Giving Flexibility To The Nelson-Siegel Class Of Term Structure Models

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Abstract

This paper compares the interpolation abilities of nonparametric and parametric term structure models which are widely used by the main Central Banks of the world. Seeking the combination of smoothness and flexibility, a new Nelson-Siegel class model is introduced. It emerges as an extension of the Svensson (1994) and the five factor model proposed by De Rezende & Ferreira (2008) and Christensen *et al.* (2008). It is shown the superiority of the smoothing spline model in interpolating the spot and forward rates as well as the advantage of the proposed model over the other Nelson-Siegel models. The superiority of the smoothing spline, however, comes with a cost: its instability in fitting the initial vertices of the term structure. The proposed model, on the other hand, exhibits the desirable properties of smoothness and flexibility, especially for the forward rates and the spot rates of medium and long terms.

***Keywords*:** spot curve; forward curve; Nelson-Siegel models; smoothing spline.

JEL codes: E43; G12.

Resumo

Este artigo compara as habilidades de interpolação de modelos não-paramétricos e paramétricos da estrutura a termo amplamente utilizados pelos principais bancos centrais do mundo. Buscando a combinação de alisamento e flexibilidade, um novo modelo da classe Nelson-Siegel é introduzido. Ele surge como uma extensão do modelo de Svensson (1994) e do modelo de cinco fatores proposto por De Rezende & Ferreira (2008) e Christensen *et al.* (2008). É mostrada a superioridade do modelo smoothing spline na interpolação das curvas spot e forward, bem como a vantagem do modelo proposto em relação a outros modelos Nelson-Siegel. A superioridade do smoothing spline, porém, vem com um custo: sua instabilidade ao interpolar os vértices iniciais da estrutura a termo. O modelo proposto, por outro lado, apresenta as propriedades desejáveis de alisamento e flexibilidade, especialmente ao interpolar as taxas forward e as taxas spot de médio e longo prazos.

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1. Introduction

In the last decades the use of the term structure of interest rates has been one of the most important topics of research in macroeconomics and finance. For macroeconomists, in a monetary policy context, forward rates are potentially useful as indicators of market expectations of future interest rates, inflation rates and exchange rates as discussed by Svensson (1994) and Söderlind & Svensson (1997), while the yield curve carries information about future GDP growth as shown by Estrella & Mishkin (1996, 1998). In finance, fixed income portfolio managers make use of the yield curve to mark to market, while risk managers use it for pricing derivatives and performing hedging operations. However, the market does not provide us securities at all the desired maturities and what we observe is only an incomplete set of yields across the maturity spectrum. Hence, to overcome this problem, some interpolation method is necessary, which leads to the estimation of the yield curve.

Basically, the literature on term structure interpolation can be separated in the parametric and nonparametric methods. The parametrics, with the Nelson & Siegel (1987) and the Svensson (1994) models as the most well known models in this class, exhibit at least three reasons for their popularity. First, they are easy to estimate. In fact, if the so-called time-decaying parameters are fixed, their curves are obtained by linear regression techniques. If not, one has to resort to non-linear regression methods. Second, adapting them in a time series context, it is possible to obtain accurate yield curve forecasts, and their estimated factors can assume economic interpretations of yield curve's level, slope, curvature and double curvature (see Diebold & Li (2006), De Pooter (2007) and Almeida *et al.* (2009)). Third, their functional forms impose more smoothness on the shapes of the curves, as desirable by macroeconomists (see Gürkaynak *et al.* (2006)). However, parametric methods are not immune to problems. First, they do not impose the presumably desirable theoretical restriction of absence of arbitrage (Filipovic, 1999, Diebold *et al.*, 2005). And second, they are not flexible enough to fit well both noisy curves and curves with a long maturity spectrum.

