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Some properties of (A, B)-fuzzy positive implicative ideals in BCK-algebras

Some properties of (α, β)-fuzzy positive implicative ideals in BCK-algebras

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ABSTRACT. In this paper, by using the concept of belongingness ($\in$) and quasi-coincidence (q) between fuzzy points and fuzzy sets, we introduce (α, β)-fuzzy positive implicative ideals in BCK-algebras where α, β are any of $\{\in, q, \in \vee q, \in \wedge q\}$ with $\alpha \neq \in \wedge q$.

Keywords: BCK-algebra; belongingness ($\in$), quasi-coincidence (q), (α, β)-fuzzy positive implicative ideals, ($\in$, $\in$)-fuzzy positive implicative ideal.

Propriedades de (α, β)-fuzzy positivo ideal implicativo em álgebras BCK

RESUMO. O presente estudo utilizou o conceito de pertença ($\in$) e de quase coincidência (q) entre pontos fuzzy e conjuntos fuzzy para apresentar o (α, β)-fuzzy positivo ideal implicativo em álgebras BCK, onde α, β são quaisquer $\{\in, q, \in \vee q, \in \wedge q\}$ com $\alpha \neq \in \wedge q$.

Palavras-chave: álgebra BCK, pertence a, quase coincidência (q), o (α, β)-fuzzy positivo ideal implicativo, $\in$ fuzzy positivo ideal implicativo.

Introduction

The concept of a fuzzy set, which was published by (ZADEH, 1965) was applied by many researchers to generalize some of the basic concepts of algebra. The fuzzy algebraic structures play a vital role in Mathematics with wide applications in many other branches such as theoretical physics, computer sciences, control engineering, information sciences, coding theory, topological spaces, logic (ZADEH, 2005), set theory, real analysis, measure theory etc. In (Xi, 1991) applied fuzzy subsets in BCK-algebras and studied fuzzy BCK-algebras. He defined the concept of fuzzy ideal and fuzzy positive implicative ideal and he got some interesting results.

The theory of BCK-algebras was initiated by (IMAI; IS'EKI, 1966). For the general development of BCK-algebras, the ideal theory and its fuzzification play an important role. The concept of implicative ideals in a BCK-algebra was first introduced by (IS'EKI, 1975) and then the fuzzification of implicative ideals is studied in (XI, 1991). In (JUN et al., 1994) developed the fuzzy positive implicative ideals in BCK-algebras.

(MURALI, 2004) defined the concept of belongingness of a fuzzy point to a fuzzy subset under a natural equivalence on a fuzzy subset. (PU; LIU, 1980), give the idea of quasi-coincidence of a fuzzy point with a fuzzy set, plays a vital role to generate some different types of fuzzy subgroups, called (α, β)-fuzzy subgroups, introduced by (BHAKAT; DAS, 1996). In particular, ($\in$, $\in \vee q$)-fuzzy subgroup is an important and useful generalization of the Rosenfeld's fuzzy subgroups (ROSENFELD, 1971). (BHAKAT, 1999, 2000) studied ($\in \vee q$)-level subsets, ($\in$, $\in \vee q$)-fuzzy normal, quasi-normal and maximal subgroups. In (JUN, 2009), introduced the concept of ($\in$, $\in \vee q$)-fuzzy subalgebras in BCK/BCI-algebras and investigated some related results. (ZHAN et al., 2009) studied ($\in$, $\in \vee q$)-fuzzy ideals in BCI-algebras. (JUN, 2004, 2005) introduced the concept of (α, β)-fuzzy subalgebras (ideals) of BCK/BCI-algebras. Recently, (SHABIR et al., 2012) studied characterizations of hemirings by $(\overline{\in}, \overline{\in} \vee \overline{q})$ -fuzzy ideals.

In this paper, we define (α, β)-fuzzy positive implicative ideals in BCK-algebras where α, β are any of $\{\in, q, \in \vee q, \in \wedge q\}$ with $\alpha \neq \in \wedge q$, by using the concept of belongingness and quasi-coincidence between fuzzy points and fuzzy sets.

