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Some extended Tauberian theorems for \((A)^{(k)}(C,\alpha)\) summability method

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ABSTRACT. In this paper, some new Tauberian conditions are introduced for \((A)^{(k)}(C,\alpha)\) summability method.

Keywords: Abel summability, \((A,C)\) summability, \((A)^{(k)}(C,\alpha)\) summability, Tauberian conditions and theorems.

Introduction

Let \(\sum s_n\) be a given infinite series of real numbers with the sequence of n-th partial sums \(s_n = (\sum_{k=0}^{n}a_k)\). For a sequence \(s_n\), we define \(\Delta s_n = s_n - s_{n-1}\), with \(\Delta s_0 = s_0\). Let \(A^{(\alpha)}\) be defined by the generating function \((1-x)^{-\alpha-1} = \sum_{n=0}^{\infty} A^{(\alpha)} x^n\) \(|x| < 1\), where \(\alpha < -1\). A sequence \(s_n\) is said to be \((C,\alpha)\) summable to \(s\) and we write \(s_n \to s(C,\alpha)\), if

\[
s_n^{(\alpha)} = \frac{1}{A^{(\alpha)}} \sum_{k=0}^{n} A^{(\alpha)}_{n-k}s_k \to s
\]

as \(n \to \infty\). Note that \((C,0)\) summability is the ordinary convergence. We write \(\tau_n = na_n\) and define \(\tau^{(\alpha)}_n\) as the \((C,\alpha)\) mean of \(\tau_n\).

A sequence \(s_n\) is said to be Abel summable to \(s\), and we write \(s_n \to s(A)\), if the series \(\sum_{n=0}^{\infty} a_n x^n\) is convergent for \(0 \leq x < 1\) and tends to \(s\) as \(x \to 1^{-}\). A sequence \(s_n\) is said to be \((A,C,\alpha)\) summable to \(s\) and we write \(s_n \to s(A)(C,\alpha)\), if \((1-x)^{\alpha} \sum_{n=0}^{\infty} a_n x^n\) is convergent for \(0 \leq x < 1\) and tends to \(s\) as \(x \to 1^{-}\). If we take \(\alpha = 0\), then \((A,C)\) summability reduces to Abel summability.

A generalization of Abel summability is introduced by (LITTLEWOOD, 1967) as follows. Let \(f(x) = \sum_{n=0}^{\infty} a_n x^n\), \(0 \leq x < 1\). Let

\[
f_1(x) = \frac{1}{1-x} \int_0^1 f(t) dt,
\]

and suppose that \(\int_0^1 f_1(t) dt\) exists as \(\lim_{x \to 1^{-}} \int_0^1 f(t) dt\). Let

\[
f_2(x) = \frac{1}{1-x} \int_0^1 f_1(t) dt,
\]

an so on. We write

\[
f_k(x) = \frac{1}{1-x} \int_0^1 f_{k-1}(t) dt
\]

for positive integer \(k\). The \(f_k(x)\) is called the \(k\)-tuple average of \(f\) as \(x \to 1^{-}\) by (LITTLEWOOD, 1967). If \(\lim_{x \to 1^{-}} f_k(x) = s\) for some positive integer \(k\), we say that \(s_n\) is \((A)^{(k)}\) summable to \(s\).

Let \(g(x) = (1-x) \sum_{n=0}^{\infty} a_n x^n\), \(0 \leq x < 1\), \(\alpha > -1\). If \(\lim_{x \to 1^{-}} g_k(x) = s\) for some positive integer \(k\), we say that \(s_n\) is \((A)^{(k)}(C,\alpha)\) summable to \(s\).
A sequence \((s_n)\) is said to be slowly oscillating (STANOJEVIĆ, 1998) if,

\[
\lim_{n \to \infty} \limsup_{n \to \infty} |s_{n+k} - s_n| = 0.
\]

A sequence \((s_n)\) is said to be \((C, \alpha)\) slowly oscillating if \((s_n^\alpha)\) is slowly oscillating.

