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Transitioning from "It Looks Like" to "It Has To Be" in Geometrical Workspaces: affect and near-to-me attention

Transición entre "Se ve como" a "Tiene que ser" en un Espacio de Trabajo Geométrico: afecto y atención

Melissa Rodd*

Abstract

Within a practitioner researcher framework, this paper draws on a particular mathematics education theory and aspects of neuroscience to show that, from a learner's perspective, moving to a deductive reasoning style appropriate to basic Euclidean geometry, can be facilitated, or impeded, by emotion and/or directed attention. This shows that the issue of a person's deductive reasoning is not a merely cognitive one, but can involve affective aspects related to perception – particularly perception of nearby sense data – and emotion. The mathematics education theory that has been used is that of the *Espace de Travail Mathématique*, the English translation of which is known as Mathematical Working Spaces (MWS). The aspects of neuroscience that have been used pertain to the distinct processing streams known as top-down and bottom-up attention. The practitioner research perspective is aligned with Mason's teaching-practice-based 'noticing'; qualitative data analysed in this report include individual interviews with school teachers on in-service courses and reflective notes from teaching. Basic Euclidean geometry is used as the medium for investigating transition from 'it looks like' to a reasoned 'it has to be'.

Keywords: School Geometry. Mathematical Working Spaces. Top-Down Attention. Affect. Practitioner Research.

Resumen

Este artículo, desde un marco de referencia práctico, bosqueja desde las teorías de educación matemática y de neurociencia el paso del razonamiento del alumno al razonamiento deductivo idóneo en *la Geometría Euclídea*. Cómo este paso puede ser facilitado (o impedido) por la emoción y/o dirigido por la atención. Se pone de manifiesto que la cuestión de razonamiento deductivo de una persona no es una meramente cognitivo, pero puede involucrar a los aspectos afectivos relacionados con la percepción - en particular la percepción de sentido de los datos- y la emoción. El modelo utilizado es el de *Espace de Travail Mathématique*, la traducción en inglés de lo que se conoce como *Mathematical Working Spaces* (MWS). Los aspectos de neurociencia que se han utilizado pertenecen a las corrientes de tratamiento de la atención (as top-down and bottom-up attention). La perspectiva de la investigación profesional está basada en la enseñanza práctica 'noticing' de Mason. Los datos cualitativos analizados en este informe incluyen entrevistas individuales con los profesores en servicio y notas reflexivas de la enseñanza. La Geometría euclidiana se utiliza como un medio para estudio de la transición de 'parece' a una razonada tiene 'que ser'.

Palabras clave: Geometría. Espacio de Trabajo Geométrico. Arriba-abajo Atención. Afecto. Investigación práctica.

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1 Introduction

Central to geometry education in secondary school mathematics curricula throughout the world is the transition from empirical reasoning based on perceptual givens to deductive reasoning based on accepted premises. Van Hiele (e.g. 1986) recognised this in his conceptualisation of learners' progression; the typical child starts with shape recognition, proceeds to description and analysis of properties of shapes and then to reasoning deductively concerning properties of geometrical concepts (like shapes) with increasing levels of rigour and abstraction. This paper draws on teaching in-service courses, 'Learning Geometry for Teaching' and 'Mathematics Subject Knowledge Enhancement', in order to offer a theorisation for aspects of this transition from perceptual geometry to deductive geometry.

The Mathematical Working Space (MWS) framework (e.g., KUZNIAK, 2006; KUZNIAK; RAUSCHER, 2011; GÓMEZ-CHACÓN; KUZNIAK, 2015) offers a structure that theorises links (*geneses*) between a learner's environment and his/her cognition. This MWS framework gives a way to track transition from perceptual geometry to deductive geometry through identifying how a learner's experience is processed through the links (*geneses*) of tool use, imagery or language (i.e., respectively: instrumental genesis, figural genesis or discursive genesis; GÓMEZ-CHACÓN; KUZNIAK 2015). A contribution to this framework presented in this paper is to specialise the notion of *top-down* attention (e.g., PINTO et al., 2013; AUSTIN 2009) for circumstances in which tool use, imagery and language are processed together in 'near-to-me' geometrical work. By conceptualising 'near-to-me' attention in terms of attention-processing pathways, an explanation for a learner's transition from "it looks like" to "it has to be" can be offered which integrates influences of emotion and other affects into their *spaces* of geometrical work.

The main claim of the paper is that transition from perceptual to deductive thinking in geometry, can be facilitated by emotional stimulation of 'near-to-me' attention together with employment of senses that contribute to holding that sort of attention (and vice versa: senses stimulate the attention which is sustained by suitable emotion). Furthermore, school geometrical thinking involves, in ways to be explained further below, a combination of the MWS *geneses* that link a learner's environment with his/her cognition. The paper uses data in the form of responses to and progression in geometrical problem-solving situations which have been collected from in-service courses mentioned above, provided by myself and my colleagues, during which school teacher participants engaged in mathematical work.