Preliminaries

Throughout this paper, X always means a BCK-algebra unless otherwise specified. We also include some basic aspects that are necessary for this paper.

By a BCK-algebra (IS'EKI; TANAKA, 1978), we mean an algebra $(X, *, 0)$ of type $(2, 0)$ satisfying the axioms:

(BCK-I) $((x * y) * (x * z)) * (z * y) = 0$
(BCK-II) $(x * (x * y)) * y = 0$
(BCK-III) $x * x = 0$
(BCK-IV) $0 * x = 0$
(BCK-V) $x * y = 0$ and $y * x = 0$ imply $x = y$
$\forall$ x, y, z $\in$ X.

We can define a partial order "$\leq$" on X by $x \leq y$ if and only if $x * y = 0$.

Proposition 2.1. (MENG, 1997) In any BCK-algebra X, the following are true:

(1) $(x * y) * z = (x * z) * y$
(2) $(x * z) * (y * z) \leq x * y$
(3) $(x * y) * (x * z) \leq z * y$
(4) $x * 0 = x$
(5) $x * (x * (x * y)) = x * y$
$\forall$ x, y, z $\in$ X.

A nonempty subset I of a BCK-algebra X is called an ideal (IS'EKI; TANAKA, 1976) of X if it satisfies (I1) and (I2), where

(I1) $0 \in I$,
(I2) $x * y \in I$ and $y \in I$ imply $x \in I$,
$\forall$ x, y $\in$ X.

A nonempty subset S of a BCK-algebra X is called a subalgebra (MENG, 1994) of X if it satisfies

$x * y \in S$, $\forall$ x, y $\in$ S.

A fuzzy subset λ of a universe X is a function from X to the unit closed interval [0, 1], that is $\lambda : X \rightarrow [0, 1]$. For a fuzzy set λ of a BCK-algebra X and $t \in (0, 1]$, the crisp set

$$\lambda_t = \{x \in X \mid \lambda(x) \geq t\}$$

is called the level subset of λ (DAS, 1981).

A fuzzy set λ of a BCK-algebra X (JUN, 2005) having the form

$$\lambda(y) = \begin{cases} t \in (0, 1] & \textit{if } y = x, \\ 0 & \textit{if } y \neq x, \end{cases}$$

is said to be a fuzzy point with support x and value t and is denoted by x_t.

For a fuzzy point x_t and a fuzzy set λ in a set X, (PU; LIU, 1980), gave meaning to the symbol $x_t \alpha \lambda$, where $\alpha \in \{\in, q, \in \vee q, \in \wedge q\}$. A fuzzy point x_t is said to belong to (resp., be quasi-coincident with) a fuzzy set λ, written as $x_t \in \lambda$ (resp., $x_t q \lambda$) if $\lambda(x) \geq t$ (resp., $\lambda(x) + t > 1$).

To say that $x_t \in \vee q \lambda$ ($x_t \in \wedge q \lambda$) means that $x_t \in \lambda$ or $x_t q \lambda$ ($x_t \in \lambda$ and $x_t q \lambda$). For all $t_1, t_2 \in [0, 1]$, $\min\{t_1, t_2\}$ and $\max\{t_1, t_2\}$ will be denoted by $t_1 \wedge t_2$ and $t_1 \vee t_2$, respectively.

A fuzzy set λ of a BCK-algebra X is called a fuzzy ideal (MENG, 1997) of X if it satisfies (F1) and (F2), where

(F1) $\lambda(0) \geq \lambda(x)$,
(F2) $\lambda(x) \geq \lambda(x * y) \wedge \lambda(y)$,
$\forall$ x, y $\in$ X.

Let X be a BCK-algebra. A fuzzy set λ in X is said to be a fuzzy subalgebra (XI, 1991) of X if it satisfies

$$\lambda(x * y) \geq \lambda(x) \wedge \lambda(y), \forall x, y \in X \qquad (1)$$

Theorem 2.2. (JUN, 2005) Let λ be a fuzzy set in X. Then λ is a fuzzy subalgebra of X if and only if $\lambda_t = \{x \in X \mid \lambda(x) \geq t\}$ is a subalgebra of X for all $t \in (0, 1]$, for our convenience, the empty set ϕ is regarded as a subalgebra of X.

