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New type surfaces in terms of B-Smarandache Curves in Sol^3

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ABSTRACT. In this work, new type ruled surfaces in terms of B- Smarandache TM_1 curves of biharmonic B- slant helices in the SOL^3 are studied. We characterize the B- Smarandache TM_1 curves in terms of their Bishop curvatures. Additionally, we express some interesting relations.

Keywords: bienergy, B-slant helix, sol space, curvature, ruled surface.

Novos tipos de superfícies conforme as Curvas de B-Smarandache em Sol^3

RESUMO. Analisa-se novos tipos de superfícies conforme as curvas B- Smarandache TM_1 das hélices com inclinação B bi-harmonica. Caracterizam-se as curvas B- Smarandache TM_1 conforme as curvaturas de Bishop, acresentando outras relações interessantes.

Palavras-chave: bi-energia; hélice com inclinação B; espaço SOL; curvatura, superfície reguada.

Introduction

Ruled surfaces have been popular in architecture. Structural elegance, these and many other contributions are in contrast to recent free-form architecture. Applied mathematics and in particular geometry have initiated the implementation of comprehensive frameworks for modeling and mastering the complexity of today's architectural needs shapes in an optimal sense by ruled surfaces, (CARMO, 1976; LIUA et al., 2007).

A smooth map $\phi : N \to M$ is said to be biharmonic if it is a critical point of the bienergy functional (CADDEO et al., 2004):

$$E_2(\phi) = \int_N \frac{1}{2} \left| \tau(\phi) \right|^2 dv_h,$$

where $\tau(\phi) := \mathrm{tr}\nabla^{\phi} d\phi$ is the tension field of ϕ.

The Euler--Lagrange equation of the bienergy is given by $\tau_2(\phi) = 0$. Here the section $\tau_2(\phi)$ is defined by

$$\tau_2(\phi) = -\Delta_{\phi}\tau(\phi) + \mathrm{tr}R\left(\tau(\phi), d\phi\right)d\phi, \qquad (1.1)$$

and called the bitension field of ϕ. Non-harmonic biharmonic maps are called proper biharmonic maps, (ARSLAN et al., 2005; DIMITRIC, 1992; EELLS; LEMAIRE, 1978; EELLS; SAMPSON, 1964; JIANG, 1986).

New methods for constructing a canal surface surrounding a biharmonic curve in the Lorentzian Heisenberg group $Heis^3$ were given, (KORPINAR; TURHAN, 2010; 2011; 2012). Also, in (TURHAN; KORPINAR, 2010a; 2010b) they characterized biharmonic curves in terms of their curvature and torsion. Also, by using timelike biharmonic curves, they give explicit parametrizations of canal surfaces in the Lorentzian Heisenberg group $Heis^3$.

This study is organised as follows: Firstly, we study B- Smarandache TM_1 curves of biharmonic B- slant helices in the SOL^3. Additionally, we characterize the B- Smarandache TM_1 curves in terms of their Bishop curvatures. Finally, we express some interesting relations.

Riemannian Structure of Sol Space SOL^3

Sol space, one of Thurston's eight 3-dimensional geometries, can be viewed as $\mathbb{R}^3$ provided with Riemannian metric

$$g_{SOL^3} = e^{2z}dx^2 + e^{-2z}dy^2 + dz^2,$$

where (x, y, z) are the standard coordinates in $\mathbb{R}^3$.

Note that the Sol metric can also be written as:

$$g_{SOL^3} = \sum_{i=1}^{3} \boldsymbol{\omega}^i \otimes \boldsymbol{\omega}^i,$$

where

$$\boldsymbol{\omega}^1 = e^z dx,\ \boldsymbol{\omega}^2 = e^{-z} dy,\ \boldsymbol{\omega}^3 = dz,$$

and the orthonormal basis dual to the 1-forms is

$$\mathbf{e}_1 = e^{-z}\frac{\partial}{\partial x},\ \mathbf{e}_2 = e^{z}\frac{\partial}{\partial y},\ \mathbf{e}_3 = \frac{\partial}{\partial z}. \tag{2.1}$$

Proposition 2.1. For the covariant derivatives of the Levi-Civita connection of the left-invariant metric g_{SOL^3}, defined above the following is true:

$$\nabla = \begin{pmatrix} -\mathbf{e}_3 & 0 & \mathbf{e}_1 \\ 0 & \mathbf{e}_3 & -\mathbf{e}_2 \\ 0 & 0 & 0 \end{pmatrix}, \tag{2.2}$$

where the (i, j)-element in the table above equals $\nabla_{\mathbf{e}_i}\mathbf{e}_j$ for our basis

$$\{\mathbf{e}_k, k = 1,2,3\} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}.$$