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1.1 Orientation: Teaching for transition from "it looks like" to "it has to be"

(Extract from audio recorded and field-note annotated teaching interaction between author (I) and a school teacher (P) who was attending the in-service course mentioned in the introduction (plain text)).

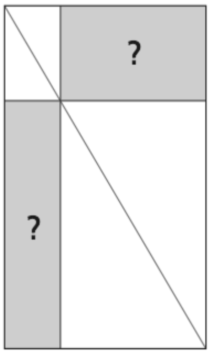

Figure 1 – [NB: the rectangles represented as grey in Fig. 1. were actually yellow.]

[…] indicates a cut in the transcript.

I What do you see?
PAUSE
P I see a rectangle
I Put your finger round the rectangle
P The big rectangle which is divided into four sections large and three smaller rectangles and diagonal, I see two yellow rectangles. with question marks on them and I'm thinking: something to do with perfect numbers? PAUSE something to do with areas?

(This right handed student's right hand fingers remain on the worksheet, moving slightly back and forth within the (big) rectangle.)

I Keep thinking areas
P So the area of the two yellow [rectangles] relates somehow to the biggest rectangle. And I am thinking: is the area of the two yellow rectangles the same? and it looks like it is
I It looks like it is
The participant paused
I Explain that to me again, using your fingers maybe […]
The participant paused
P Why is that diagonal here? PAUSE so those big triangles have to be equal, the areas have to be equal, now this triangle and this triangle are PAUSE congruent so the areas are the same. Those two little white triangles are congruent, so what's left of this triangle Ah I've got it! (big smile, thump on desk) so what's left of this triangle has to be equal with what's left of this triangle here […]
I You've a way of explaining it by subtraction…do they look like they have the same area?
P They don't look like it, PAUSE because they're a different shape, they've different dimensions, but once you've subtracted all the other shapes they have to be. […]
I And have you proved it?
P If you cut those shapes, cut them out you can prove it, as there are two pairs of triangles which are congruent. (smiling, sitting more upright)
I You moved from visual to logical.

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The dialogue above is part of a transcript in which a participant is working on solving the geometry problem represented in Fig.1. The geometry problem was presented to the participant as a visual stimulus (an A4 worksheet consisting solely of the diagram of Fig. 1 with the shaded rectangles coloured yellow) which was placed on the table directly in front of him. This extract illustrates the main claim of the paper, that is, when working on a particular geometry problem, a person's transition from perceptual to deductive thinking, can be stimulated and reinforced by emotions – here pleasure, satisfaction – and that information is incorporated and integrated from more than one sense – here touching, seeing, listening, speaking – that is, attention is focused 'near-to-me'.

1.2 Outline of the rest of the paper

There follows two literature sections that respectively introduce the foundational ideas used in the rest of the paper: The first of these sections presents a small subset of literature pertaining to the MWS framework and explains how this framework is used here (this section is quite brief given its centrality in the special edition). The second section gives a background for 'near-to-me' attention. This section includes mathematics education literature that focuses on emotion and perception together with an introduction to a small amount (relative to the knowledge base) of work in the neuroscience of attention that relates to ideas of emotion, perception that can be applied to learning geometry and a brief argument for neuroscience having a role in education research. This is followed by a methodology section after which data from participants working on geometric problems and the analysis thereof is presented. The paper concludes with a discussion.

2 Literature

The MWS framework and the *geometrical paradigms* (KUZNIAK; RAUSCHER, 2011) are presented in the first part of this review and the second part is in three subsections: (1) mathematics education literature that pertains particularly to emotion and perception; (2) a brief discussion and critique of the potential of neuroscience applied to education; (3) a focused introduction to the neuroscience of attention that relates to ideas of emotion, perception and geometry learning.

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2.1 Theoretical framework: MWS for geometric problem solving

A child's introduction to Euclid is one of the well-trodden ways to initiate his/her curiosity about importance of axioms, (i.e., 'what can be assumed?' or 'where are we starting from?') and associated rules of inference. This Euclidean geometry tradition is foundational with respect to this paper's central query, which is: how does a learner change their reasoning styles to include deductive reasoning, and what can stimulate transition from empirical to deductive thinking?

The notion of a Mathematical Working Space (MWS) was introduced to help organise thinking about the many practices that go on in a mathematical problem-solving or learning situation. The general MWS has furthermore been specialised to the geometry education context, (e.g., KUZNIAK, 2006; GÓMEZ-CHACÓN; KUZNIAK, 2015) in which case it can be referred to a *geometrical working space*, *espace de travail géométrique* or GWS, but in this paper the term MWS is used throughout. The MWS diagram, in the geometry context has two (*horizontal*) layers of geometrical work: *personal environment* (lower layer) and *personal cognition* (upper layer); personal environment and personal cognition are linked by *geneses* mentioned above (i.e., tool use, imagery or language). That is, personal environment and personal cognition are connected (diagrammatically *vertically*) by that which is respectively, manipulated, seen, and languaged (GÓMEZ-CHACÓN; KUZNIAK, 2015). Furthermore, these horizontal layers of the formal diagram are structured: personal cognition in the *cognitive plane* consists of *visualization*, *construction* and *proving*. And personal environment in the *epistemological plane* consists of *real space*, *artefacts* and *reference* (ibid. 3). This diagram is, of course, a schema with which to engage, as learner, teacher or researcher, and itself is a tool for helping to think about geometry learning and teaching.