3. Fuzzy positive implicative ideals in BCK-algebras

In this section we obtain some properties of fuzzy positive implicative ideals in BCK-algebras and investigate their properties.

Definition 3.1. (MENG, 1994) A nonempty subset I of a BCK-algebra X is called a positive implicative ideal if it satisfies (I1) and (I3), where

(I1) $0 \in I$,
(I3) $(x * y) * z \in I$ and $y * z \in I$ imply $x * z \in I$,
$\forall$ x, y, z $\in$ X.

If we put $z = 0$, then it follows that I is an ideal. Thus, every positive implicative ideal is an ideal.

Definition 3.2. (JUN, 1994) A fuzzy set λ of a BCK-algebra X is called a fuzzy positive implicative ideal of X if it satisfies (F1) and (F3), where

(F1) $\lambda(0) \geq \lambda(x)$,
(F3) $\lambda(x * z) \geq \lambda((x * y) * z) \wedge \lambda(y * z)$,
$\forall$ x, y, z $\in$ X.

Clearly $z = 0$ gives λ is a fuzzy ideal of X.

Example 3.3. Let X = $\{0, a, b, c\}$ in which $*$ is given by Table 1.

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	a	0	b
c	c	c	c	0

Then X is a BCK-algebra (MENG, 1994). Let s_0, s_1, $s_2 \in [0, 1]$ be such that $s_0 > s_1 > s_2$. We define a map $\lambda : X \rightarrow [0, 1]$ by $\lambda(0) = s_0$, $\lambda(a) = \lambda(b) = s_1$ and $\lambda(c) = s_2$. Simple calculations show that λ is a fuzzy ideal of X. But it is not a fuzzy positive implicative ideal of X, because

Put x = b, y = a, z = a in (F3) we get

$$\lambda(x * z) \geq \lambda((x * y) * z) \wedge \lambda(y * z)$$
$$\lambda(b * a) \geq \lambda((b * a) * a) \wedge \lambda(a * a)$$
$$\lambda(a) \geq \lambda(a * a) \wedge \lambda(0)$$
$$s_1 \geq \lambda(0) \wedge \lambda(0)$$
$$\geq s_0 \wedge s_0$$
$$\geq s_0$$
$$s_1 \not\geq s_0.$$

The Theorem 3.4 is a simple consequence of the transfer principle described in (KONDO; DUDEK, 2005).

Theorem 3.4. A fuzzy set λ in BCK-algebras X is a fuzzy positive implicative ideal of X if and only if for every t ∈ (0, 1], λ_t = {x ∈ X | λ(x) ≥ t} is a positive implicative ideal of X, where $\lambda_t \neq \phi$.

Corollary 3.5. A fuzzy set λ of a BCK-algebra X is a fuzzy positive implicative ideal if and only if for any $x_0 \in$ X,

$$P_{x_0} = \{x \in X \mid \lambda(x) \geq \lambda(x_0)\}$$

is a positive implicative ideal of X.

Theorem 3.6. If λ is a fuzzy positive implicative ideal of a BCK-algebra X, then

$$P = \{x \in X \mid \lambda(x) = \lambda(0)\}$$

is a positive implicative ideal of X.

Proof. Straightforward.

4. (α, β)-fuzzy positive implicative ideals in BCK-algebras

In what follows let α and β denote any one of ∈, q, ∈ ∨ q, ∈ ∧ q unless otherwise specified. To say that $x_t \overline{\alpha} \lambda$ means that $x_t \alpha \lambda$ does not hold.

Proposition 4.1. (JUN, 2005) For any fuzzy set λ in X, the condition (1) is equivalent to the following condition

$$x_{t_1}, y_{t_2} \in \lambda \implies (x * y)_{t_1 \wedge t_2} \in \lambda \quad (2)$$

for all x, y ∈ X and $t_1, t_2 \in$ (0, 1].

A fuzzy set λ in a BCK-algebra X is said to be an (α, β)-fuzzy subalgebra of X, where α ≠ ∈ ∧ q, if it satisfies the following condition (JUN, 2005):

$$x_{t_1} \alpha\lambda, y_{t_2} \alpha\lambda \implies (x*y)_{t_1 \wedge t_2} \beta\lambda \quad (3)$$

for all $t_1, t_2 \in$ (0, 1].