The nonparametric methods, which have the spline models of McCulloch (1971, 1975), Vasicek & Fong (1982) and Fisher *et al.* (1995) as its main representatives, also show some desirable properties. First, since they do not assume a particular functional form, they are robust to misspecification errors. Second, they exhibit great flexibility, fitting almost perfectly all kinds of curves. The flexibility, however, comes with costs. The models usually exhibit great instability in fitting, especially, at the curves' extremes and the estimation involves a large number of parameters. Another problem is the location and number of the knot points that must be chosen.[1]

[1] Althought this is a typical problem in linear, quadratic, cubic and exponential regression splines which in general rely on ad hoc choice criteria, some methods, like the penalized smoothing splines of Jarrow *et al.* (2004) use generalized cross validation (GCV). Other approaches use economic interpretations of short, intermediate and long-term money, like the one suggested by Litzenberger & Rolfo (1984) and employed by Barzanti & Corradi (1999). Koenker *et al.* (1994) make use of the Schwartz

Hence, we conclude that when one must decide which interpolation method is going to be used, basically, one is confronted by the issue: how much flexibility to allow in the curve estimation. If a spline-based method is chosen, a very flexible curve could be estimated, but it would be done with considerable variability in the spot and forward rates. On the other hand, through the parametric methods, more smoothness could be imposed on the shapes of the curves, while some of the fit would be sacrificed.

The choice in this dimension depends on the purpose that the curves are intended to serve. A trader looking for small pricing anomalies may be very concerned with how a specific security is priced relative to those securities immediately around it and would, probably, choose the more flexible method to estimate the yield curve. By contrast, a macroeconomist may be more interested in understanding the fundamental determinants of the yield curve and the expectations of some economic variables indicated by the forward curve, preferring then the smoothest method.

Trying to solve this puzzle this paper proposes a parametric method of six factors (SF in the remainder of the paper) which is smooth and also flexible enough to fit accurately a pool of spot and forward curves' shapes. Adding only two parameters in the estimation procedure, it is shown that the proposed model, which can be included in the Nelson-Siegel class,[2] presents a great flexibility gain, fitting very well all the yields through the maturity spectrum, but, specially, the longest ones. The results are compared to those obtained by the popular models of Fisher *et al.* (1995) (SS in the remainder of the paper), Nelson & Siegel (1987) (NS in the remainder of the paper), Svensson (1994) (SV in the remainder of the paper) and by the five factor parametric model (FF in the remainder of the paper) proposed by De Rezende & Ferreira (2008) and Christensen *et al.* (2008).[3] This choice was based on the conclusion that these models are widely used in Central Banks and industry, including the Federal Reserve Board (see Gürkaynak *et al.* (2006)), the European Central Bank (see Coroneo *et al.* (2008)) and many other Central Banks (see Bank for International Settlements (2005)).

Since the NS, SV, FF and SF models can be considered nested, the addition of a third curvature term necessarily leads to lower approximation errors and better in-sample fitting. It is then natural to argue that the inclusion of additional terms will always improve the fitting of this type of models, what gives rise to criticisms to the model we propose. To overcome this problem we calculate statistics along the time dimension that penalize for extra covariates showing that the third curvature

Criteria for the quantile smoothing spline.

[2]This class includes the models of Nelson & Siegel (1987), Svensson (1994), Bliss (1997), Björk & Christensen (1999) and De Rezende & Ferreira (2008).

[3]Since in the Nelson-Siegel family each slope factor must be accompanied by a curvature factor to impose no-arbitrage consistency (see Björk & Christensen (1999)), Christensen *et al.* (2008) include a double-curvature factor in Svensson's model in order to impose no-arbitrage. In their model an additional term is also added to allow for no-arbitrage corrections. In this paper we estimate the five-factor model without this last term.

term indeed adds information to the models, especially when the forward curve is estimated. We see this result as an important contribution of the paper that also serves as an argument against the possible criticism of over fitting.

The remainder of the paper is organized as follows. The second section presents the models which will be analyzed in the paper; the third discusses the data used in the estimation; in the fourth section the estimation procedures of the models are addressed; the fifth section presents the results; and the sixth section concludes the paper.

2. Term Structure Models

2.1 Basic Definitions

The term structure of interest rates can be described in terms of the spot (or zero-coupon) rate, the discount rate and the forward rate. The forward curve determines rates as a function of maturities. A forward rate is the interest rate of a forward contract on an investment which will be initiated $\bar{\tau}$ periods in the future and which will mature τ^* periods beyond the start date of the contract. We obtain the instantaneous forward rate $f(\bar{\tau})$ by letting the maturity of such forward contract go to zero: $\lim_{\tau^* \to 0} f(\tau^*, \bar{\tau}) = f(\bar{\tau})$.