Lie brackets can be easily computed as, (BLAIR, 1976), (OU; WANG, 2008):

$$[\mathbf{e}_1, \mathbf{e}_2] = 0,\ [\mathbf{e}_2, \mathbf{e}_3] = -\mathbf{e}_2,\ [\mathbf{e}_1, \mathbf{e}_3] = \mathbf{e}_1.$$

The isometry group of SOL^3 has dimension 3. The connected component of the identity is generated by the following three families of isometries:

$$(x, y, z) \to (x + c, y, z),$$
$$(x, y, z) \to (x, y + c, z),$$
$$(x, y, z) \to (e^{-c}x, e^{c}y, z + c).$$

$\mathbb{B}-$ Smarandache $\mathbf{TM}_1$ Curves of Biharmonic $\mathbb{B}-$ Slant Helices in Sol Space SOL^3

Assume that $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ be the Frenet frame field along γ. Then, the Frenet frame satisfies the following Frenet--Serret equations:

$$\begin{aligned} \nabla_{\mathbf{T}}\mathbf{T} &= \kappa\mathbf{N}, \\ \nabla_{\mathbf{T}}\mathbf{N} &= -\kappa\mathbf{T} + \tau\mathbf{B}, \\ \nabla_{\mathbf{T}}\mathbf{B} &= -\tau\mathbf{N}, \end{aligned} \tag{3.1}$$

where κ is the curvature of γ and τ its torsion and

$$\begin{aligned} &g_{\mathrm{SOL}^3}(\mathbf{T}, \mathbf{T}) = 1, g_{\mathrm{SOL}^3}(\mathbf{N}, \mathbf{N}) = 1, g_{\mathrm{SOL}^3}(\mathbf{B}, \mathbf{B}) = 1, \\ &g_{\mathrm{SOL}^3}(\mathbf{T}, \mathbf{N}) = g_{\mathrm{SOL}^3}(\mathbf{T}, \mathbf{B}) = g_{\mathrm{SOL}^3}(\mathbf{N}, \mathbf{B}) = 0. \end{aligned} \tag{3.2}$$

The Bishop frame or parallel transport frame is an alternative approach to defining a moving frame that is well defined even when the curve has vanishing second derivative. The Bishop frame is expressed as

$$\begin{aligned} \nabla_{\mathbf{T}}\mathbf{T} &= k_1\mathbf{M}_1 + k_2\mathbf{M}_2, \\ \nabla_{\mathbf{T}}\mathbf{M}_1 &= -k_1\mathbf{T}, \\ \nabla_{\mathbf{T}}\mathbf{M}_2 &= -k_2\mathbf{T}, \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} &g_{\mathrm{SOL}^3}(\mathbf{T}, \mathbf{T}) = 1, g_{\mathrm{SOL}^3}(\mathbf{M}_1, \mathbf{M}_1) = 1, g_{\mathrm{SOL}^3}(\mathbf{M}_2, \mathbf{M}_2) = 1, \\ &g_{\mathrm{SOL}^3}(\mathbf{T}, \mathbf{M}_1) = g_{\mathrm{SOL}^3}(\mathbf{T}, \mathbf{M}_2) = g_{\mathrm{SOL}^3}(\mathbf{M}_1, \mathbf{M}_2) = 0. \end{aligned} \tag{3.4}$$

Here, we shall call the set $\{\mathbf{T}, \mathbf{M}_1, \mathbf{M}_2\}$ as Bishop trihedra, k_1 and k_2 as Bishop curvatures and $\mathrm{U}(s) = \arctan\frac{k_2}{k_1}$, $\tau(s) = \mathrm{U}'(s)$ and $\kappa(s) = \sqrt{k_1^2 + k_2^2}$.