Theorem 4.2. Let λ be a fuzzy set of a BCK-algebra X. Then λ_t is a positive implicative ideal of X for all t ∈ (0.5, 1] if and only if it satisfies

(a) ∀ x ∈ X, λ(0) ∨ 0.5 ≥ λ(x)
(b) ∀ x, y, z ∈ X, λ(x * z) ∨ 0.5 ≥ λ((x * y) * z) ∧ λ(y * z).

Proof. Suppose that λ_t is a positive implicative ideal of X for all t ∈ (0.5, 1]. If there is $a \in$ X such that the condition (a) is not valid, that is, there exist $a \in$ X such that

$$\lambda(0) \vee 0.5 < \lambda(a)$$

then λ(a) ∈ (0.5, 1] and $a \in \lambda_{\lambda(a)}$. But λ(0) < λ($a$) implies 0 ∉ $\lambda_{\lambda(a)}$, a contradiction. Hence (a) is valid. Suppose that

$$\lambda(a * c) \vee 0.5 < \lambda((a * b) * c) \wedge \lambda(b * c) = v$$

for some a, b, c ∈ X. Then v ∈ (0.5, 1] and (a * b) * c ∈ λ_v, b * c ∈ λ_v. But a * c ∉ λ_v since λ(a * c) < v. This is a contradiction, and therefore (b) is valid. Conversely, suppose that λ satisfies conditions (a) and (b). Let t ∈ (0.5, 1]. For any x ∈ λ_t, we have

$$\lambda(0) \vee 0.5 \geq \lambda(x)$$
$$\geq t$$
$$> 0.5$$

and so

$$\lambda(0) \geq t$$

Thus 0 ∈ λ_t. Let x, y, z ∈ X be such that

$$(x * y) * z \in \lambda_t, y * z \in \lambda_t$$

Then

$$\lambda(x * z) \vee 0.5 \geq \lambda((x * y) * z) \wedge \lambda(y * z)$$
$$\geq t \wedge t$$
$$\geq t$$
$$> 0.5$$

and thus

$$\lambda(x * z) \geq t,$$

that is,

$$x * z \in \lambda_t.$$

Hence is λ_t is a positive implicative ideal of X.

Definition 4.3. A fuzzy set λ of a BCK-algebra X is called an (α, β)-fuzzy positive implicative ideal of X, where α ≠ ∈ ∧ q, if it satisfies

(c) $x_t\alpha\lambda \Rightarrow 0_t\beta\lambda$,

(d) $((x * y) * z)_{t_1}\alpha\lambda$, $(y * z)_{t_2}\alpha\lambda$ $\Rightarrow$
$(x * z)_{t_1 \wedge t_2}\beta\lambda$,

$\forall$ t, t_1, $t_2 \in$ (0, 1].

Theorem 4.4. For any fuzzy set λ in BCK-algebra X, the condition (F1) and (F3) are equivalent to the conditions

(e) $x_t \in \lambda$ $\Rightarrow$ $0_t \in \lambda$,

(f) $((x * y) * z)_{t_1} \in \lambda$, $(y * z)_{t_2} \in \lambda$ $\Rightarrow$
$(x * z)_{t_1 \wedge t_2} \in \lambda$,

$\forall$ t, t_1, $t_2 \in$ (0, 1].

Proof. Suppose that (F1) is valid and let $x \in$ X and t $\in$ (0, 1] be such that $x_t \in \lambda$. Then $\lambda(0) \geq \lambda(x) \geq t$, and so $0 \in \lambda_t$. Assume that (e) is true. Since

$x_{\lambda(x)} \in \lambda$, $\forall x \in$ X,

it follows from (e) that $0_{\lambda(x)} \in \lambda$ so that

$\lambda(0) \geq \lambda(x)$, $\forall$ $x \in$ X.

Suppose that the condition (F3) holds. Let x, y, z $\in$ X and t_1, $t_2 \in$ (0, 1] be such that

$((x * y) * z)_{t_1} \in \lambda$, $(y * z)_{t_2} \in \lambda$.

