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Universidade Estadual de Maringá
Maringá, Brasil

Available in: http://www.redalyc.org/articulo.oa?id=303241163011
On $\alpha$- $\tau$-disconnectedness and $\alpha$- $\tau$-connectedness in topological spaces

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ABSTRACT. The aims of this paper is to introduce new approach of separate sets, disconnected sets and connected sets called $\alpha$- $\tau$-separate sets, $\alpha$- $\tau$-disconnected sets and $\alpha$- $\tau$-connected sets of topological spaces with the help of $\alpha$-open and $\alpha$-closed sets. On the basis of new introduce approach, some relationship of $\alpha$-$\tau$-disconnected and $\alpha$-$\tau$-connected set with $\alpha$-$\tau$-separate sets have been investigated thoroughly.

Keywords: $\alpha$-open set, $\alpha$-closed set, $\alpha$-closure, $\alpha$- $\tau$-separate sets, $\alpha$- $\tau$-disconnected sets and $\alpha$- $\tau$-connected sets.

Introduction

There are several natural approaches that can take to rigorously the concepts of connectedness for a topological spaces. Two most common approaches are connected and path connected and these concepts are applicable Intermediate Mean Value Theorem and use to help distinguish topological spaces. These concepts play a significant role in application in geographic information system studied by Egenhofer and Franzosa (1991), topological modelling studied by Clementini et al. (1994) and motion planning in robotics studied by Farber et al. (2003). The generalization of open and closed set as like $\alpha$- $\tau$-open and $\alpha$- $\tau$-closed sets was introduced by Njastad (1965) which is nearly to open and closed set respectively. These notion are plays significant role in general topology. In this paper, the new approaches of separate sets, disconnected sets and connected sets called $\alpha$- $\tau$-separate set, $\alpha$- $\tau$-disconnected sets and $\alpha$- $\tau$-connected set with the help of $\alpha$-open and $\alpha$- $\tau$-closed set are firstly introduced. Further, some relationship concerning $\alpha$- $\tau$-disconnected and $\alpha$- $\tau$-connected sets with $\alpha$- $\tau$-separate sets are also investigated.

Throughout this paper $(X,\tau)$ and $(X,\tau_\alpha)$ will always be topological spaces. For a subset $A$ of topological space $X$, $\text{Int}(A),\text{Cl}(A),\text{Cl}_\alpha(A)$ and $\text{Int}_\alpha(A)$ denote the interior, closure, $\alpha$- closure and $\alpha$ – interior of $A$ respectively and $G_\alpha$ is the $\alpha$ – open set for topology $\tau_\alpha$ on $X$.

Preliminaries

We shall requires the following definitions and results.

Definition 2.1. Levine (1963), defined a subset $A$ of $(X,\tau)$ is semi-open if $A \subseteq \text{Cl}(\text{Int}(A))$ and its complement is called semi-closed set.

The family $SO(X,\tau)$ of semi-open sets is not a topology on $X$.

Definition 2.2. Mashhour et al. (1982), defined a subset $A$ of $(X,\tau)$ is called pre-open locally dense or nearly open if $A \subseteq \text{Int}(\text{Cl}(A))$ and its complement is called pre-closed set.

Theorem 2.3. According to Mashhour et al. (1982), the family $PO(X,\tau)$ of pre-open sets is not a topology on $X$.

Definition 2.4. Maheshwari and Jain (1982), defined a subset $A$ of $(X,\tau)$ is called $\alpha$ – open if
A ⊂ Int(Cl(Int(A))). The family τa of α-open sets of (X, τ) is a topology on X which is finer than τ and the complement of an α-open set is called an α-closed set.

**Theorem 2.5.** According to Njastad (1965), for a subset A of (X, τ), the following are equivalent: 

A ⊂ τa; 

A is semi open and pre-open; 

A ∩ B ∈ SO(X, τ), for all B in So(X, τ); 

A = O/N, where O ∈ τa and N is nowhere dense. 

There exists U contained in τa such that U ⊂ A ⊂ scI(U) = Int(Cl(A)).

**Theorem 2.6.** The intersection of semi-open (resp. pre-open) set and an α-open set is semi-open (resp. pre-open).

**Theorem 2.7.** According to Njastad (1965), 

SO(X, τ) = SO(X, τa) and PO(X, τ) = PO(X, τa). 

**Theorem 2.8.** According to Njastad (1965), the α-open sets A and B are disjoint if and only if 

Int(Cl(Int(A))) and Int(Cl(Int(B))) are disjoint.

**Definition 2.9.** A point x in X is called an α-interior point of a set A in X if there exists A ∋ Ga ∈ τa such that y ∈ Ga with y ≠ x, i.e. x in X is called an α-interior point of a set A in X if for every Ga ∈ τa with x ∈ Ga and Ga ⊂ A.

