1 Projection: Generating a Set of Registers

In nearly 20% of the total of 355 answers, the pupils created from the situation some type of set of registers, but had difficulty in coordinating the information available. Here, we consider *a register* the parts of answers that can be used to infer the students' cognitive activity. In *The job offer* problem, 5.6% of the pupils used particular cases to obtain information that might help in making a decision. A typical example of the procedure employed to create a set of registers was the following:

- *For 10 pizzas delivered, Earnings A = 66•* → *Earnings B = 33•* → *A is better.*
- *For 20 pizzas delivered, Earnings A = 72•* → *Earnings B = 42•* → *A is better.*
- ...

Initially pupils centred their attention exclusively on the information provided by the set of particular cases. This kind of behaviour, using very low numbers of pizzas delivered, or focusing attention on only part of the account in the situation prevents the more or less explicit appearance of the existence of a change in the profitability of the offers as the number of pizzas increases. The following protocol shows an example of this kind of procedure.

13E: What else did you do? In the end, what conclusion did you come to?

14A: Well, I saw that in pizza takeaway A they pay better because you are guaranteed the 60 euros, so you don't have to worry about delivering one pizza more or one less.

The consequence of using very low quantities is that in all cases job-offer A is considerably better than job-offer B. Pupil E19's attention was centred on the six particular cases considered instead of on the information that could have been obtained by comparing the difference in earnings as the number of pizzas delivered increased.

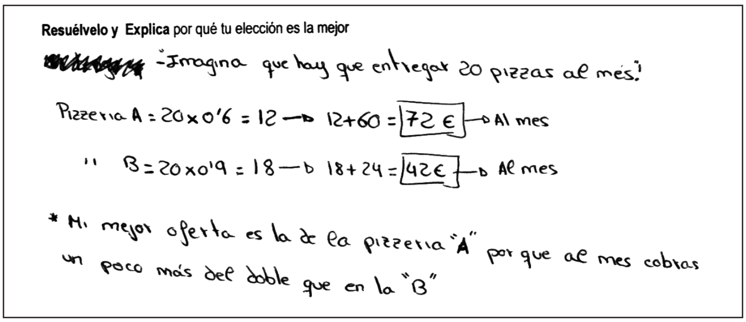
These characteristics were also observed in *The dance floor* problem. In this problem, 3 pupils (4.2%) used particular cases in an attempt to find some kind of regularity between the length of a side of the dance floor and the dimensions of the tiles, taking into consideration the two conditions imposed in the problem *without cutting any of the tiles* and *the floor must be square*. These pupils restricted the information derived from the set of registers to the consideration of the particular cases, without actually perceiving that it is possible to lay a square floor with tiles of this size.

This kind of approach also occurred in problems requiring a generalisation of patterns. In problem 5, *The sequence*, for instance, the first question requires the pupils to continue the sequence by drawing the appropriate number of stars. In order to do this, the pupils would have to notice the recursive relationship between the different terms in the sequence (each term requires the addition of two more stars to the previous one). The question *Express in numbers the first five terms in the progression* requires the construction of a set of registers. The pupils receive information from the set of registers, which is restricted to the particular terms under consideration, with no clear evidence of the establishment of a relationship between the number of stars and the position that number occupies. In problem 5, 53.5% of the pupils answered in this manner. E07, for example, followed a recursive strategy in order to calculate the values at terms 10, 25 and 50, by starting at the first term (3) and adding 2 to move to the next one. E07 thus set up a large number of registers, but this strategy was not an efficient way to determine the value of higher terms in the progression. The pupils who showed this kind of behaviour were not successful in answering the questions which referred to high-term values, and did no better in answering leading questions during the interview. Behaviour of this kind is a characteristic of the Projection moment, in which learners do not perceive the *regularity* which would enable them to calculate high values. This kind of strategy (*short-cut methods*, ORTON; ORTON, 1999) which attempts to speed up the calculation of high values without using the pattern is a consequence of the difficulty experienced in finding it.