Then

$\lambda((x * y) * z) \geq t_1$ and $\lambda(y * z) \geq t_2$.

It follows from (F3) that

$\lambda(x * z) \geq \lambda((x * y) * z) \wedge \lambda(y * z)$
$\geq t_1 \wedge t_2$

So $(x * z)_{t_1 \wedge t_2} \in$ λ. Finally suppose that (f) is valid. Note that for every x, y, z $\in$ X,

$((x * y) * z)_{\lambda((x * y) * z)} \in \lambda$ and $(y * z)_{\lambda(y * z)} \in \lambda$

Hence

$(x * z)_{\lambda((x * y) * z) \wedge \lambda(y * z)} \in \lambda$ by (f),

and thus

$\lambda(x * z) \geq \lambda((x * y) * z) \wedge \lambda(y * z)$.

Theorem 4.5. Every $(\in \vee q, \in \vee q)$-fuzzy positive implicative ideal is an $(\in, \in \vee q)$-fuzzy positive implicative ideal.

Proof. Straightforward.

Theorem 4.6. A fuzzy set λ in a BCK-algebra X is an $(\in, \in \vee q)$-fuzzy positive implicative ideal of X if and only if it satisfies condition

(g) $\lambda(0) \geq \lambda(x) \wedge 0.5$,

(h) $\lambda(x * z) \geq \lambda((x * y) * z) \wedge \lambda(y * z) \wedge 0.5$,
$\forall$ x, y, z $\in$ X.

Proof. Assume that λ is an $(\in, \in \vee q)$-fuzzy positive implicative ideal of X. Let x $\in$ X and suppose that $\lambda(x) < 0.5$. If $\lambda(0) < \lambda(x)$, then $\lambda(0) < t < \lambda(x)$ for some t $\in$ (0, 0.5) and $x_t \in \lambda$ and $0_t \overline{\in} \lambda$. Since $\lambda(0) + t < t$, we have $0_t \overline{q} \lambda$. It follows that $0_t \overline{\in \vee q} \lambda$, a contradiction. Hence

$\lambda(0) \geq \lambda(x)$.

Now if $\lambda(0) \geq 0.5$, then $x_{0.5} \in \lambda$ and thus $0_{0.5} \in \vee q \lambda$. Thus $\lambda(0) \geq 0.5$. Otherwise

$\lambda(0) + 0.5 < 0.5 + 0.5 = 1$,

a contradiction. Consequently,

$\lambda(0) \geq \lambda(x) \wedge 0.5$, $\forall$ x $\in$ X.

Let x, y, z $\in$ X and suppose that
$\lambda((x * y) * z) \wedge \lambda(y * z) < 0.5$.
Then
$\lambda(x * z) \geq \lambda((x * y) * z) \wedge \lambda(y * z)$.
If not, then $\lambda(x * z) < t < \lambda((x * y) * z) \wedge \lambda(y * z)$ for some t $\in$ (0, 0.5). It follows that
$((x * y) * z)_t \in \lambda$, $(y * z)_t \in \lambda$ but $(x * z)_{t \wedge t} = (x * z)_t \overline{\in \vee q} \lambda$,
a contradiction. Hence
$\lambda(x * z) \geq \lambda((x * y) * z) \wedge \lambda(y * z)$.
Whenever
$\lambda((x * y) * z) \wedge \lambda(y * z) < 0.5$.
If
$\lambda((x * y) * z) \wedge \lambda(y * z) \geq 0.5$,
then
$((x * y) * z)_{0.5} \in \lambda$ and $(y * z)_{0.5} \in \lambda$.
This implies that
$(x * z)_{0.5} = (x * z)_{0.5 \wedge 0.5} \in \vee q \lambda$.
Therefore $\lambda(0) \geq 0.5$ because if $\lambda(x) < 0.5$, then
$\lambda(x) + 0.5 < 0.5 + 0.5$
$= 1$,
a contradiction. Hence
$\lambda(x * z) \geq \lambda((x * y) * z) \wedge \lambda(y * z) \wedge 0.5$.
Conversely, assume that λ satisfies condition (g) and (h). Let x $\in$ X and t $\in$ (0, 1] be such that $x_t \in \lambda$. Then $\lambda(x) \geq t$. Suppose that $\lambda(0) < t$. If $\lambda(x) < 0.5$, then
$\lambda(0) \geq \lambda(x) \wedge 0.5$
$= \lambda(x)$
$\geq t$
a contradiction. Hence we know that $\lambda(x) \geq 0.5$ and so