Bishop curvatures are defined by (BISHOP, 1975)

$$k_1 = \kappa(s)\cos\mathrm{U}(s),$$
$$k_2 = \kappa(s)\sin\mathrm{U}(s).$$

The relation matrix may be expressed as

$$\begin{aligned} \mathbf{T} &= \mathbf{T}, \\ \mathbf{N} &= \cos\mathrm{U}(s)\mathbf{M}_1 + \sin\mathrm{U}(s)\mathbf{M}_2, \\ \mathbf{B} &= -\sin\mathrm{U}(s)\mathbf{M}_1 + \cos\mathrm{U}(s)\mathbf{M}_2. \end{aligned}$$

On the other hand, using above equation we have

$$\begin{aligned} \mathbf{T} &= \mathbf{T}, \\ \mathbf{M}_1 &= \cos\mathrm{U}(s)\mathbf{N} - \sin\mathrm{U}(s)\mathbf{B} \\ \mathbf{M}_2 &= \sin\mathrm{U}(s)\mathbf{N} + \cos\mathrm{U}(s)\mathbf{B}. \end{aligned}$$

With respect to the orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ we can write

$$\mathbf{T} = T^1 e_1 + T^2 e_2 + T^3 e_3,$$

$$\mathbf{M}_1 = M_1^1 \mathbf{e}_1 + M_1^2 \mathbf{e}_2 + M_1^3 \mathbf{e}_3, \tag{3.5}$$

$$\mathbf{M}_2 = M_2^1 \mathbf{e}_1 + M_2^2 \mathbf{e}_2 + M_2^3 \mathbf{e}_3.$$

Theorem 3.1. $\gamma : I \to \mathrm{SOL}^3$ is a biharmonic curve according to Bishop frame if and only if

$$k_1^2 + k_2^2 = \text{constant} \neq 0,$$

$$k_1'' - \left[k_1^2 + k_2^2\right] k_1 = -k_1\left[2M_2^3 - 1\right] - 2k_2 M_1^3 M_2^3, \tag{3.6}$$

$$k_2'' - \left[k_1^2 + k_2^2\right] k_2 = 2k_1 M_1^3 M_2^3 - k_2\left[2M_1^3 - 1\right].$$

Definition 3.2. A regular curve $\gamma : I \to \mathrm{SOL}^3$ is called a slant helix provided the unit vector $\mathbf{M}_1$ of the curve γ has constant angle E with unit vector u, that is

$$g_{\mathrm{SOL}^3}\left(\mathbf{M}_1(s), u\right) = \cos \mathrm{E} \ \text{ for all } s \in I. \tag{3.7}$$

The condition is not altered by reparametrization, so without loss of generality we may assume that slant helices have unit speed. The slant helices can be identified by a simple condition on natural curvatures.

To separate a slant helix according to Bishop frame from that of Frenet- Serret frame, in the rest of the paper, we shall use notation for the curve defined above as B-slant helix.

Theorem 3.3. Let $\gamma : I \to \mathrm{SOL}^3$ be a unit speed non-geodesic biharmonic $\mathrm{B}-$slant helix. Then, the parametric equations of γ are

$$\begin{aligned}
\mathbf{x}(s) &= \frac{\cos \mathrm{E} e^{\sin \mathrm{E} s - \mathrm{C}_3}}{\mathrm{C}_1^2 + \sin^2 \mathrm{E}}[\sin \mathrm{E} \cos[\mathrm{C}_1 s + \mathrm{C}_2] + \mathrm{C}_1 \sin[\mathrm{C}_1 s + \mathrm{C}_2]] + \mathrm{C}_4, \\
\mathbf{y}(s) &= \frac{\cos \mathrm{E} e^{-\sin \mathrm{E} s + \mathrm{C}_3}}{\mathrm{C}_1^2 + \sin^2 \mathrm{E}}[\mathrm{C}_1 \cos[\mathrm{C}_1 s + \mathrm{C}_2] + \sin \mathrm{E} \sin[\mathrm{C}_1 s + \mathrm{C}_2]] + \mathrm{C}_5, \\
\mathbf{z}(s) &= -\sin \mathrm{E} s + \mathrm{C}_3,
\end{aligned} \tag{3.8}$$

where $\mathrm{C}_1, \mathrm{C}_2, \mathrm{C}_3, \mathrm{C}_4, \mathrm{C}_5$ are constants of integration.

Corollary 3.4. Let $\gamma : I \to \mathrm{SOL}^3$ be a unit speed non-geodesic biharmonic $\mathrm{B}-$slant helix. Then, the position vector of γ is