3.2 Reflection: The Coordination of Information during the Interview

In the course of the interview some of the pupils coordinated the information derived from particular cases in response to prompts from the interviewer. Sometimes they made inferences of a general kind from the situation, with no written trace of the activity carried out. On other occasions however the pupils wrote down registers which enabled them to investigate how to compare and relate the particular data, or generated a search for new information; in both cases they were coordinating the information.

For example, in *The job offer* problem, E11 perceived during the interview the change of profitability in total earnings, basing the conclusion on a single particular case he had constructed on the written answer paper. On paper, E11 calculated the monthly earnings at each of the pizza takeaways in the case of *20 pizzas delivered*, concluding that the better-paid job *is the one at pizza takeaway A because you earn just over twice as much as at B* (Figure 2). We had considered this kind of answer a manifestation of the Projection moment.


Resuélvelo y Explica por qué tu elección es la mejor

"Imagina que hay que entregar 20 pizzas al mes."

Pizzeria A = 20x0'6 = 12 → 12+60 = 72 € → Al mes

" B = 20x0'9 = 18 → 18+24 = 42€ → Al mes

* Mi mejor oferta es la de la pizzeria "A" por que al mes cobras un poco más del doble que en la "B"

Figure 2 - Protocol E11_P1. Written answer.

During the interview, however, he indicated the following:

3E: OK. Let's start with the first one. Do you remember what it was about?

4A: Yes, here it is … you have two job offers, in one it is 6 cents for each pizza, and a fixed amount every month. In the other, the amount for … what they pay for each pizza you deliver, and then the fixed amount every month. And the other, the amount they pay for each pizza delivered is quite high, but the amount they pay every month is lower. I've given an example. I mean, imagine you have to deliver about 20 pizzas a month. So you multiply the 20 pizzas, the pizzas by 6 cents, which is the same as 12 plus 12 and then the 60 euros you get every month, that's 72 altogether. In the other case 20 by 0.9 [by 9 cents] is 18, plus 24, that's 42. So the difference is bigger. So my better offer was A. A was much better.

5E: You'd take A, then?

6A: Yes.

As the interview continued, the researcher asked him what would happen if a greater number of pizzas were delivered.

7E: And what do you think would happen if more pizzas were sold or…?

8A: Yeah, that's what I was going to tell you, that probably as the number of pizzas increased you would earn more with option B. But with the example I've given you the better offer is A. Maybe with 200 pizzas B is a better offer.

This reply seems to show that E11 perceives the existence of a change of profitability in the offers as the number of pizzas delivered increases. To find out how he managed to perceive this change, the interviewer asked him to explain why he thought it might be possible to earn more in job B.

17E: Why do you think, then, why do you think you might be able to earn more in job B?

18A: Because … because for each pizza, eh, you get more than 3 cents at the end of that … as you deliver more and more pizzas, you get, like, 3 cents for each pizza. I mean, after a lot, that's more, more money.

34A:In the end, in the end … the more pizzas you deliver the more you get back the difference you've got here.

In his answer E11 refers to the difference in the money paid by each pizza takeaway for each pizza delivered, saying, "because for each pizza, eh, you get 3 cents more than at the end of that … as you deliver more and more pizzas, you get, like, 3 cents for each pizza. I mean, after a lot, that's more, more money" (line 18). He therefore perceives that the difference between the fixed amounts offered by pizza takeaways A and B can be compensated by selling a large number of pizzas. This is possible due to the difference in payment for each pizza delivered, and E11 comes to this conclusion via a qualitative analysis of the data. The regularity lies in the fact that the difference between the two offers diminishes as the number of pizzas delivered increases (the earnings in A get closer and closer to those in B) and therefore there comes a point at which B is better than A (there has been a change of tendency in the profitability of the two offers).