$\lambda(0) + t > 2 \lambda(0)$

$\qquad \geq 2 (\lambda(x) \wedge 0.5)$

$\qquad = 1$

Thus $0_t \in \vee q \lambda$. Let x, y, z $\in$ X and t_1, $t_2 \in$ (0, 1] be such that

$((x * y) * z)_{t_1} \in \lambda$ and $(y * z)_{t_2} \in \lambda$.

Then

$\lambda((x * y) * z) \geq t_1$ and $\lambda(y * z) \geq t_2$.

Suppose

$\lambda(x * z) < t_1 \wedge t_2$.

If $\lambda((x * y) * z) \wedge \lambda(y * z) < 0.5$, then

$\lambda(x * z) \geq \lambda((x * y) * z) \wedge \lambda(y * z) \wedge 0.5$

$\qquad = \lambda((x * y) * z) \wedge \lambda(y * z)$

$\qquad \geq t_1 \wedge t_2$.

This is a contradiction, and so

$\lambda((x * y) * z) \wedge \lambda(y * z) \geq 0.5$.

It follows that

$\lambda(x * z) + t_1 \wedge t_2 > 2 \lambda(x * z)$

$\qquad\qquad \geq 2 (\lambda((x * y) * z) \wedge \lambda(y * z) \wedge$

0.5)

$\qquad\qquad = 1$

So

$(x * z)_{t_1 \wedge t_2} \in \vee q \lambda$.

Hence λ is an ($\in$, $\in \vee$ q)-fuzzy positive implicative ideal of X.

Theorem 4.7. Let λ be an ($\in$, $\in \vee$ q)-fuzzy positive implicative ideal of X such that $\lambda(x) < 0.5$ for all x $\in$ X. Then λ is an ($\in$, $\in$)-fuzzy positive implicative ideal of X.

Proof. Straightforward.

Theorem 4.8. A fuzzy set λ in BCK-algebra X is an ($\in$, $\in \vee$ q)-fuzzy positive implicative ideal of X if and only if the set $\lambda_t = \{x \in X \mid \lambda(x) \geq t\}$ is a positive implicative ideal of X for all t $\in$ (0, 0.5].

Proof. Suppose λ is an ($\in$, $\in \vee$ q)-fuzzy positive implicative ideal of X and t $\in$ (0, 0.5]. Using Theorem 4.6(g), we have

$\lambda(0) \geq \lambda(x) \wedge 0.5$ for any x $\in \lambda_t$

It follows that

$\lambda(0) \geq t \wedge 0.5$

$\qquad = t$

So that $0 \in \lambda_t$. Let x, y, z $\in$ X be such that

$(x * y) * z \in \lambda_t$ and $y * z \in \lambda_t$.

Then

$\lambda((x * y) * z) \geq t$ and $\lambda(y * z) \geq t$.

Using Theorem 4.6(h), we get

$\lambda(x * z) \geq \lambda((x * y) * z) \wedge \lambda(y * z) \wedge 0.5$

$\qquad \geq t \wedge t \wedge 0.5$

$\qquad \geq t \wedge 0.5$

$\qquad = t$

and so x * z $\in \lambda_t$. Hence λ_t is a positive implicative ideal of X.

Conversely, assume that λ is a fuzzy set in X such that

$\lambda_t = \{x \in X \mid \lambda(x) \geq t\}$

is a positive implicative ideal of X. If there is $a \in$ X such that

$\lambda(0) < \lambda(a) \wedge 0.5$.

Then

$\lambda(0) < t < \lambda(a) \wedge 0.5$

for some t $\in$ (0, 0.5], and so $0 \notin \lambda_t$. This is a contradiction. Hence

$\lambda(0) \geq \lambda(x) \wedge 0.5$, $\forall$ x $\in$ X.