$$\begin{aligned}
\gamma(s) = {} & [\frac{\cos \mathrm{E}}{\mathrm{C}_1^2 + \sin^2 \mathrm{E}}[\sin \mathrm{E} \cos[\mathrm{C}_1 s + \mathrm{C}_2] + \mathrm{C}_1 \sin[\mathrm{C}_1 s + \mathrm{C}_2]] + \mathrm{C}_4 e^{-\sin \mathrm{E} s + \mathrm{C}_3}]\mathbf{e}_1 \\
& + [\frac{\cos \mathrm{E}}{\mathrm{C}_1^2 + \sin^2 \mathrm{E}}[\mathrm{C}_1 \cos[\mathrm{C}_1 s + \mathrm{C}_2] + \sin \mathrm{E} \sin[\mathrm{C}_1 s + \mathrm{C}_2]] + \mathrm{C}_5 e^{\sin \mathrm{E} s - \mathrm{C}_3}]\mathbf{e}_2 \\
& + [-\sin \mathrm{E} s + \mathrm{C}_3]\mathbf{e}_3,
\end{aligned} \tag{3.9}$$

where $\mathrm{C}_1, \mathrm{C}_2, \mathrm{C}_3, \mathrm{C}_4, \mathrm{C}_5$ are constants of integration.

To separate a Smarandache $\mathbf{TM}_1$ curve according to Bishop frame from that of Frenet- Serret frame, in the rest of the paper, we shall use notation for the curve defined above as $\mathrm{B}-$ Smarandache $\mathbf{TM}_1$ curve.

Definition 3.5. Let $\gamma : I \to \mathrm{SOL}^3$ be a unit speed $\mathrm{B}-$slant helix in the Sol Space SOL^3 and $\{\mathbf{T}, \mathbf{M}_1, \mathbf{M}_2\}$ be its moving Bishop frame. $\mathrm{B}-$ Smarandache $\mathbf{TM}_1$ curves are defined by

$$\gamma_{\mathbf{TM}_1} = \frac{1}{\sqrt{2k_1^2 + k_2^2}}\left(\mathbf{T} + \mathbf{M}_1\right). \tag{3.10}$$

Theorem 3.6. Let $\gamma : I \to \mathrm{SOL}^3$ be a unit speed biharmonic $\mathrm{B}-$slant helix in the Sol Space SOL^3. Then, the equation of $\mathrm{B}-$ Smarandache $\mathbf{TM}_1$ curves of biharmonic $\mathrm{B}-$ slant helix is given by

$$\begin{aligned}
\gamma_{\mathbf{TM}_1}(s) = {} & \frac{1}{\sqrt{2k_1^2 + k_2^2}}[\cos \mathrm{E} \cos[\mathrm{C}_1 s + \mathrm{C}_2] + \sin \mathrm{E} \cos[\mathrm{C}_1 s + \mathrm{C}_2]]\mathbf{e}_1 \\
& + \frac{1}{\sqrt{2k_1^2 + k_2^2}}[\sin \mathrm{E} \sin[\mathrm{C}_1 s + \mathrm{C}_2] + \cos \mathrm{E} \sin[\mathrm{C}_1 s + \mathrm{C}_2]]\mathbf{e}_2 \\
& + \frac{1}{\sqrt{2k_1^2 + k_2^2}}[\cos \mathrm{E} - \sin \mathrm{E}]\mathbf{e}_3,
\end{aligned} \tag{3.11}$$

where $\mathrm{C}_1, \mathrm{C}_2$ are constants of integration, [(KORPINAR; TURHAN, 2012) 12].

Then, the obtained parametric equations are illustrated in Figure 1 and 2:

We can use Mathematica in Theorem 3.3, yields

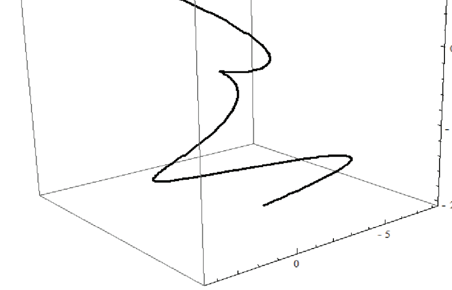

Figure 1. For $\mathrm{E} = 0$.

Also, we may use Mathematica in Theorem 3.6, yields

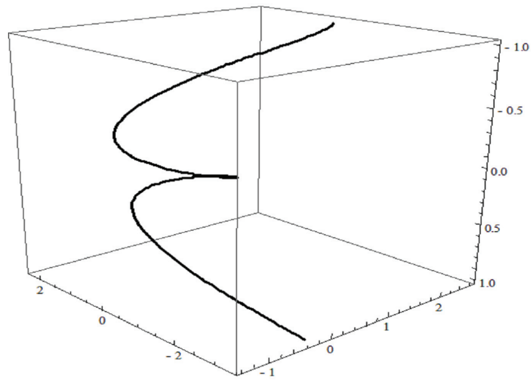

Figure 2. For $\mathrm{E} = \pi/2$.