Collection of all α-interior points of A is called α-interior of A which is denoted by Inta(A).

Alternatively, we can define as Inta(A) by 

Inta(A) = ∪{Ga ∈ τa : Ga ⊂ A}.

**Main Results**

**Definition 3.1.** Let (X, τ) and (X, τa) be topological spaces. Then the subsets A and B of (X, τ) are said to be α−τ−separate sets if and only if 

A and B are non-empty set. 

A ∩ Cla(B) and B ∩ Cla(A) are non-empty. 

Remark 3.2 If A and B are α−τ−separate sets, then both of them are also disjoint sets.

**Definition 3.3.** Let (X, τ) and (X, τa) be topological spaces. Then the subsets A of X is said α−τ−disconnected, if there exists Ga and Ha in τa such that 

A ∩ Ga and A ∩ Ha ≠ φ. 

(A ∩ Ga) ∩ (A ∩ Ha) = φ. 

(A ∩ Ga) ∪ (A ∩ Ha) = A. 

(X, τa) is said to be α−τ−disconnected if there exists non-empty Ga and Ha in τa such that Ga ∩ Ha ≠ φ and Ga ∪ Ha ≠ X.

**Definition 3.4.** Let (X, τ) be a topological space and A be non-empty subset of X. let Ga be arbitrary in τa, then collection 

τa = {Ga ∩ A : Ga ∈ τa} is a topology on A, called the subspace or relative topology of topology τa.

**Theorem 3.5.** If (X, τ) a disconnected space and (X, τa) is a topological space, then (X, τa) is α−τ−disconnected.

**Proof.** As (X, τ) disconnected and τa is finer than, hence by Theorem 3.1 (X, τ) is disconnected.

**Theorem 3.6.** Let (X, τ) and (X, τa) are spaces, then (X, τ) is α−τ−disconnected if and only if there exists non-empty proper subset of X which is both α−open and α−closed.

**Proof.** Necessity: Let (X, τa) be α−τ−disconnected. Then, by Definition 3.3 there exist non-empty sets Ga and Ha in τa such that Ga ∩ Ha is non-empty and Ga ∪ Ha = X. Since Ga ∩ Ha = φ and Hα is open in τa show that Ga = X − Hα, but it is α-closed. Hence Ga is non-empty proper subset of X which is α-closed as well α-open.

Sufficiency: Suppose A is non-empty proper subset of X such that it is α-open as well α-closed. Now A is nonempty α-closed show that X − A is non-empty and α-open. Suppose B = X − A, then A ∪ B = X and A ∩ B = φ. Thus A and B are non-empty disjoint α-open as well as α-closed subset of X such that A ∪ B = X. Consequently X is α−τ−disconnected.

**Theorem 3.7.** Every (X, τa) discrete space is α−τ−disconnected if the space contains more than one element.

**Proof.** Let (X, τa) be discrete space such that X = {a, b} contains more than one element. But τ is discrete topology so τ = {φ, X, {a}, {b}} and family of all α-open sets is τa = {φ, X, {a}, {b}}.
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All $\alpha$-closed sets are $\phi, X, \{a\}, \{b\}$. Since $\{a\}$ is non-empty proper subset of $X$ which is both $\alpha$-open and $\alpha$-closed in $X$. Finally, we can say that $(X,\tau_a)$ is $\alpha-\tau$-disconnected by Theorem 3.6.

**Theorem 3.8.** A topological space $(X,\tau_a)$ is $\alpha-\tau$-connected if and only if one non-empty subset which is both $\alpha$-open and $\alpha$-closed in $X$ is $X$ itself.

**Proof. Necessity:** Assume that $(X,\tau_a)$ is $\alpha-\tau$-connected topological space. So, our assumption show that $(X,\tau_a)$ is not $\alpha$-disconnected i.e. there does not exist a pair of non-empty disjoin $\alpha$-open and $\alpha$-closed $A$ and $B$ such that, $A \cup B = X$. This shows that there exist non-empty subsets (other than $X$) which are both $\alpha$-open and $\alpha$-closed in $X$.

**Sufficiency:** Suppose that $(X,\tau_a)$ is topological space such that the only non-empty subsets of $X$ which is $\alpha$-open as well as $\alpha$-closed in $X$ is $X$ itself. By hypothesis, there does not exist a partition of the space $X$. Hence $X$ is not $\alpha-\tau$-disconnected i.e. $\alpha-\tau$-connected.

**Theorem 3.9.** Let $A$ be a non-empty subset of topological space $(X,\tau)$. Let $\tau_a^d$ be the relative topology on $A$, then $A$ is $\alpha-\tau$-disconnected if and only if $A$ is $\alpha-\tau_a^d$-disconnected.