Another characteristic of this procedure is the way in which the identification of the regularity is triggered by the researcher's prompt "What do you think would happen if more pizzas were sold?" (line 7). From a theoretical viewpoint the question functioned as a prompt which moved the pupil's focus of attention from a single case of what a pizza-deliverer might earn towards a consideration of "how the difference between the two amounts earned might vary" depending on the number of pizzas delivered. We have called this change of attention-focus Reflection, which makes it possible to identify the regularity by coordinating certain types of information as a consequence of the interviewer's prompts.

3.3 Local Anticipation: Identification of Regularities

In 33.8% of the answers, the pupils formed a provisional anticipation of regularity. This construction process allows the learner to reason why the effects follow the activity. The identification of regularities, seen as a consequence of a student's awareness of structure, allows the student to generate a new mathematical conception for this situation and to use it to solve the problem. For instance, in problem 5 "The sequence", in order to obtain terms 10, 25 and 50 in the sequence the pupils abandoned the recursive strategy they had used in the first place, in order to calculate any given term without having to know the one before it. E34 provides a revealing example of this, when she uses a regularity she has identified while trying to "shortcut" the recursive strategy in order to calculate terms 25 and 50, thus extending the pattern to new particular cases.

229E: Let's see then. In the end, how did you calculate that 101?
230A: Ah, 101.
231E: Yes.
232A: Well, I don't really know if it's right. 50 minus 5. That gives me 45.

233E: 45.
234A: Now with that 5, I know it's ….
235E: You know it's 11.
236A: Then 45 by 2 gives me 90.
237E: Right.
238A: If I add 90 and 11 I get …
239E: 101.
240A: 101.

To do this, she starts from the 5th term (which is 11), and multiplies by 2 the difference between the term she wants to calculate and term 5 (lines 230-239), as follows: the fifth term is 11, and from there to the 50th term she would have to add 2 forty-five times, so the value at term 50 will be $11 + 45 \cdot 2$. Pupil E34 is also able to express this relation in general terms (line 250).

250A: Oh, yes. Two multiplied by n, minus 5 plus 11.
251E: Right.
252A: That's how it's done, that's what I wrote. That's what I did with it.

This kind of process should be considered to be evidence of a major leap forward, involving the cognitive coordination of information that can be used in this local context. E34 is therefore able to anticipate distant terms in the sequence by using the regularity perceived from the set of registers. In this case, the student's awareness of regularity should be understood as a manifestation of seeing the general in the particular. This way of using a perceived regularity in the problem 1 enabled the pupils to justify and use the change of profitability (effect) in the two job-offers (the activity being to calculate the difference between the two offers with regard to the number of pizzas). In this problem the characteristic of Local Anticipation lies in the *adjustment* of the decision and is revealed when the pupil considers particular cases approaching 120 (which is the number of pizzas delivered that makes the two offers the same in earnings). The following protocol (Figure 3) shows how the information derived from the set of registers is used to anticipate new particular cases until 120 is reached.

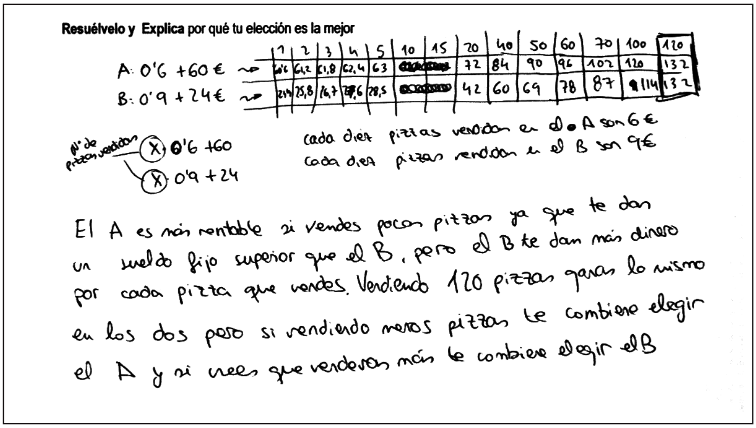


Figure 3 – Protocol E22_P1. Written answer.