The term *geometry* in a mathematics education context will be used here as articulated by Kuzniak and colleagues (e.g., KUZNIAK, 2006; KUZNIAK; RAUSCHER, 2011; GÓMEZ-CHACÓN; KUZNIAK, 2015). The three *geometrical paradigms* (KUZNIAK; RAUSCHER, 2011) for geometry education are: GI – the geometry of perception; GII – *natural axiomatic geometry* (ibid. p. 134), typically the planar geometry of Euclid but could just as well be the positive-curvature geometry of the sphere with its *natural* axiomatisation; (GIII, *formal axiomatic geometry* (ibid. p. 134), is not relevant to discussion in this paper). By clarifying the sources of validation for propositions related to each of the paradigms, students' reasoning styles can be classified. In particular, in GI, the role of perception is particularly

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important and, for GII, being able to draw inferences from within the given axiomatic model (e.g., Euclid's) is the appropriate mode of reasoning. The mathematics education issue of interest concerns how a learner changes his or her mode of reasoning (here, from perceptual GI to reasoned GII geometry). The MWS framework helps an investigator analyse how a learner's environment (*lower plane*) affords a change in cognition (*upper plane*) from GI reasoning to GII reasoning through the manipulation, visual perception and language experienced in that environment; this assertion will be exemplified in the analysis below. Furthermore, Kuzniak and Rauscher's recognition that "mathematics [i]s a social activity that is carried out by a human brain" (ibid. p. 134) suggests that current and developing understandings from neuroscience might inform, in particular, how individuals' brains might develop the capacity to shift paradigm from GI to GII.

2.2 'Near-to-me attention'

Before reviewing potential contributions from neuroscience and related sources, the issue of affect, in particular, emotion in mathematics education is addressed.

2.2.1 Emotion and perception in mathematics education

Affect in mathematics education is a wide field that investigates attitude and participation, beliefs and values, motivation and self-regulation as well as emotion and feeling in mathematics learning and teaching. Gerald Goldin has recently contributed an extremely comprehensive review of affect in mathematics education (GOLDIN, 2014) which contains much detail and many further references. The aspect of affect in mathematics education research relevant to this study is that of emotion and feeling for mathematical work, (e.g., GÓMEZ-CHACÓN, 2000; PRESMEG; BALDERAS-CAÑAS, 2001) and in particular (Euclidean) geometry (e.g., BARRANTES; BLANCO, 2006; GAL; LINCHEVSKI, 2010) and aims to draw attention to the potential role of 'near-to-me' attention in linking emotion with perception in a learner's geometrical reasoning.

In mathematics education, the juxtaposition of emotion and perception occurs in some 'Aha!' situations (GARDNER, 1978). 'Aha!' insight occurs when where a solution has become, in an instant, *obvious* to the problem solver (LILJEDAHL 2005); an emotional transition signals or stimulates insight/deduction given some perception (which, in general,

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could be an internal image or insight as well as an external stimulus). In this context, consider the utterance – "Ah I've got it!" – from the participant in the orientation. His "Ah I've got it!" was concurrent with his directed gaze on the geometry problem stimulus and his big smile with his thump on the desk. His gaze, his smile, his thump, his utterance together marked a *point of affect* (RODD, 2015) in his experience of 'aha!'.

Different sorts of emotion and perception are associated with the use of manipulatives in mathematics learning: "educators have observed that numerous students get engaged with materials that they manipulate with their hands and move them physically around, with an intensity and insight that do not seem to be present when they just observe a visual display on a blackboard, a screen, or a textbook" (NEMIROVSKY; BORBA, 2004). Nemirovsky and Borba continue their discussion by wondering what it is about manipulatives that is valuable to student learning even in the digital age. An explanation may include the importance of integrating the near-to-me senses, as humans have evolved to use fingers to represent ideas and to communicate through gesture as well as manipulation of material and these actions have important functions in developing imagination (MOORE; BARESSI, 2013). A learner using manipulatives integrates perception of space, including their proprioceptive awareness (INGRAM et al., 2000), as well as sight, sound, touch and smell (maybe!) and this bodes well for stimulating engagement.

There is now a considerable literature on mathematical embodiment (e.g., TALL, 2013) that addresses some of the same issues as those of concern here. However, in this paper, the analysis does not employ *embodiment* lenses, but uses a mathematics educational framework, the MWS, together with a construct – near-to-me attention – derived from neurological understandings of attention processing.