From the instantaneous forward rates, we get the forward curve, $f(\tau)$.

We can then determine the spot rate implicit in a zero-coupon bond with maturity $\bar{\tau}$, $z(\bar{\tau})$. Under continuous compounding, taking an average of forward rates, we get the spot rate:

$$z(\bar{\tau}) = \frac{1}{\tau} \int_0^{\tau} f(x)dx \tag{1}$$

Then, from the spot rates, we get the spot or zero-coupon yield curve, $z(\tau)$.

The discount curve is made by rates which give the present value of a zero-coupon bond that pays a nominal value of \$1.00 after τ periods. It can be obtained from the spot curve through the following relationship:

$$d(\tau) = e^{-z(\tau)\tau} \tag{2}$$

From the equations above we can then relate the discount and the forward curves by the following formulas:

$$d(\tau) = \exp\left[-\int_0^{\tau} f(x)dx\right] \tag{3}$$

$$f(\tau) = -\frac{d'(\tau)}{d(\tau)} \tag{4}$$

We can move from a curve to the other using the relationships specified above.

The next two sub-sessions describe the interpolation models analyzed in the paper. They may have as object variables the spot, forward, discount rates or the prices of the bonds. In this paper the spot and the forward rates were chosen as the object variables for the minimization procedures and there are at least three reasons for this choice. As pointed out by Svensson (1994), minimizing price errors sometimes results in fairly large yield errors for bonds and bills with short maturities while minimizing yield errors generates a substantially better fit for short maturities. The two procedures seem to perform equally well for long maturities. Furthermore, Fisher *et al.* (1995) show that splining the forward rate function with a smoothing spline produces, in general, the most accurate and least biased results.[4] We also make use of the forward rates because one of the purposes of the paper is to show the usefulness of the six factor model in interpolating forward curves.

2.2 Nonparametric model

The smoothing spline method for yield curve estimation was introduced in the analysis of yield curve estimation by Fisher *et al.* (1995). In general, for an explanatory variable x_i and a response variable y_i, the smoothing spline tries to find the unique function $\hat{g}(.)$ over the class of all twice differentiable functions which minimizes the *penalized sum of squares*,

$$SS_{\omega}(g) = \sum_{i=1}^{n} [y_i - g(x_i)]^2 + \omega \int_{x_{(1)}}^{x_{(n)}} [g''(t)]^2 dt \tag{5}$$

on the interval $\left| x_{(1)}, x_{(n)} \right|$. $\hat{g}(.)$ is a natural cubic spline with knots at the distinct observed values of x. The first term on the right hand side reflects the expectation that the values of the data should be close to those of the estimates, an idea which is common to that of the ordinary least squares. The second term, called "roughness penalty", imposes the condition that the estimates should vary smoothly. ω is a positive constant called a smoothing parameter which plays the role of adjusting the degree of smoothness. A large value of ω results in highly smooth estimates.

Let $Y = (y_1, ..., y_n)^{'}$ be a column-vector of either zero-coupon or instantaneous forward rates and $\Gamma = (\tau_1, ..., \tau_n)$ be a vector of maturities (knot points) with $0 < \tau_1 < ... < \tau_n < M$. Let also the function $g(\tau) \in \{f(\tau), z(\tau)\}$,[5] be expressed as a linear combination of cubic B-splines: $g_S(\tau, \beta) \equiv (\phi_1(\tau), ..., \phi_\kappa(\tau)) \, (\beta_1, ..., \beta_\kappa)^{'} \equiv \phi(\tau)\beta$, where $\phi(\cdot)$ is a cubic B-spline basis, β is a column-vector, κ is the number of knot points plus 2, that is $n + 2$, and

[4]The exercise of splining the discount rates was also made, but the results were poor compared to splining both the spot and forward rates. We decided not to show these results in the paper.

[5]Recall that $f(\tau)$ and $z(\tau)$ are referred as the forward and zero-coupon curves, respectively, as functions of the maturity $\tau \in \Gamma$.