In his written answer, E22 drew up a table showing various particular cases and the earnings corresponding to each one for both job-offers. In the interview he explains the process he followed.

6A: Look, in the first one they say there are two pizza takeaways, right? A and B, so in takeaway A they give you 60 euros a month, a fixed sum every month, and in B they give you 24, right? So if they give you more in one than in the other, but in… in the first one they give you 0.6 for every pizza you deliver, and in the second one 0.9, right? So that means that for every 10 pizzas you sell it'll be 0.6 times 10, six euros, you move the decimal point, and here it's 9 euros. So for every 10 pizzas you sell… I mean, look, it's here. From 20 to 40 that's 20, right? Well, you go on adding on, and here it says which will pay you better, right? Well, in the first one as it's 60 euros, in the first one if you don't sell many pizzas the chance is you'll get quite a bit of money, right? I mean it's quite a lot, a lot, a lot of money every month. But it is not on the second one. But in the second one you take more of a risk because you have to sell more pizzas. In the second one they give you more, less money every month, but they give you more money for every pizza you sell.

7E: Yes.

8A: So when you get to 120 pizzas …

9E: What did you do? Did you keep trying it, going up and up, and seeing how many deliveries…

10A: Sure, I went 1, 2, 3, 4, 5, right? I kept on multiplying it.
11E: Is that the number of pizzas? [pointing to the first row in the table]
12A: The number of pizzas sold. 5, right? But I saw it was not enough, so I went on adding more and more.
13E: Fine.
14A: I went on multiplying, and here I wrote an equation, right?
15E: Yes.
16A: Say x is the number of pizzas you sell at 0.60, at 0.60 cents plus the money they give you every month, then you multiply, it might only be two pizzas. Two times 0.60, 1.20 plus 60 euros maybe, and so on.

The particular cases used are organised in a table beginning with the case of “1 pizza delivered”, and increasing by one pizza at a time for the subsequent cases up to the case of “5 pizzas delivered”. From 10 pizzas onwards, he uses the relation “for every 10 pizzas you sell it’ll be 0.6 times 10, six euros [Job-offer A], you move the decimal point, and here it’s 9 euros [Job-offer B]” (line 6). This regularity is perceived from the comparison between the amounts paid for each pizza delivered. As he states in his written answer:

- “Every ten pizzas sold in A mean 6•”
- “Every ten pizzas sold in B mean 9•”

The coordination of the information is revealed in the way he looks at the amounts earned for pizzas delivered (going up in tens of pizzas), together with the comparison between the fixed monthly amounts, which lead E22 to realise that job-offer B can be better than job-offer A (i.e. the regularity in the situation seen as a change of tendency). He is searching for the number of pizzas which will make the two offers the same by setting up new registers of particular cases, ten by ten. This *directed* search for the number that will indicate the change of tendency is a manifestation of the coordination of information, in which the particular cases are used as an iterative activity towards a pre-established goal. After calculating the case of 120 pizzas, E22 states that “If you sell 120 pizzas you earn the same in both places, but if you are going to sell fewer pizzas you should choose A and if you think you will sell more you should choose B”. (line 18A)

18A: And in the end I went on doing that and with 120 pizzas you earn the same in both. So if 120 pizzas are sold you would earn the same in

both. So you could take either. But from 120 onwards you'd earn more in B. So…
19E: So which of the two would you choose?
20A: Personally, I'd take A, because it's difficult to sell 120 pizzas. The thing is … but if you want to take a risk and you think you'll sell more, you'd take B.

E22 therefore discerns the change of tendency which occurs as the number of pizzas delivered increases, and is able to use it to discover at what number of pizzas the two job-offers pay the same. At the end of the interview he states that "Personally, I'd take A, because it's difficult to sell 120 pizzas. The thing is…but if you want to take a risk and you think you'll sell more, you'd take B" (line 20). The perception of the change of tendency and the use of this insight into the structure of the situation to find the number of pizzas at which the change occurs enables the pupil to make a decision and justify it appropriately.