2.2.2 Neuroscience in education? Brief introduction and caveats

Can results from neuroscience improve education? Such open questions cannot be answered directly but are part of current public debate. Advocates of neuroscience's potential positive impact include Mareschal, Butterworth and Tolmie (2013) who claim neuroscience research can contribute both to educational practices and policy making. The Royal Society (2011), furthermore, argues that as understanding the way the brain works should contribute to understanding how learning takes place, neuroscience, or results there from, should be included in teacher education. This view is endorsed by Dubinsky, Roehrig and Varma (2013) who specify explicit “neuroscience learning concepts” (ibid. p. 319). However, cautions about over-enthusiastic application of neuroscience to education include those from neuroscientists. For instance, Blakemore and Frith (2005) in their introduction to educationally relevant neuroscience book observe that “much of the research is not yet ready for implications to be drawn” (ibid. p. 15). Nevertheless, Willingham and Lloyd (2007) argue that well-established results from neuroscience can offer a guide for cognitive theories that underpin educational practice. They cite as example a hot debate in the 1970s between those who claimed memory used visual as well as linguistic representation and those who claimed that all memories were in some sense linguistic representations. Brain location studies a decade or so later showed “decisively” (WILLINGHAM; LLOYD, 2007) that the visual and linguistic parts of the brain were used for memory. Willingham and Lloyd also report that “visual processing operates differently if one is naming objects versus simply observing them” (ibid. p. 145) and that visual acuity is thwarted by distraction from producing language. This neuroscience-based observation that visual and linguistic can be processed sub-consciously differently, contributes to curiosity about how it is learners progress in geometry a discipline in which there are visual and linguistic aspects.

2.2.3 Neuroscience of attention: outline of theory of sensory processing

The aspect of neuroscience considered for this paper concerns brain activity relative to types of attention. Neuroscientific understanding is that there are two distinct sensory processing pathways that exist in primate brains which are correlated with different forms of attention and these forms of attention are known as *top-down* (focused) and *bottom-up* (receptive) (e.g., AUSTIN, 2009; PINTO et al., 2013). These processing pathways are physical neural circuits located, respectively, in the dorsal and ventral regions of the brain (e.g., AUSTIN, 2009; KRAVITZ et al., 2011). Activity in the processing pathways can be tracked and recorded by brain scans. Top-down, focused attention is experienced by humans when using tools like hammers; when hammering a nail we have to attend to where the hammer and nail are quite precisely, employing ‘near-to-me’ sensing. Top-down attention is also experienced when we are care-giving, say, trying to feed a poorly baby while attending to the infant’s reactions closely, again all the ‘near by’ senses are on alert, the smell, sound, feel and look of the baby are all processed together in attentive care-giving. In both these scenarios there are affective components of the subject’s attention. Complimentarily, *bottom-*

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up attention is experienced when we are receptive to wider aspects of our environment through sight or sound inputs that alert us to pleasure (sound of friends' voices), warn us of danger (ferocious animal seen approaching!) or prompt a new behaviour (sun rises) (AUSTIN, 2013).

These two evolved forms of attention are experienced throughout life, including in mathematics education situations. For instance, top-down attention is appropriate for geometrical problem-solving: our diagrams, rulers, compasses, Dynamic Geometry Software (DGS), models etc. are attended to with coordinated near-to-me vision and touch via this *top-down* sensory processing. Bottom-up attention is appropriate for teaching: in the context of a (mathematics) classroom, the teacher is receptively aware of the atmosphere of the classroom, the feedback from students and to what is happening generally through the processing of sounds and signs coming from the environment (RODD, 2013).

3 Methodology

In this section, an argument for the pertinence of practitioner research is given, followed by a description of the data collection and analysis methods used.

3.1 Practitioner research

Part of the practice of a teacher of mathematics is to induct students into deductive reasoning and to assess progress in their learning. Assessment of this learning poses a difficult challenge as formal, deductive proofs can be rehearsed and produced by students without the students having changed their belief structures (RODD, 2000). The enquiry of this study is centred on how and when a learner is convinced by deduction rather than empirical reasoning. This is a different enquiry from that of assessing whether the student has the skill to produce a standard proof. From the orientation transcript, there is reasonable evidence of the participant's shift in reasoning from: "it looks like" the shaded rectangles' areas are equal, to: "they have to be" equal due to pairs of triangles' congruence. If the transcript does offer evidence for such change, and it is accepted that assessment of a student's written standard proofs is not sufficient evidence of their shift in reasoning from empirical to deductive, then this suggests that an appropriate methodological approach should be qualitative research in which participants' spontaneous verbal, diagrammatic/written and bodily responses are

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construed as data. So, how can a researcher get such data? In the study reported in this paper, a practitioner research approach (HATCH; SHIU, 1998; MASON, 2002) was employed because, as a teacher educator and teacher of mathematics I was in a position to invite students to come for an interview or to track data from my own teaching. I studied cases in order to give meaning to the concepts as well as justify the claims through specification of instances. Justifying appropriateness of case study work is a standard challenge in qualitative research (HODKINSON; HODKINSON, 2001) and here, how reasoning changes from GI to GII style, would be considered by Hodkinson and Hodkinson as a lived reality that includes unexpected aspects so is “not generalisable in the conventional sense” (ibid.).