$\tau \in [0, M]$. Then the smoothing spline $g_S(\tau, \beta*)$ solves the following problem for a given parameter ω:

$$\min_{\beta(\omega)} \left[\sum_{i=1}^{n} [y_i - \phi(\tau_i)\beta]^2 + \omega \int_0^M [g''(\tau)]^2 \, d\tau \right] \tag{6}$$

where, in terms of the spline $g_S(\tau, \beta)$, the penalty term can be written as

$$\omega \int_0^M \left(\frac{\partial^2 g_S(\tau, \beta)}{\partial \tau^2} \right)^2 d\tau = \omega \beta' \left(\int_0^M \phi''(\tau)' \phi''(\tau) d\tau \right) \beta = \omega \beta' H \beta \tag{7}$$

The OLS solution is given by $\beta * (\omega) = (X'X + \omega H)^{-1} X'Y$, where $X = \phi(\tau)$ is a $n \times \kappa$ matrix.

In order to choose ω, Fisher *et al.* (1995) suggested using the value which minimizes the generalized cross validation (GCV). That is, ω is chosen as the solution to the following problem:

$$\min_{\omega} GCV(\omega) = \frac{\sum_{i=1}^{n} [Y_i - g(\tau_i, \beta * (\omega))]}{(n - \theta.tr(A(\omega)))^2} \tag{8}$$

where $A(\omega) \equiv X (X'X + \omega H)^{-1} X'$, n is the dimension of the implicit smoother matrix A and $tr(A)$ denotes the trace of A which is usually used as the measure of the effective number of parameters. The parameter θ is called the cost and it controls the entire parameterization of the spline. As θ gets bigger, the curves appear smoother at the expense of goodness of fit. Following Fisher *et al.* (1995) θ was preset in the value of 2.

2.3 Parametric models

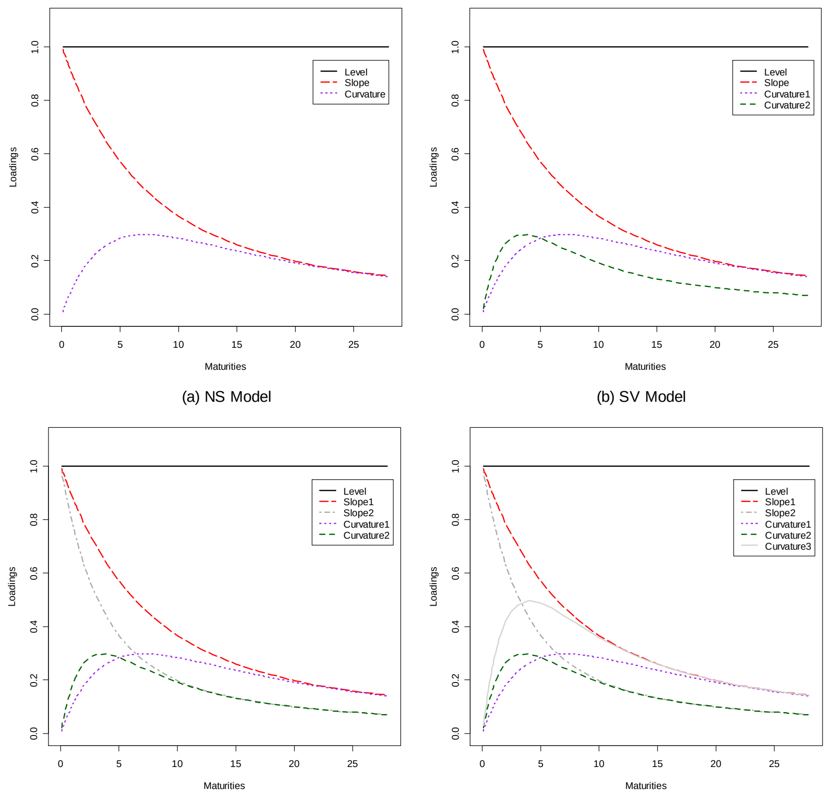
Nelson & Siegel (1987) suggest fitting the forward curve at a particular point in time using the following parametric model:

$$f(\tau) = \beta_1 + \beta_2 e^{-\frac{\tau}{\lambda}} + \beta_3 \frac{\tau}{\lambda} e^{-\frac{\tau}{\lambda}} \tag{9}$$