4 Discussion

The written answers and the interviews provided us with information regarding different manifestations of how students coordinate the information in different situations, and how sometimes they changed the focus of attention and were able to identify the underlying structure, and how sometimes they made use of mathematical conceptions. Our data provide empirical support on how the construction phase (Participatory Stage) and the use of mathematical conceptions constructed in novel problem-situations (Anticipatory Stage) operate and we identified two different moments in the Participatory Stage from the mechanism of reflection on the activity-effect relationship. The use of different mathematical conceptions in the same study, such as the comparison of linear functions, the minimum value of a rational function, the conception of an odd number as an even number minus/plus 1, the lowest common multiple, and arithmetical progressions, together with a broad sample of pupils and the combination of questionnaire and post-reflection interviews made it possible to amplify and complement previous characterisations of the process of coordination of information that supported the abstraction process. Our findings have enabled us to generate two ideas which may help to explain some aspects of how mathematical conceptions have been constructed by students. In the first place, the way in which activity-effect reflection reveals what route is followed from

Projection to Local Anticipation and, secondly, the two manifestations of reflective abstraction in the process of problem solving.

4.1 The Participatory Stage: From Projection to Local Anticipation through Reflection

Progress from Projection to Local Anticipation is based on the capacity to observe regularities (the effect of the activity understood as a manifestation of regularities in one's mental activity) and to coordinate information in the set of particular cases. The way in which learners use particular cases is evidence of the steps they take when they have not identified a previously-constructed mathematical structure (Participatory Stage). The use of particular cases is linked to the performance of cognitive actions such as comparing, relating or searching. This kind of action leads the student to notice the effect of his/her activities and coordinate the information which in turn leads to a change in the learner's attention-focus. Coordination and change of focus are evidence of mental activities at some stage of the construction process of mathematical conceptions. Such prompted attention-changes, linked to cognitive actions, are what Reflection consists of. A process of this nature has also been identified by Ellis (2007), via different kinds of generalisation tasks in which learners related and associated two situations or properties discernible in two situations, or used repeated acts to search for a relation. In these cases, the prompts proceed from the design of the task or from the interviewer. Our data have shown that in certain cases the existence of some kind of prompt or stimulus (made by the teacher/researcher or the task design) allow to student change through Reflection and accede to anticipation (mathematical conception). These prompts favour the change of focus which is itself the beginning of the recognition of some kind of regularity in the set of data (effect of activity).

4.2 Two Manifestations of Reflective Abstraction

The findings of this study support the distinction between the process of abstraction of a mathematical conception (participatory stage) and the use made of previously-constructed conceptions (anticipation stage). Furthermore, our results indicate the existence of two cognitive moments in the participatory stage characterized by how students coordinate information and gain awareness of the structure underlying the data. These cognitive states are differentiated by how students manage the information that they generate from particular cases and how they coordinate it by searching, comparing or relating. When a student *searches*, he or she repeats an action in an attempt to force some element of similarity or relation (the activity-effect reflection). These cognitive actions can be visualised by the researcher when observing how a student repeats them in different problem-solving contexts. At this point, for reflection to occur, a change of attention-focus is needed from the particular data to the structure of properties (GARCIA-CRUZ; MARTINON, 1997; ROIG; LLINARES; PENALVA, 2010; SRIRAMAN, 2004). Therefore, the process of cognitive construction referred to as the Participatory Stage is characterised by cognitive actions such as sorting the activity-effect records, or combining elements of mental structures in order to achieve a given goal, and the coordination of information through comparing, searching or relating to generate a new structure. This cognitive process repeated in different situations leads to the possibility of consolidating mathematical conceptions. Therefore, the iteration of these phases may help in the consolidation of such mathematical structures (MONAGHAN; OZMANTAR, 2006). From this perspective, the consolidation process of mathematical conceptions is another manifestation of reflective abstraction that leads to a new cognitive state. In this new cognitive state the student is able to recognise some mathematical structure (object, property, relation etc) as anticipation in the situation, which shows a certain similarity to some previously-experienced situation allowing its use (Anticipatory Stage) (Figure 6). We argue that it is possible to identify different aspects of the abstraction process using problems from different mathematical domains and using a broad sample of students. The relationship between the Participatory and Anticipatory Stages in the Projection stage (PIAGET, 2001) gives greater strength to this way of understanding the abstraction process when learners think mathematically, and locates the focus of attention on the relation between the learner's mental actions while abstracting, the outcome of these acts and their subsequent use.