3.1.1 Project ethic

In my job as a university teacher educator, following Hatch and Shiu (ibid.), it is quite typical to both set up small research projects, like the one reported in this paper, in which the research participants are students I teach. This affords development both for my students, as learners or teachers of mathematics or as potential educational researchers, and for myself in terms of reflection on teaching practice. Therefore, a characteristic of such practitioner research is that the boundaries between the roles of teacher educator, researcher and mathematics teacher are fuzzy. Fuzzy boundaries, while conceptually interesting, need ethical protocols (BERA, 2014) as well as a non-judgemental attitude to that which the participants are offering.

3.1.2 ‘Noticing’ within practice

Within mathematics education and consonant with accepting without judgement participants’ responses to stimuli designed as part of the research, is John Mason’s ‘Discipline of Noticing’ (MASON, 2002). Mason’s approach includes distinguishing carefully between ‘what happened’ and the attribution of ‘reasons for observations’. The practice of ‘noticing’, into which I was inducted as a student of Mason’s, therefore pervades my approach as a researcher and contributes to the foundational ethic for the study. Mason (ibid.) argues that that which is noticed may be given status as ‘data’ and further investigated. Such investigation could be through application of theories that have explanatory power or frameworks that help structure thinking about this issue. In this study, research participants’

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reasoning about elementary Euclidean problems is investigated by noticing data from teaching scenarios, looking through a MWS framework, and using theories that explain issues to do with participants' perception, attention and emotion. Noticing what happens without judgement is therefore central to both ethic and practical research methodology.

3.2 Methods: data collection and analysis

Data from two sorts of teaching scenarios are presented below.

3.2.1 Case (a)

In case (a) masters students were invited to take part in an interview "to do with visualisation, equipment and mathematical reasoning" (quotation from email sent) and that would include working on geometry problems. The subsequent *clinical interviews* with volunteers were audio recorded and transcribed. The design of these interviews incorporated aspects of the neuroscience of attention as well as employing the MWS framework, specifically:

- I used a task, shown in Fig 2. with which I was very familiar (KÜCHEMANN; RODD, 2012), so I had no curiosity about the geometrical problem *qua* problem, thus focus on the research participant's responses could be done without distraction;
- I provided extra tools (e.g., DGS, scissors), so that the MWS tool use (*instrumental genesis*, GÓMEZ-CHACÓN; KUZNIAK, 2011) might be observed and also so that the participant's attention might be directed by the work of his/her hands;
- I used spoken language minimally, so not to distract the participant's visual attention (WILLINGHAM; LLOYD, 2007);
- I wrote down actions or other observations (e.g., "Sounds of Paul drumming fingers on table" 19:02 below), so that data about such possible attention-focusers should be available for analysis (MASON, 2002).

In case (a) I present a fairly long extract from Paul's interview, discuss this, then Paul's response is compared with that of another participant, Su. Sections of Su's transcript have not been presented like that of Paul's, instead data from Su are used for comparison.

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3.2.2 Case (b)

In case (b), data came from my teaching of a class of twelve in-service teachers on day 9 of a 20 day in-service course that was designed to prepare them to contribute to teaching mathematics in the high schools or colleges where they are employed within greater London. This lesson was deliberately planned to stimulate the *geneses* links between learner environment and learner cognition in order to facilitate transition from GI to GII reasoning. I recorded what happened in field notes in the form “account of” followed by “account for” (MASON; PIMM, 1991).

In case (b), the serendipitous observation of unexpected affect together with near-to-me attention exhibited by participants is analysed in terms of the MWS framework and the neuroscience of attention. In both case (a) and case (b) analysis is in section 4 below.

4 Data and analysis

In this section, data from cases (a) and (b) are presented and analysed.

4.1 Case (a): individuals working on a geometry problem

Case (a) research participants were recruited from in-service courses they were currently attending. In their interviews, described above, they were invited to investigate geometrical relationships given the visual prompt of Fig.2. The responses of two participants, Paul and Su (pseudonyms), have been selected for presentation here. The choice of this selection is based on (i) clear evidence of each participant’s emotional response, and (ii) considerable difference both in their task-attention, reasoning style and the type of emotional response exhibited.

4.1.1 Paul

Paul is an experienced school teacher and a part-time Masters student. He initially qualified, about twenty years ago, to teach physical education and then retrained three years ago to teach mathematics. This section is an extract from the transcript of Paul’s interview

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and the following subsection offers a commentary on what was noticed and an interpretation with respect to the MWS framework and the theories concerning affect and attention.

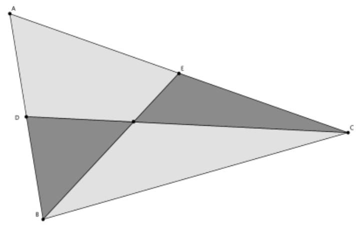

Figure 2 – Triangle, two medians and triangles formed with medians' point of intersection and half edges.