We use the symbols \(s_n = o(1), s_n = O(1)\) to mean respectively that \(s_n \to 0\) as \(n \to \infty\) and that \((s_n)\) is bounded for large enough \(n\). We also write \(s_n = o(1)(C, \alpha)\) to mean that \(s_n^\alpha = o(1)\).

Hardy (1910) proved that \(na_n = O(1)\) is a Tauberian condition for \((C, \alpha), \alpha > 0\), summability of \((s_n)\). Later, Littlewood (1911) proved that \((C, \alpha)\) summability of \((s_n)\) in Hardy’s theorem (HARDY, 1910) can be replaced by the Abel summability of \((s_n)\). (HARDY, LITTLEWOOD, 1913) replaced the condition \(na_n = O(1)\) by the one-sided Tauberian condition \(na_n \geq -H\) for some positive constant \(H\). Littlewood (1911) proved that if \((s_n)\) is Abel summable to \(S\) and \(s_n = O(1)\), then \((s_n)\) is \((C, 1)\) summable to \(S\). Szasz (1935) generalized Littlewood’s theorem (LITTLEWOOD, 1911) which states that if \((s_n)\) is Abel summable to \(S\) and \(s_n^\alpha \geq -H\) for some positive constant \(H\), then \((s_n)\) is \((C, 1)\) summable to \(S\). Pati (2002) obtained a more general theorem which states that if \((s_n)\) is \((A)\) summable for some \(\alpha \geq 0\) to \(S\) and \(s_n^\alpha \geq -H\) for some positive constant \(H\), then \((s_n)\) is \((C, \alpha)\) summable to \(S\). Quite recently, several new Tauberian conditions for \((A(C, \alpha))\) summability method have been obtained in Çanak et al. (2010), Erdem and Çanak (2010), and Çanak and Erdem, (2011).

Littlewood (1967) proved that \(na_n \geq -H\) for some positive constant \(H\) is a Tauberian condition for \((A(k))\), where \(k\) is a positive integer \(k\), summability of \((s_n)\). Pati (2007) established two Tauberian theorems which are more general than a theorem of Pati (2002) and a theorem of Littlewood (1967).

Our aim in this paper is to introduce some new conditions in terms of \(\tau_n^\alpha\) to recover \((C, \alpha)\) convergence of \((\tau_n)\) from its \((A(k))(C, \alpha)\) summability. Namely, we prove the following Tauberian theorems.

**Theorem 1.1**

If, for some positive integer \(k\) and \(\alpha \geq 0\), \((\tau_n)\) is \((A(k))(C, \alpha)\) summable to \(S\) and

\[
n\Delta \tau_n^\alpha = o(1)
\]

then \((\tau_n)\) is \((C, \alpha - 1)\) summable to \(S\) and \((s_n)\) is \((C, \alpha - 1)\) slowly oscillating.

**Theorem 1.2**

If, for some positive integer \(k\) and \(\alpha \geq 0\), \((\tau_n)\) is \((A(k))(C, \alpha)\) summable to \(S\) and for some positive constant \(H\)

\[
n\Delta \tau_n^\alpha \geq -H
\]

then \((\tau_n)\) is \((C, \alpha)\) summable to \(S\) and \((s_n)\) is \((C, \alpha)\) slowly oscillating.

**Theorem 1.3**

If, for some positive integer \(k\) and \(\alpha \geq 0\), \((\tau_n)\) is \((A(k))(C, \alpha)\) summable to \(S\) and for some positive constant \(H\)

\[
n\Delta \tau_n^\alpha = O(1)
\]

then \((\tau_n)\) is \((C, \alpha + \delta - 1)\) summable to \(S\) for every \(\delta > 0\).

Proofs of our Theorems depend on the following Tauberian theorem due to Littlewood (1967).

**Theorem 1.4**

If for some positive integer \(k\), \((s_n)\) is \((A(k))\) summable to \(S\), then \(na_n \geq -H\) for some positive constant \(H\) is a Tauberian condition for the convergence of \((s_n)\) to \(S\).

**Lemmas**

For the proof of our theorems, we need the following lemmas.