From (1) we can get the spot yield curve:

$$z(\tau) = \beta_1 + \beta_2 \left(\frac{1 - e^{-\frac{\tau}{\lambda}}}{\frac{\tau}{\lambda}} \right) + \beta_3 \left(\frac{1 - e^{-\frac{\tau}{\lambda}}}{\frac{\tau}{\lambda}} - e^{-\frac{\tau}{\lambda}} \right) \tag{10}$$

where the constant λ governs the decaying speed of the β_2's exponential component and the maximum point of the β_3's exponential component. Thus λ governs the decay rate of the whole curve. The exponential components of the spot and forward NS curves can be viewed in Figure 1 (a) and Figure 2 (a), respectively.



(a) NS Model (b) SV Model

(c) FF Model (d) SF Model

Note: this figure exhibit the loadings of the NS models for the spot curve.

Figure 1
Loadings of the NS class models – spot curve

Although the basic model captures many curves shapes, it cannot deal with all the shapes that the term structure assumes over time, especially the ones with a long maturity spectrum and those that are very twisted, with more than one inflection point. Trying to remedy this problem, several more flexible parametric models of the NS class have been proposed in the literature, adding factors, including other decaying parameters, or combining both of them.

A popular term structure approximation model is the four factor SV model. Svensson (1994) proposes to increase the NS flexibility through the inclusion of a fourth exponential component that recalls the third one, but shows a different parameter λ. The model that fits the forward curve is given by:

$$f(\tau) = \beta_1 + \beta_2 e^{-\frac{\tau}{\lambda_1}} + \beta_3 \frac{\tau}{\lambda_1} e^{-\frac{\tau}{\lambda_1}} + \beta_4 \frac{\tau}{\lambda_2} e^{-\frac{\tau}{\lambda_2}} \tag{11}$$

And the model that approximates the zero-coupon yield curves:

$$\begin{aligned} z(\tau) &= \beta_1 + \beta_2 \left(\frac{1 - e^{-\frac{\tau}{\lambda_1}}}{\frac{\tau}{\lambda_1}} \right) + \beta_3 \left(\frac{1 - e^{-\frac{\tau}{\lambda_1}}}{\frac{\tau}{\lambda_1}} - e^{-\frac{\tau}{\lambda_1}} \right) \\ &+ \beta_4 \left(\frac{1 - e^{-\frac{\tau}{\lambda_2}}}{\frac{\tau}{\lambda_2}} - e^{-\frac{\tau}{\lambda_2}} \right) \end{aligned} \tag{12}$$

The fourth component differs from the third only because of the decaying parameter λ. It can be interpreted as a double curvature component, as well as its factor. The SV model usually fits the various spot and forward curves shapes better then the three factor model. The SV's exponential components can be viewed in Figure 1 (b) and Figure 2 (b), respectively.

The five factor model introduced by De Rezende & Ferreira (2008) and by Christensen *et al.* (2008) emerges as a natural extension of the SV. Seeking a greater flexibility they included another term, which recalls the second NS exponential component. It differs because of the decaying parameter. The following model fits the forward curve:

$$f(\tau) = \beta_1 + \beta_2 e^{-\frac{\tau}{\lambda_1}} + \beta_3 e^{-\frac{\tau}{\lambda_2}} + \beta_4 \frac{\tau}{\lambda_1} e^{-\frac{\tau}{\lambda_1}} + \beta_5 \frac{\tau}{\lambda_2} e^{-\frac{\tau}{\lambda_2}} \tag{13}$$

And the one that models the spot curve is given by:

$$\begin{aligned} z(\tau) &= \beta_1 + \beta_2 \left(\frac{1 - e^{-\frac{\tau}{\lambda_1}}}{\frac{\tau}{\lambda_1}} \right) + \beta_3 \left(\frac{1 - e^{-\frac{\tau}{\lambda_2}}}{\frac{\tau}{\lambda_2}} \right) \\ &+ \beta_4 \left(\frac{1 - e^{-\frac{\tau}{\lambda_1}}}{\frac{\tau}{\lambda_1}} - e^{-\frac{\tau}{\lambda_1}} \right) + \beta_5 \left(\frac{1 - e^{-\frac{\tau}{\lambda_2}}}{\frac{\tau}{\lambda_2}} - e^{-\frac{\tau}{\lambda_2}} \right) \end{aligned} \tag{14}$$

The third component of both the curves can be interpreted as a double slope component and can be visualized in Figure 1 (c) and Figure 2 (c).