4.1.1.1 Paul's personal geometrical working space (i.e. his MWS specialised to geometry)

Extracts from my audio recorded interview with Paul (P) are indexed in the table by the time from the beginning of the recording in mins.secs.

8.15 P: Those blue [shaded dark grey in Fig. 2] triangles also make me think 'is there something to do with the area?' but then if it is area I would have to. PAUSE

13.47 P: So PAUSE could they have the same area? …PAUSE. (A Geogebra file (Geogebra webref) is brought up on the computer in front of Paul. He drags point A, making the triangle look isosceles.)

14:30 P: PAUSE. Oh yes they are! PAUSE ok, must be something with triangles it must be something with the properties of triangles.

14.52 I: When you moved the mouse and you went 'ah yes!' (Paul laughs) how did you see it just at that moment?

15.03 P: I could move them I could make them look congruent. …So it doesn't matter how I manipulate them the area will still be the same because it is the same area, whatever transformation you go through except for enlargement.

17.24 P: I have to have more information about properties of triangles to be able to, I dunno, I've got the feeling that I am right, but I cannot put it into words, I cannot PAUSE prove it

17.40 P: (Paul laughs) sounds of cutting (triangle on worksheet cut up)

18.42 P: Oh! (Sounds of moving cut out pieces around) This is height?

19.02 P: (Sounds of Paul drumming fingers on table)

19.34 P: PAUSE. Does mean, the areas are the same (upward inflection suggesting 'result' but softly not triumphant)

19.46 P: So ok those areas are the same. … Cut this one out. Ah! (sounds of cutting).

20.21 P: So? PAUSE I am lost. PAUSE, Sigh. PAUSE.

20.53 P: If I cut, AAH! OK! So that means that those two are the same. Those two are the same, and if I cut through here, those two are the same …

23.30 P: I know they are the same area but I'm not seeing them as the same area

(Softly) This area is the same as that area and this area is the same as that area (Loudly) I know just one step away ….

26.40 P: Oh Oh I think I know it …. I can see it now but I don't know how to explain it.

I: I bet you do …

28.19 P: May be I can do it the other way round (Sounds of moving cut out pieces around)

28.56 P: I jumped in my mind I jumped I'm trying to find which way lead me to it. PAUSE.

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29.14 P: Something to do with this half and this is half so those are two halves … this triangle is the same as that triangle PAUSE. OOH and I know that those two are half of that, the same as that and this is the same as that, so those two have to be the same!
29.50 P: Feels good (laughs and rubs hands vigorously!) I needed a lot of manipulation I would not have been able to see it in my head without having those triangles on the table.
34.00 P: Now it is enough for me to look at those (points to pieces of cut out triangles).
34.11 I: Look how confident you are! Something's changed. (And it was just in a split second.)
34.40 P: It does not matter what tool you use, it is emotional.

4.1.1.2 Commentary and interpretation

Paul took a while looking at Fig. 2 on the worksheet then was offered a Dynamic Geometry Software (DGS) representation (using Geogebra). After manipulating the figure to an isosceles triangle (AB=AC) he refers says (14.30) "Oh yes they are!", referring to the areas of the blue triangles (shaded dark gray in Fig. 2). This is a *point of affect* (RODD, 2013), that is a strong emotional moment related to the task he is engaged with that is indicated by his gazing at the screen and smiling. Paul then turns back to the paper representation and says that he does not know why they should be equal. At this stage, he has incorrectly generalised from the isosceles triangle situation and could be said to be working perceptually, within GI. Nevertheless, his near-to-me attention is being held by the screen's image.

Paul works in various ways before cutting two copies of the triangle in half in different ways (cut through BE to get two halves BEC and BEA and through CD to get CDB and CDA). He spends over 11 minutes (17.40-28.56) moving the four triangle halves around. His gaze is on his hands, his utterances are not in full sentences but repeats "this is the same as that" frequently. Then, he refers to "half of" (29.14) and thence to "OOH I know" – another *point of affect* – and gives his explanation which was delivered by pointing to parts of the cut out triangles. Paul's attention was downward towards his hands cutting and moving the tangible triangle pieces and he reported that he had an internal *conversation* that kept him focused. This exemplifies Paul's 'near-to-me' attention, which was sustained by the availability of the manipulables as well as his internal narration, and that around a point of affect his reasoning transitioned from perceptual (GI) to deductive (GII). The claim that Paul used GII reasoning here is made because there is evidence from the transcript and field notes that Paul gave a deductive argument based on premises (e.g., (1) triangles were Euclidean, and (2) midpoints of line segments were given). These premises were abstracted from perception, clarified and made significant to him through discussion. The philosopher Resnik

(1999) refers to *templating* abstract mathematics through such alignments towards an ideal; in this case the ideal is a pure deduction and the template is a structure of reasoning that Paul is grasping. Paul did no *writing up*. Does this matter? Not to me at the time, as I understood how his actions (moving the halves of the triangle) together with his verbal communication constituted a deductive argument.