New Type Ruled Surfaces of $\mathrm{B}-$Smarandache $\mathbf{TM}_1$ Curves of Biharmonic $\mathrm{B}-$Slant Helices in Sol Space SOL^3

The purpose of this section is to construct new type ruled surfaces of $\mathrm{B}-$Smarandache $\mathbf{TM}_1$ curves of biharmonic $\mathrm{B}-$slant helices in Sol Space SOL^3.

We define new type ruled surface

$$\mathrm{A}_{\mathbf{TM}_1}(s,u)=\gamma(s)+u\gamma_{\mathbf{TM}_1}(s). \tag{4.1}$$

Theorem 4.1. Let $\gamma: I \to \mathrm{SOL}^3$ be a unit speed biharmonic $\mathrm{B}-$slant helix and A its new type ruled surface in the Sol Space SOL^3. Then, the equation of new type ruled surface of $\mathrm{B}-$Smarandache $\mathbf{TM}_1$ curves of biharmonic $\mathrm{B}-$slant helix is given by

$$\begin{aligned}\mathrm{A}_{\mathbf{TM}_1}(s,u)=&[[\frac{\cos\mathrm{E}}{\mathrm{C}_1^2+\sin^2\mathrm{E}}[\sin\mathrm{E}\cos[\mathrm{C}_1s+\mathrm{C}_2]+\mathrm{C}_1\sin[\mathrm{C}_1s+\mathrm{C}_2]]+\mathrm{C}_4e^{-\sin\mathrm{E}s+\mathrm{C}_3}\\&+\frac{u}{\sqrt{2k_1^2+k_2^2}}[\cos\mathrm{E}\cos[\mathrm{C}_1s+\mathrm{C}_2]+\sin\mathrm{E}\cos[\mathrm{C}_1s+\mathrm{C}_2]]]\mathbf{e}_1\\&+[[\frac{\cos\mathrm{E}}{\mathrm{C}_1^2+\sin^2\mathrm{E}}[\mathrm{C}_1\cos[\mathrm{C}_1s+\mathrm{C}_2]+\sin\mathrm{E}\sin[\mathrm{C}_1s+\mathrm{C}_2]]+\mathrm{C}_5e^{\sin\mathrm{E}s-\mathrm{C}_3}]\\&+\frac{u}{\sqrt{2k_1^2+k_2^2}}[\sin\mathrm{E}\sin[\mathrm{C}_1s+\mathrm{C}_2]+\cos\mathrm{E}\sin[\mathrm{C}_1s+\mathrm{C}_2]]]\mathbf{e}_2\\&+[[-\sin\mathrm{E}s+\mathrm{C}_3]+\frac{u}{\sqrt{2k_1^2+k_2^2}}[\cos\mathrm{E}-\sin\mathrm{E}]]\mathbf{e}_3,\end{aligned} \tag{4.2}$$

where $\mathrm{C}_1,\mathrm{C}_2$ are constants of integration.

Proof. Assume that γ is a non geodesic biharmonic B-slant helix according to Bishop frame.

For non-constant u, we obtain

$$\mathbf{M}_1=\sin\mathrm{E}\cos[\mathrm{C}_1s+\mathrm{C}_2]\mathbf{e}_1+\sin\mathrm{E}\sin[\mathrm{C}_1s+\mathrm{C}_2]\mathbf{e}_2+\cos\mathrm{E}\mathbf{e}_3, \tag{4.3}$$

where $\mathrm{C}_1,\mathrm{C}_2\in\mathrm{R}$.

Using Bishop frame, we have

$$\mathbf{T}=\cos\mathrm{E}\cos[\mathrm{C}_1s+\mathrm{C}_2]\mathbf{e}_1+\cos\mathrm{E}\sin[\mathrm{C}_1s+\mathrm{C}_2]\mathbf{e}_2-\sin\mathrm{E}\mathbf{e}_3. \tag{4.4}$$

Substituting (4.3) and (4.4) in (4.1) we have (4.2), which completes the proof.