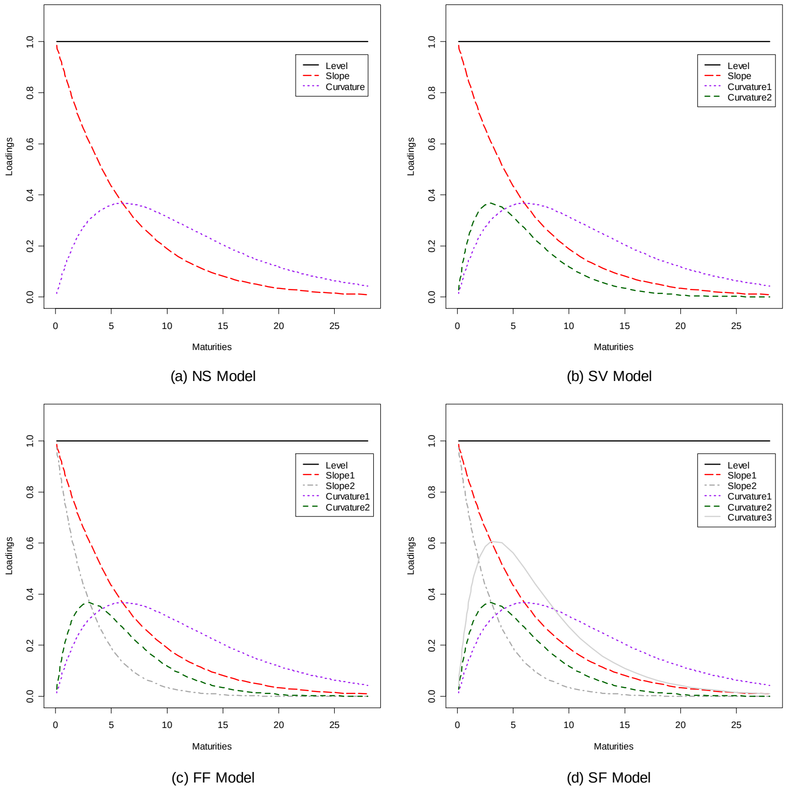
The proposed six factor model is also an extension of the other ones described above. Seeking a greater flexibility it was included another term which is a modification of the third. It differs because of the decaying parameter. The following model was proposed to fit the forward curve:

$$\begin{aligned} f(\tau) &= \beta_1 + \beta_2 e^{-\frac{\tau}{\lambda}_1} + \beta_3 e^{-\frac{\tau}{\lambda}_2} + \beta_4 \frac{\tau}{\lambda_1} e^{-\frac{\tau}{\lambda}_1} + \beta_5 \frac{\tau}{\lambda_2} e^{-\frac{\tau}{\lambda}_2} \\ &+ \beta_6 \left[e^{-\frac{\tau}{\lambda_1}} + \left(\frac{2\tau}{\lambda_1} - 1 \right) e^{-\frac{2\tau}{\lambda_1}} \right] \end{aligned} \tag{15}$$

And the one that models the zero-coupon curve:

$$\begin{aligned} z(\tau) &= \beta_1 + \beta_2 \left(\frac{1 - e^{-\frac{\tau}{\lambda}_1}}{\frac{\tau}{\lambda_1}} \right) + \beta_3 \left(\frac{1 - e^{-\frac{\tau}{\lambda}_2}}{\frac{\tau}{\lambda_2}} \right) \\ &+ \beta_4 \left(\frac{1 - e^{-\frac{\tau}{\lambda}_1}}{\frac{\tau}{\lambda_1}} - e^{-\frac{\tau}{\lambda}_1} \right) + \beta_5 \left(\frac{1 - e^{-\frac{\tau}{\lambda}_2}}{\frac{\tau}{\lambda_2}} - e^{-\frac{\tau}{\lambda_2}} \right) \\ &+ \beta_6 \left(\frac{1 - e^{-\frac{\tau}{\lambda}_1}}{\frac{\tau}{\lambda_1}} - e^{-\frac{2\tau}{\lambda}_1} \right) \end{aligned} \tag{16}$$

The sixth component can be interpreted as a triple curvature component. However it presents a higher maximum point. The exponential components of both SF curves can be visualized in Figure 1 (d) and Figure 2 (d). We expect that the six factor model fits better more complex and twisted yield and forward curves, like those with two or more inflection points. We also expect that the greater flexibility allows for a better fit at the short and long term maturities of the term structure.