A natural query at this juncture is to ask ‘To what extent is deduction now Paul’s way of being convinced?’ (RODD, 2000). There is not sufficient evidence to answer this question affirmatively for Paul, however, for this one small geometry problem, the transcript and notes above indicate that it was deductive reasoning that provided satisfying completion for him.

4.1.2 Su

Su is an experienced primary mathematics specialist initially trained in her home country in East Asia more than a decade prior to her coming to London where she is now resident. At the time of the interview she was a full-time Masters student. This section presents a narrative based on contrasting Su’s responses with those of Paul.

4.1.2.1 Su’s personal geometrical working space

While Paul enjoyed the *light bulb* moment when he solved this problem, Su did not. Su took about the same length of time to solve the problem (about 20 minutes) and also cut up copies of the triangle to manipulate. She showed that the triangles were of equal area, using half triangles like Paul did. During the solving process, Su had been in a good humour, chatting to me pleasantly and saying that she liked the problem. However, when asked how she felt when she conceptualised a solution to the problem, Su reported experiencing strong emotion: “stupid! I should notice it! [in a shorter time]”. When asked how she conceptualised her solution, Su said “manipulation”. When asked “Why are you sure?” she repeated the word “reasoning” several times while showing her manipulation of the cut out triangles that communicated a deductive proof.

The triangle on the worksheet was coloured blue and yellow and Su remarked: “I was obsessed with the colour” indicating her distraction from the task. And though Su was in a good mood during the interview, she was irritated with herself when she saw how simple the problem was and she attributed her lack of quick problem solving to a pleasant relaxation:

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"Maybe I was distracted by cutting which I enjoyed, I didn't recognise what I did. We love manipulations, in [my home country in East Asia] geometry is kind of rest time."

4.1.2.2 Commentary and interpretation

A conjecture, given the neuroscience of attention: Su was insufficiently top-down focused during the start of her working on the problem of Fig. 2. (suggested by her good humoured chat as well as her remarks concerning colour). Although Su thought that doing cutting up distracted her, she did manipulate the cut up triangle pieces. And while doing this manipulating she did get a solution, whereas the use of colour in the visual prompt had put her on the wrong track. In her personal geometrical working space, colour on the Artefact (worksheet) disturbed her (upward) progression to proving (discursively), whereas the Artefact de-constructed (cut up) afforded a tool (instrument) for her to manipulate and thence to prove. I conjecture that Su's positive affective state during the beginning of the interview insufficiently focused her attention on the task, when she became involved with working with her hands her attention turned to *top-down* and this near-to-me attention facilitated her deductive insight.

4.2 Case (b): a class working on learning Euclidean geometry

This section reports on case (b), which is a class teaching scenario where the students were in-service teachers. These teachers had a wide variety of backgrounds, for example, their teaching qualification subjects ranged from literacy to electronics, they formed an ethnically and culturally diverse group, most of them had obtained their initial teaching qualification in different countries outside the UK. The group was lively, keen and cooperative and there was usually lots of talk as they worked on mathematical tasks.

4.2.1 Teaching for transition from GI to GII

In the lesson reported on, I introduced Euclidean geometry. The teaching plan was influenced by the MWS framework: I aimed to generate learning using artefacts (from the environment) and having, as a learning outcome, the MWS associated conceptual construction (in their cognition). In particular, order to provoke the linking through tool use

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(genèse expérimentale/instrumental genesis) of the environmental and cognitive planes, digital resources as well as traditional manipulables were provided. I relied on my reflections noted down during and after the lesson to reconstruct the spontaneous events that took place. This Euclidean geometry lesson followed on from a session in which the participating teachers had been using Geogebra to work on graphs, hence they were familiar with the DGS's *drag* function. To begin the geometry session they sat at a central table, away from their computers, but with Geogebra still open, and after a brief introduction to Euclid's historical context and his geometrical system, they were given the Euclidean tools of rulers and compasses, artefacts of Euclidean real space. This was their introduction to Euclidean geometry, as none of them had systematically studied Euclid as a deductive system before. Their first task was to construct an equilateral triangle with the rulers and compasses, as in Euclid I proposition1, and then to figure out how to construct a square using these same tools.

4.2.2 Analysis of what happened

Like in many planned lessons, unexpected – pleasurable – affect was evidenced. The relevance to this report is the witnessed relationships between 'near-to-me' attention, affect, the participants' learning and their potential for transition from GI to GII. I recorded what happened in field notes, reported below, using the form "account of" (•) followed by "account for" (○) (MASON; PIMM, 1991).

1. After giving out the rulers and compasses, gazes went down, hands manipulated the tools and chatter decreased to near silence.
    - Only one of the twelve teachers had used compasses much before so most of them had to concentrate and coordinate mental attention with physical manipulation to make an acceptable looking circle, several had practical difficulties with manipulating their compasses.
    - o Use of the manipulables engaged *top down* attention. If drawing a circle with compasses is very easy to do there is not the attention focus and limited affective engagement. The too familiar does not stimulate affect and concentrated eye-hand activity stills chatter (AUSTIN, 2009; 63).
    - The tools were integral to formation of the concept of a Euclidean circle – they all knew perceptual circles. For most of them, this was the first time they had been asked to use compasses to make a structure (equilateral triangle) that had

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properties by virtue of the constructing tools considered abstractly, rather than the marks made by the tools.