**Lemma 2.1**

Kogbetliantz (1925, 1931) For \(\alpha > -1\),

\[
\tau_n^\alpha = n\Delta s_n^\alpha = n(s_n^\alpha - s_{n-1}^\alpha)
\]
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**Lemma 2.2**

Çanak et al. (2010) For
\[
\alpha > -1, \ n\Delta r_n^\alpha = (\alpha + 1)(r_n^\alpha - r_{n-1}^\alpha)
\]
\[(1)\]

**Lemma 2.3**

(HARDY, 1991) If \(s_n^\alpha \to s \) as \(n \to \infty\), \(\alpha > -1\), then \(s_n^{\alpha + \delta} \to s \) as \(n \to \infty\) for every \(\delta > 0\).

**Lemma 2.4**

(HARDY, 1991) If \(s_n^\alpha \to s(C, \beta)\), then \(s_n^{\alpha + \beta} \to s\) for \(\alpha \geq 0, \ \beta \geq 0\), and conversely.

**Lemma 2.5**

(PEYERIMHOFF, 1969) All the Cesàro methods of positive order are equivalent for bounded sequences. More precisely, if \(s_n^\alpha \to s\) as \(n \to \infty\) for some \(\alpha > 0\), then \(s_n^\beta \to s\) as \(n \to \infty\) for some \(\beta > 0\).

**Proofs of Theorems**

**Proof of Theorem 1.1**

By hypothesis, we have \(f_k(x) \to s\) as \(x \to 1^-\), where \(f_k(x)\) is the \(k\)-tuple average of:
\[
f(x) = (1-x)\sum_{n=0}^{\infty} r_n^\alpha x^n = \sum_{n=0}^{\infty} (r_n^\alpha - r_{n-1}^\alpha)x^n, \ 0 \leq x < 1, (r_0^\alpha = 0).
\]
\[(4)\]

The condition (1) implies that \(n\Delta r_n^\alpha \geq -H\) for some positive constant \(H\). By Theorem 1.4, we get
\[
\sum_{n=0}^{\infty} (r_n^\alpha - r_{n-1}^\alpha), (r_0^\alpha = 0)
\]
\[(5)\]
is convergent to \(s\), i.e.,
\[
r_n^\alpha \to s, n \to \infty.
\]
\[(6)\]

This means that \((r_n)\) is \((C, \alpha)\) summable to \(s\).

By Lemma 2.2, we have
\[
n\Delta r_n^\alpha = \alpha (s_n^{\alpha - 1} - r_n^\alpha).
\]
\[(7)\]

It follows from (1) and (6) that
\[
r_n^{\alpha - 1} \to s, n \to \infty,
\]
\[(8)\]

which means that \((r_n)\) is \((C, \alpha - 1)\) summable to \(s\). By Lemma 2.1, we have
\[
S_n^{\alpha - 1} = \sum_{k=1}^{n} \frac{r_k^{\alpha - 1}}{k}.
\]
\[(9)\]

Since \((r_n^{\alpha - 1})\) converges to \(s\), there exists \(M > 0\) such that
\[
|r_n^{\alpha - 1}| \leq M
\]
\[(10)\]

for all \(n\). For any \(n < k < \infty\), we have
\[
|s_k^{\alpha - 1} - s_n^{\alpha - 1}| \leq \frac{M}{k} \leq M \frac{|\lfloor k/2 \rfloor| - n}{n},
\]
\[(11)\]

whence we conclude that
\[
\limsup_n \max_{n+1 \leq k \leq |\lfloor k/2 \rfloor|} |s_k^{\alpha - 1} - s_n^{\alpha - 1}| \leq M(\lambda - 1).
\]
\[(12)\]

Letting \(\lambda \to 1^+\), we obtain \((s_n)\) is \((C, \alpha - 1)\) slowly oscillating. This completes the proof of Theorem 1.1.

**Corollary 3.1**

If, for some positive integer \(k\), \((r_n)\) is \((A)\)\(^{(k)}\)\((C, \alpha)\) summable to \(s\), and (1) holds, then \((r_n)\) is convergent to \(s\) and \((s_n)\) is slowly oscillating.