Assume that there exist a, b, c $\in$ X such that

$\lambda(a * c) < \lambda((a * b) * c) \wedge \lambda(b * c) \wedge 0.5$

Taking

$t = \dfrac{1}{2} (\lambda(a * c) + (\lambda((a * b) * c) \wedge \lambda(b * c) \wedge 0.5))$

We get t $\in$ (0, 0.5] and $\lambda(a * c) < t < \lambda((a * b) * c) \wedge \lambda(b * c) \wedge 0.5$. Thus

$(a * b) * c \in \lambda_t$ and b * c $\in \lambda_t$ but $a * c \notin \lambda_t$.

This is a contradiction. Hence

$\lambda(x * z) \geq \lambda((x * y) * z) \wedge \lambda(y * z) \wedge 0.5$

It follows from Theorem 4.6 that λ is an ($\in$, $\in \vee$ q)-fuzzy positive implicative ideal of X.

Theorem 4.9. Let P be an positive implicative ideal of X and let λ be a fuzzy set in BCK-algebra X such that

(i) $\lambda(x) = 0$ for all x $\in$ X\P,

(j) $\lambda(x) \geq 0.5$ for all x $\in$ P.

Then λ is a (q, $\in \vee$ q)-fuzzy positive implicative ideal of X.

Proof. Let x $\in$ X and t $\in$ (0, 1] be such that $x_t q \lambda$. Then $\lambda(x) + t > 1$ and so x $\in$ P. Thus $\lambda(x) \geq$ 0.5 and t > 0.5. Since $0 \in$ P, it follows that

$\lambda(0) + t > 0.5 + 0.5$

$\qquad = 1$

So $0_t \in \vee q \lambda$. Let x, y, z $\in$ X and t_1, $t_2 \in$ (0, 1] be such that

$((x * y) * z)_{t_1} \in \vee q \lambda$ and $(y * z)_{t_2} \in \vee q \lambda$.

Then

$\lambda((x * y) * z) + t_1 > 1$ and $\lambda(y * z) + t_2 > 1$.

Thus

$(x * y) * z \in$ P and $y * z \in$ P.

For, $(x * y) * z \notin$ P (resp. $y * z \notin$ P), then

$\lambda((x * y) * z) = 0$ (resp. $\lambda(y * z) = 0$)

and so $t_1 > 1$ (resp. $t_2 > 1$), a contradiction. Since P is a positive implicative ideal of X, it follows that $x *$ $z \in$ P so that $\lambda(x * z) \geq 0.5$. If $t_1 \leq 0.5$ or $t_2 \leq 0.5$, then

$\lambda(x * z) \geq 0.5 \geq t_1 \wedge t_2$

Hence $(x * z)_{t_1 \wedge t_2} \in \lambda$. If $t_1 > 0.5$ and $t_2 > 0.5$, then

$$\lambda(x * z) + t_1 \wedge t_2 > 0.5 + 0.5$$
$$= 1$$

and so

$(x * z)_{t_1 \wedge t_2}$ q λ.

Consequently

$(x * z)_{t_1 \wedge t_2} \in \vee$ q λ.

Hence λ is a (q, $\in \vee$ q)-fuzzy positive implicative ideal of X.

Conclusion

In the study of fuzzy algebraic system, we see that the fuzzy positive implicative ideals with special properties always play a central role.

In this paper, we define (α, β)-fuzzy positive implicative ideals in BCK-algebras and give several properties of fuzzy positive implicative ideals in BCK-algebras in terms of these notions.

We believe that the research along this direction can be continued, and in fact, some results in this paper have already constituted a foundation for further investigation concerning the further development of fuzzy BCK-algebras and their applications in other branches of algebra. In the future study of fuzzy BCK-algebras, perhaps the following topics are worth to be considered:

(1)   To characterize other classes of BCK-algebras by using this notion;

(2)   To apply this notion to some other algebraic structures;

(3)   To consider these results to some possible applications in computer sciences and information systems in the future.

Acknowledgements

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