- In MWS terms: *tools* (ruler and compasses) linked environmental *artefacts* (pencil and paper) with an abstract *construction* (the locus of points equidistant to a fixed point). This round *real space* item is linked by its *image* to the Euclidean circle understood cognitively.

2. I used Geogebra's basic geometry to go through the equilateral triangle construction on the screen and drag it about. And there were audible gasps of delight as the students witnessed the triangle moving and yet keeping its equilateral shape.

    - There had been in their environment a surprising and pleasurable event: that which was awkward to manipulate on paper had been animated on the whiteboard using the DGS.
    - This *bottom up* attention – coloured by positive affect of surprise and delight, was directed to the nature and purpose of the DGS: it is a digital representation of the Euclidean tools.
    - It was an important teaching point for me to get them to grasp the abstractness of GII. My *explanation* that Geogebra does digitally what the compasses do in your hand prompted them towards understanding that the software was not only to explore shapes, GI style, but also was a means to work in GII.
    - In MWS terms: I assessed that for at least some of the participating teachers, they did get that GII was reasoning based. In other words, their MWS had a strengthened connection between *construction* and *proving* in their *cognitive planes* which had been stimulated by surprise and which had been reinforced within the teaching-learning environment.

4.3 Summary

In case (a) the individual interviews were designed to capture detail and variation of the participants' attention to 'near-to-them' sense data by noticing what they touched, what they said they were looking at, their reactions to listening to what I said or their internal narration and what they said to me. And data to indicate this design intention have been presented. Case (b) comes from authentic teaching and aspects of the students' near-by attention was noticed and recorded. In both cases (a) and (b), with respect to the MWS

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framework, the near-to-me attention focusing function of the artefacts, that is, the manipulable tools is noteworthy. The role of language can be seen to be important in different ways: in case (a) one participant used internal narration, that was sometimes heard mumbled but also used subvocally, to keep himself *top-down* focused and was coordinated with his use of the tools, whereas another participant engaged in pleasant chat tuning in to what was expected of her that indicated her *bottom-up* attention to the social situation. In case (b) there was an unusual silence during which time geometrical constructions were done with various degrees of accuracy in a class atmosphere of peaceful concentration.

5 Concluding discussion

Mathematics education research has as its central purpose the further understanding of mathematics learning in instructional environments. The practitioner research reported here noted how change of reasoning style, from GI to GII, was facilitated by visual and tactile materials that stimulated near-to-me attention and was concomitant with positive emotion. Nevertheless, it would be naïve to claim that one instance of GII reasoning would mean the participant reasoned thus from then on. Nor would observation of the collective delight of a small group of non-specialist teachers in witnessing Euclidean geometry brought alive for them with the DGS mean that the collective was au fait with Euclidean geometry either in terms of its nature as a deductive system or in having reasoning skills to solve Euclidean geometry problems.

- A tack for further research would be to engage longer term with research participants, like in-service teachers, who are ready to transition to reasoning deductively and to collect data from their own reflections as well as participate in more formal tests of reasoning. The longitudinal proof project (PROOF PROJECT, 2003) gives a model for such research for school students.

The near-to-me attention was afforded by an environment where social language use was minimal (as seen in the cases of one individual interview and one class setting); in an environment where social language was on-going, such effective near-to-me attention was not observed (in the other individual case). In contemporary pedagogy, the practice of student-student discussion is prevalent as is the valuing of student articulation of reasoning. There are good reasons for these approaches to be valued, for instance, a student who feels that s/he has opportunities to speak, is positioned to have a positive relationship with mathematics

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(BLACK; MENDICK; SOLOMON, 2009). Nevertheless, the research presented here suggests transitioning reasoning from GI to GII might be better afforded by an environment where attention is kept near-to-me through pleasurable use of artefacts that are designed to represent mathematical theorems, concepts or processes, rather than an environment in which student talk is more prominent.

- Clearly, further research is needed in order to refine understanding of this conjecture; the use of the notion attention processing pathways used here, indicates that the field of neuroscience can have a function in giving structural explanations of such phenomena.

Although the examples used in this paper were on Euclidean geometry, the issue of top-down attention and the centrality of ‘near-to-me attention’ can apply to other geometries: for example, in 2-dimensional spherical geometry, a useful *artefact* for *construction* is a hand-held write-on ball. Geometry-specific artefacts (i.e., rulers and compasses for Euclidean geometry, hand-held balls for spherical geometry) are top-down attention focusers for the respective geometry. They are also pleasant to play with and offer potential for surprises that are shareable. These affective aspects of engagement with geometrical ideas bode well for sustaining students’ presence in their environments for mathematical work, thus deductive geometrical visualisation is sensitive to affective influences and perceptual support.

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