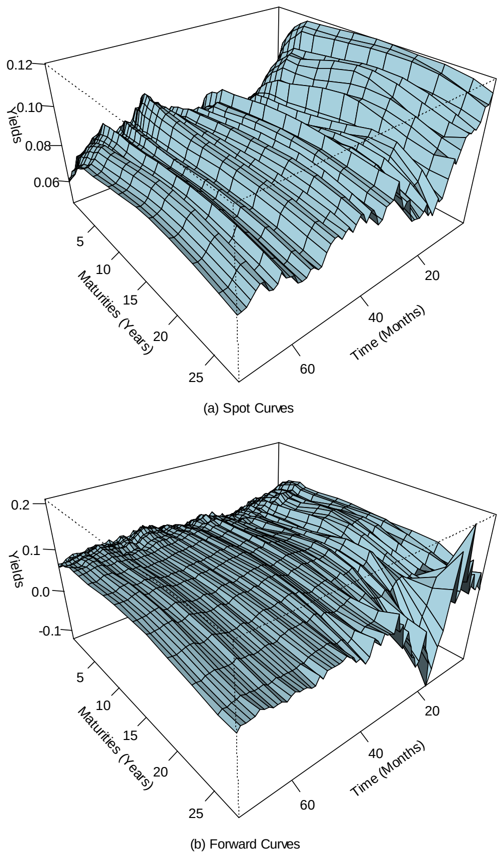
Note: this figure exhibits the loadings of the NS models for the forward curve.

Figure 2
Loadings of the NS class models – forward curve

3. Data

The data set used in the estimations is composed by the monthly spot interest rates and the corresponding instantaneous forward rates of the McCulloch U.S. Treasury term structure data.[6] All rates are end-of-month, given as percentages per annum, and are on a continuous-compounding basis. They are derived from a tax-adjusted cubic spline discount function, as described in McCulloch (1975). We considered a data with 73 curves with 19 maturities (in years): 0.083, 0.25, 0.5, 0.75, 1, 1.5, 2, 3, 4, 5, 7, 10, 12, 15, 17, 20, 23, 26, 28. The sample ranges from February 1985 to February 1991 and the Figure 3 shows the spot and forward curves along it.

[6]The data can be downloaded from `http://www.econ.ohio-state.edu/jhm/ts/mcckwon/mccull.htm`.



Note: figure 3 (a) and the figure 3 (b) show the US Treasury Spot and Forward Curves, respectively, for the McCulloch data. The sample ranges from February 1985 to February 1991.

Figure 3
Spot and forward curves

The McCulloch data was chosen because one of the interests of the paper is to verify the fitting of the models to spot and forward curves with a long-end. It permits us to verify how flexible they are. As known, this is one of the main difficulties of the Nelson-Siegel type models.

4. Estimation Procedures

For the parametric models, for each t, we need to estimate the parameters $\theta = \{\beta_t, \lambda_t\}$ by a nonlinear method. Nevertheless, following Nelson & Siegel (1987) and Diebold & Li (2006), we decided to fix λ_t for the whole sample.

Despite this method facilitates the estimations, the choice of λ is a very important and difficult issue to solve. Diebold & Li (2006) decided to fix it at the 30 months maturity,[7] but we have adopted a less arbitrary strategy. For the NS (SV, FF and SF) model(s) we created a vector (matrix) of possible optimal λ (combinations of λs) and chose the one that provided the lowest term structure fitting error over time, measured by the average of the Root Mean Squared Error (RMSE). The idea is to search for a λ (or combination of λ's) under which the model generates its best in-sample fitting by considering the whole panel of yield observations in both time and maturity dimensions.