In terms of Eqs. (2.1) and (4.2), we may give:

Corollary 4.2. Let $\gamma: I\to\mathrm{SOL}^3$ be a unit speed biharmonic $\mathrm{B}-$slant helix in the Sol Space SOL^3. Then, the parametric equations of new type ruled surface of $\mathrm{B}-$Smarandache $\mathbf{TM}_1$ curves of biharmonic $\mathrm{B}-$slant helix are given by

$$\begin{aligned}x_{\mathrm{A}_{\mathbf{TM}_1}}(s,u)=&[\frac{\cos\mathrm{E}e^{\sin\mathrm{E}s-\mathrm{C}_3}}{\mathrm{C}_1^2+\sin^2\mathrm{E}}[\sin\mathrm{E}\cos[\mathrm{C}_1s+\mathrm{C}_2]+\mathrm{C}_1\sin[\mathrm{C}_1s+\mathrm{C}_2]]+\mathrm{C}_4\\&+\frac{e^{-\frac{1}{\sqrt{2k_1^2+k_2^2}}[\cos\mathrm{E}-\sin\mathrm{E}]}}{\sqrt{2k_1^2+k_2^2}}u[\cos\mathrm{E}\cos[\mathrm{C}_1s+\mathrm{C}_2]+\sin\mathrm{E}\cos[\mathrm{C}_1s+\mathrm{C}_2]]],\\y_{\mathrm{A}_{\mathbf{TM}_1}}(s,u)=&[\frac{\cos\mathrm{E}e^{-\sin\mathrm{E}s+\mathrm{C}_3}}{\mathrm{C}_1^2+\sin^2\mathrm{E}}[\mathrm{C}_1\cos[\mathrm{C}_1s+\mathrm{C}_2]+\sin\mathrm{E}\sin[\mathrm{C}_1s+\mathrm{C}_2]]+\mathrm{C}_5\\&+\frac{ue^{\frac{1}{\sqrt{2k_1^2+k_2^2}}[\cos\mathrm{E}-\sin\mathrm{E}]}}{\sqrt{2k_1^2+k_2^2}}[\sin\mathrm{E}\sin[\mathrm{C}_1s+\mathrm{C}_2]+\cos\mathrm{E}\sin[\mathrm{C}_1s+\mathrm{C}_2]]],\\z_{\mathrm{A}_{\mathbf{TM}_1}}(s,u)=&[-\sin\mathrm{E}s+\mathrm{C}_3+\frac{u}{\sqrt{2k_1^2+k_2^2}}[\cos\mathrm{E}-\sin\mathrm{E}]],\end{aligned} \tag{4.5}$$

where $\mathrm{C}_1,\mathrm{C}_2$ are constants of integration.

Proof. Substituting (2.1) to (4.2), we have (4.5) as desired.

Finally, the obtained parametric equations are illustrated in Figure 3 and 4:

We can use Mathematica in Corollary 4.3, yields

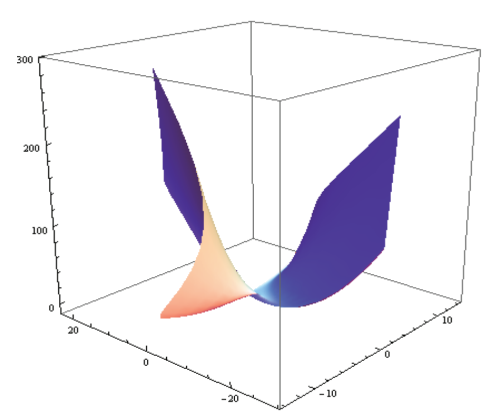

Figure 3. For $\mathrm{E} = 0$.

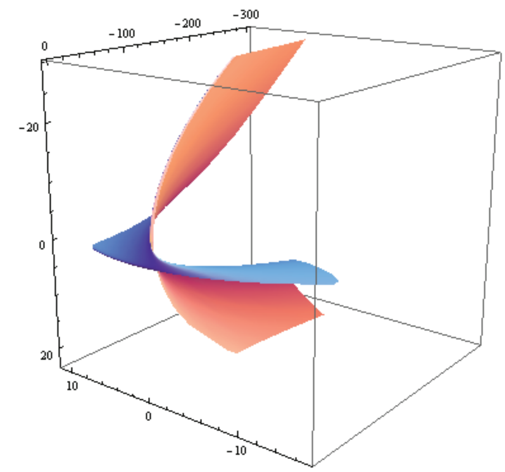

Figure 4. For $E = \pi/2$.

Conclusion

Ruled surfaces is that they can be generated by straight lines. A practical application of ruled surfaces is that they are used in civil engineering. Since building materials such as wood are straight, they can be thought of as straight lines. The result is that if engineers are planning to construct something with curvature, they can use a ruled surface since all the lines are straight.

In this paper, new type ruled surfaces in terms of $\mathbf{B}-$Smarandache $\mathbf{TM}_1$ curves of biharmonic $\mathbf{B}-$slant helices in the SOL^3 are studied.

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