**Proof. Necessity:** Let $A$ be a $\alpha-\tau$-disconnected and let $G_a \cup H_a$ be a $\alpha-\tau$-disconnection on $A$. By Definition 3.3, there exists non-empty $G_a$ and $H_a$ in $\tau_a$ such that $A \cap G_a, A \cap H_a \neq \phi$; $A \cap G_a \cap (A \cap H_a) = \phi$; $A \cap G_a \cup (A \cap H_a) = A$.

Now by the definition of relative topology, if $G_a$ and $H_a$ in $\tau_a$, then there exist $G_a^1$ and $H_a^1$ in $\tau_a^d$ such that $G_a^1 = A \cap G_a$ and $H_a^1 = A \cap H_a$.

Now by (1) $G_a^1$ and $H_a^1$ are non-empty. Hence $A \cap G_a^1$ and $A \cap H_a^1$ are non-empty. Similarly by (2) and (3), we can say that $(A \cap G_a^1 \cap (A \cap H_a^1) = \phi$ and $(A \cap G_a^1 \cup (A \cap H_a^1) = A$ respectively. Consequently $A$ is $\alpha-\tau_a^d$-disconnected.

**Sufficiency:** Suppose that $A$ is $\alpha-\tau_a^d$-disconnected and $M_a^d \cap N_a^d$ is a $\alpha-\tau_a^d$-disconnection on $A$. By definition, we can say that $M_a^d, N_a^d \neq \phi$; $M_a^d, N_a^d \in \tau_a^d$; $(A \cap M_a^d) \cap (A \cap N_a^d) = \phi$; $(A \cap M_a^d) \cup (A \cap N_a^d) = A$.

Now (2) $\Rightarrow$ there exists $M_a^d, N_a^d \in \tau_a^d$ such that $M_a^d = A \cap M_a^1, N_a^d = A \cap N_a^1$. But by (1) we can say that $A \cap M_a^1, A \cap N_a^1 \neq \phi$. Now $(A \cap M_a^d) = (A \cap (A \cap M_a^1) = (A \cap A) \cap M_a^1 = A \cap M_a^1$. Similarly we can say that $A \cap N_a^d = A \cap N_a^1$.

Now (3) $\Rightarrow$ $(A \cap M_a^d) \cap (A \cap N_a^d) = \phi$. Similarly (4) $\Rightarrow (A \cap M_a^d) \cup (A \cap N_a^d) = A$.

Finally, we can say that $A$ is $\alpha-\tau$-disconnected.

**Theorem 3.10.** The union of two non-empty $\alpha-\tau$-separate subsets of topological space $(X,\tau_a)$. Then by definition 3.1, we can say that $A$ and $B$ non-empty.

$A \cap Cl_a(B), B \cap Cl_a(A) = \phi$; $A \cap B = \phi$.

Let $X - Cl_a(A) = G_a$ and $X - Cl_a(B) = H_a$. Then $Cl_a(A)$ and $Cl_a(B)$ are non-empty $\alpha$-closed subsets of $X$ which shows that $G_a$ and $H_a$ are non-empty $\alpha$-open subsets of $X$.

Since,$G_a \cup H_a = (X - Cl_a(A)) \cup (X - Cl_a(B)) = X - Cl_a(A) \cap Cl_a(B)$

we have

$(A \cup B) \cap G_a = (A \cup B) \cap (X - Cl_a(B)) = [A \cap (X - Cl_a(B))] \cup [B \cap (X - Cl_a(B))] = \phi \cup B$.

$(A \cup B) \cap G_a = B$.

Similarly, $(A \cup B) \cap H_a = A$. Now (1) shows that $(A \cup B) \cap G_a, (A \cup B) \cap H_a \neq \phi$.

Additionally, (3) shows that
Finally, we can say that there exists $G_a$ and $H_a$ in $\tau_a$ such that

$[(A \cup B) \cap H_a] \cap [(A \cup B) \cap G_a] = \phi$ and $[(A \cup B) \cap H_a] \cup [(A \cup B) \cap G_a] = A \cup B$.

So, $G_a \cup H_a$ is a $\alpha - \tau$-disconnection of $A \cup B$. Hence $A \cup B$ is $\alpha - \tau$-disconnected.

**Theorem 3.11.** Let $(X, \tau)$ and $(X, \tau_a)$ be topological spaces and $A$ be a subset of $X$ and let $G_a \cup H_a$ be a $\alpha - \tau$-disconnection of $A$. Then $A \cap G_a$ and $A \cap H_a$ are $\alpha - \tau$-separate subsets of topological space $(X, \tau_a)$.

**Proof.** Let $G_a \cap H_a$ be a given $\alpha - \tau$-disconnection of subset $A$ of $(X, \tau_a)$. To prove $A \cap G_a$ and $A \cap H_a$ are $\alpha - \tau$-separate subsets, we must show that $A \cap G_a$ and $A \cap H_a$ are non-empty;

$[Cl_a(A \cap G_a)] \cap (A \cap H_a) = \phi$ and $[Cl_a(A \cap H_a)] \cap (A \cap G_a) = \phi$;

$[A \cap G_a] \cup (A \cap H_a) = A$.