**Proof**

Take \(\alpha = 1\) in Theorem 1.1.

**Proof of Theorem 1.2**

We have \((r_n)\) is \((C, \alpha)\) summable to \(s\) by Theorem 1.4. That \((s_n)\) is \((C, \alpha)\) slowly oscillating follows from Lemma 2.2.

**Proof of Theorem 1.3**

The condition (3) implies that
\[
n\Delta r_n^\alpha \geq -H
\]
\[(13)\]

for some positive constant \(H\). By Theorem 1.2, we have
\[
r_n \to s(C, \alpha).
\]
\[(14)\]

By Lemma 2.3,
\[
r_n \to s(C, \alpha + 1)
\]
\[(15)\]
and by Lemma 2.2,
\[ n\Delta \tau_n^{\alpha+1} = \alpha(\tau_n^{\alpha} - \tau_n^{\alpha+1}) = o(1), \tag{16} \]
which is equivalent to
\[ n\Delta \tau_n^{\alpha} = o(1)(C,1) \tag{17} \]
by Lemma 2.4. Since \( n\Delta \tau_n^{\alpha} = O(1) \) by hypothesis, we have, by Lemma 2.5,
\[ n\Delta \tau_n^{\alpha} \rightarrow 0(C,\delta) \tag{18} \]
for every \( \delta > 0 \), which is equivalent to
\[ n\Delta \tau_n^{\alpha+\delta} = o(1) \tag{19} \]
by Lemma 2.4.
By Lemma 2.2, we have
\[ n\Delta \tau_n^{\alpha+\delta} = (\alpha + \delta)(\tau_n^{\alpha+\delta-1} - \tau_n^{\alpha+\delta}) = o(1). \tag{20} \]
By Lemma 2.3,
\[ \tau_n^{\alpha+\delta} \rightarrow s, n \rightarrow \infty \tag{21} \]
It now follows from (20) that
\[ \tau_n^{\alpha+\delta-1} \rightarrow s, n \rightarrow \infty, \tag{22} \]
which is equivalent to
\[ \tau_n \rightarrow s(C,\alpha+\delta-1). \tag{23} \]
This completes the proof of Theorem 1.3.

**Corollary 3.2**
If, for some positive integer \( k \), \( \tau_n \) is \( (A)^{(k)}(C,1) \) summable to \( s \), and (3) holds, then \( \tau_n \) is \( (C,\delta) \) summable to \( s \) for every \( \delta > 0 \).

**Proof**
Take \( \alpha = 1 \) in Theorem 1.3.

**Corollary 3.3**
If, for some positive integer \( k \) and \( 0 < \alpha < 1 \), \( \tau_n \) is \( (A)^{(k)}(C,\alpha) \) summable to \( s \), and (3) holds, then \( \tau_n \) is convergent to \( s \).

**Proof**
Take \( \delta = 1 - \alpha \ (0 < \alpha < 1) \) in Theorem 1.3.

**Corollary 3.4**
If, for some positive integer \( k \), \( \tau_n \) is \( (A)^{(k)} \) summable to \( s \), and
\[ n\Delta(n\alpha_n) = O(1), \tag{24} \]
then \( \tau_n \) is \( (C,\delta-1) \) summable to \( s \) for every \( \delta > 0 \).

**Proof**
Take \( \alpha = 0 \) in Theorem 1.3.

**Conclusion**
New Tauberian theorems for the product \( (A)^{(k)} \) and \( (C,\alpha) \) summability methods have been established. Some new Tauberian conditions in terms of \( (C,\alpha) \) mean of \( \tau_n \) have been obtained to recover \( (C,\alpha) \) convergence of \( \tau_n \) and slow oscillation of \( (C,\alpha) \) mean from \( (A)^{(k)}(C,\alpha) \) summability of \( \tau_n \).

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**References**
KOGBETLIANTZ, E. Sur le séries absolument sommables par la méthode des moyennes arithmétiques.
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