Finally, the role played by prompts (in the task itself or as made by the researcher/teacher during the interview) would seem to indicate that when abstraction-centred tasks are designed they should take into account the nature of the prompts which will help the learners to coordinate the information and thus go on to Local Anticipation (the recognition of some aspect of the generality). This recommendation is compatible with that made by Tzur (2006) following a whole-class teaching experiment. Consequently, in order to give learners the opportunity to change their attention-focus and begin to see a set of activity-effect registers as a unified object (the identification of the regularity and/or the general aspect) (DÖRFLER, 2002) it will be necessary to create opportunities for the development of language-items for the new construction. This characteristic of the task has also been considered relevant in designing tasks to consolidate a new construction (GARCÍA; LLINARES; SÁNCHEZ-MATAMOROS, 2010; MONAGHAN; OZMANTAR, 2006). In any event, more research is required to provide information that will be useful in reaching a clearer theoretical understanding of task-design, with all the obvious implications for the improvement of teaching methods.

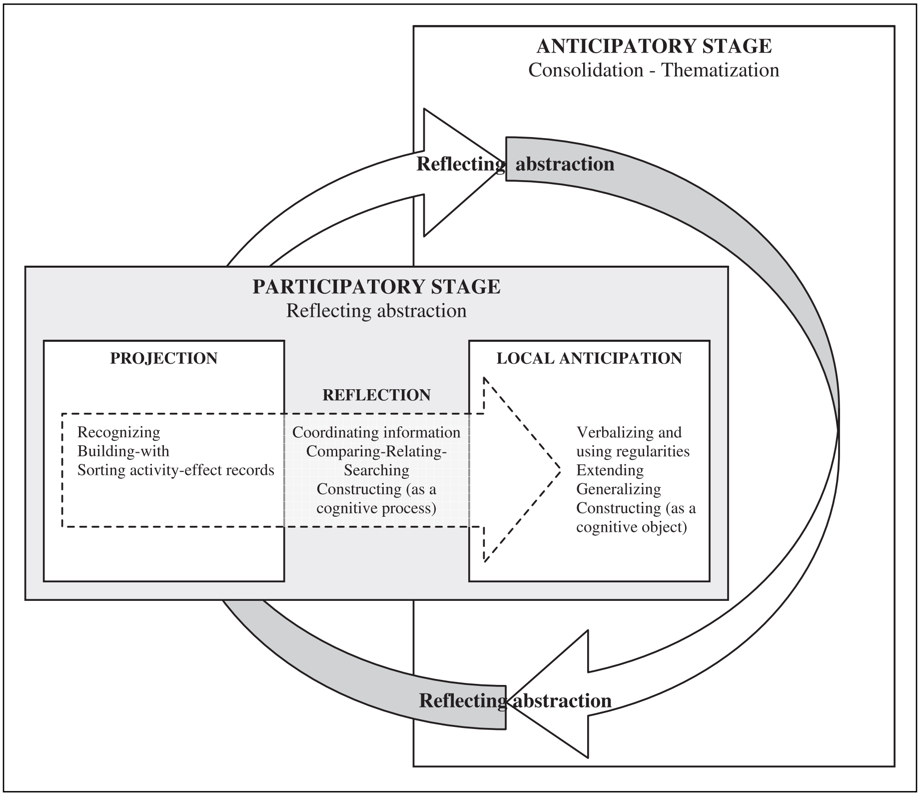


Figure 6 – Description of abstraction of mathematical conceptions: from *Participatory Stage* to *AnticipatoryStage*

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