Evidently, (4) $\Rightarrow$ (1).

To prove (2) suppose it is not possible i.e.

$Cl_a(A \cap G_a) \cap (A \cap H_a) = \phi$. Then, there exists $x \in Cl_a(A \cap G_a) \cap (A \cap H_a)$ which implies that $x \in Cl_a(A \cap G_a)$ and $x \in A \cap H_a$, that is $(A \cap G_a) \cap H_a = \phi$. Therefore $(A \cap G_a) \cap (A \cap H_a) = \phi$. But it is contrary to (5). Finally our assumption that is wrong.

**Theorem 3.12.** A subset $Y$ of a topological space $X$ is $\alpha - \tau$-disconnected if and only if it is union of two $\alpha - \tau$-separate sets.

**Proof.** Necessity: Suppose $Y = A \cup B$, where $A$ and $B$ are $\alpha - \tau$-separate sets of $X$.

By theorem 3.10, $A \cup B$ is $\alpha - \tau$-disconnected. Hence, $Y$ is $\alpha - \tau$-disconnected.

Sufficiency: Let $Y$ be $\alpha - \tau$-disconnected. To prove that there exists two $\alpha - \tau$-separate subsets of $A, B$ in $X$ such that $Y = A \cup B$. By assumption, $Y$ is $\alpha - \tau$-disconnected show that there exists a $\alpha - \tau$-disconnection $G_a \cup H_a$ of $Y$. Therefore by Definition 3.3, we can say that there exists $G_a$ and $H_a$ in $\tau_a$ such that $Y \cap G_a$ and $Y \cap H_a$ are non-empty;

$(Y \cap G_a) \cap (Y \cap H_a) = \phi$; $(Y \cap G_a) \cup (Y \cap H_a) = Y$.

Since $(Y \cap G_a)$ and $(Y \cap H_a)$ are separated sets, if we write $A = (Y \cap G_a)$ and $B = (Y \cap H_a)$, then by (3) $Y = A \cup B$. Finally, we can say that there exist two $\alpha - \tau$-separate sets $A$ and $B$ in $X$ such that $Y = A \cup B$.

**Theorem 3.13.** If $Y$ is an $\alpha - \tau$-connected subset of topological space $X$ such that $Y \subset A \cup B$, where $A$ and $B$ is $\alpha - \tau$-connected, then $Y \subset A$ and $Y \subset B$.

**Proof.** Since the inclusion $Y \subset A \cup B$ holds by the hypothesis we have $(A \cup B) \cap Y = Y$ which yields that $Y = (Y \cap A) \cup (Y \cap B)$. Now we want to prove that $(Y \cap A), (Y \cap B) = \phi$. Suppose, $(Y \cap A), (Y \cap B) \neq \phi$. Now,

$(Y \cap A) \cap Cl_a(Y \cap B) \subset (Y \cap A) \cap Cl_a(Y \cap Cl_a(Y \cap B)) = (Y \cap Cl_a(Y)) \cap Cl_a(B) = (Y \cap (A \cap Cl_a(B))) = Y \cap \phi = \phi$ i.e.,

$(Y \cap A) \cap Cl_a(Y \cap B) = \phi$. Similarly, we can prove that

$Cl_a(Y \cap A) \cap (Y \cap B) = \phi$. Hence, from the above result we can say that $Y$ is a union of two $\alpha - \tau$-separate sets $(Y \cap A)$ and $(Y \cap B)$. Consequently, $Y$ is $\alpha - \tau$-disconnected. But this contradicts the fact that $Y$ is $\alpha - \tau$-connected. Hence we can say that $(Y \cap A), (Y \cap B) = \phi$. Now if
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$$(Y \cap A) = \phi,$$ then $$Y = \phi \cup (Y \cap B) = (Y \cap B)$$
which gives that $Y \subset B$. Similarly, we can prove that $Y \subset A$ if $(Y \cap A) = \phi$.

**Conclusion**

We have introduced new approach of separate sets, disconnected sets and connected sets called $\alpha-\tau$-separate sets, $\alpha-\tau$-disconnected sets and $\alpha-\tau$-connected sets of topological spaces with the help of $\alpha$-open and $\alpha$-closed sets and investigated their properties. The results of this paper will help to study various weak and strong form of connectedness and disconnectedness in topological spaces and it can be also applied for the study of topology in robotics, topological modeling and geographical information system.

**Acknowledgements**

I am very thankful to referee and my colleague Dr. A. K. Srivastva for providing useful comments for improvement of the paper.

**Reference**


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