In summary, finding λ in the case of the NS model is the same of solving the following problem:

$$\lambda = \underset{\lambda \in \Psi}{argmin} \left\{ \frac{1}{N} \sum_{j=1}^{N} \sqrt{\frac{1}{T} \sum_{t=1}^{T} \Big(h_t(\tau_j, \lambda) - \hat{h}_t(\tau_j, \lambda) \Big)^2} \right\} \tag{17}$$

where T is the number of yield or forward curves in the sample and $h_t(\tau_j, \lambda)$ is the zero-coupon or forward rate of the $j - th$ maturity at t.

In the case of the SV, FF and SF, the problem to be solved is:

$$(\lambda_1, \lambda_2) = \underset{(\lambda_1, \lambda_2) \in \Psi}{argmin} \left\{ \frac{1}{N} \sum_{j=1}^{N} \sqrt{\frac{1}{T} \sum_{t=1}^{T} \Big(h_t(\tau_j, \lambda_1, \lambda_2) - \hat{h}_t(\tau_j, \lambda_1, \lambda_2) \Big)^2} \right\} \tag{18}$$

In both cases $\Psi = [0.08, 28]$ if $h(\cdot) = z(\cdot)$ and $\Psi = [0.15, 50]$ if $h(\cdot) = f(\cdot)$. Since the shortest time to maturity of yields in our database is 0.083 years and the greatest is 28 years, there is no reason to search for optimal values outside this interval because they correspond to the maximum of the curvature loadings at maturities a little below 0.083 years and slightly above 28 years. The optimal parameters, for each curve and model, are shown in Table 1.

[7] The authors argue that, historically, the curvature has been linked to changes of medium term yields, which have been usually represented by 2 to 3 year yields. For this reason, they decided to choose λ to maximize the curvature loadings at the average of these two maturities, that is, at 30 months.

Table 1
Optimal parameters λ

Models	Curves	
	Spot	Forward
NS	1.519	2.470
SV	0.605; 2.196	2.260; 0.624
FF	4.462; 28	2.214; 0.150
SF	1.920; 28	26.471; 16.511

Note: this table shows the optimal decaying parameters obtained in the estimation procedure of the NS class models.

After finding the optimal λ, we run an OLS regression to estimate the values of β. This estimation carried for all period results in a vector of β's for each t.

Although this procedure does not guarantee that the fixed λ is optimal for individual curves, this procedure plays a crucial role in this paper since it avoids possible identification problems that could occur if the two parameters λ_{1t} and λ_{2t} assume the same values in the SV, FF and SF models. Furthermore, it is a standard practice tracing to Nelson & Siegel (1987).

5. Results

Both the Table 2 and the Figure 4(a) provide the spot curve fitting RMSE by maturities for each estimated model. As expected, due to the greater flexibility, the SS presents a large advantage over the other models and the SF is superior to the NS models up to the maturity of twelve years. Interesting to note, however, is the good fitting of the SF in both the beginning and ending of the curve. As we know a major difficulty of this type of models is to fit well both the extremes of the curves, but the SF behaves particularly well showing an important gain over the other NS models. The superiority occurs also in the time spectrum, as shown by the Figure 4(c). The SF is superior to the NS and SV along the entire sample and, in general, it is also superior to the FF.

Table 2
Spot curve fitting – RMSE

Maturities in years	Models SS	NS	SV	FF	SF
0,083	0,00000	0,00150	0,00058	0,00181	0,00059
0,25	0,00001	0,00060	0,00054	0,00066	0,00052
0,5	0,00009	0,00075	0,00056	0,00073	0,00049
0,75	0,00019	0,00103	0,00044	0,00106	0,00039
1	0,00016	0,00104	0,00041	0,00108	0,00043
1,5	0,00028	0,00070	0,00045	0,00083	0,00043
2	0,00039	0,00058	0,00056	0,00055	0,00048
3	0,00023	0,00063	0,00055	0,00054	0,00032
4	0,00017	0,00075	0,00057	0,00079	0,00038
5	0,00008	0,00081	0,00048	0,00076	0,00055
7	0,00009	0,00073	0,00061	0,00062	0,00060
10	0,00004	0,00083	0,00075	0,00063	0,00045
12	0,00005	0,00090	0,00082	0,00050	0,00028
15	0,00003	0,00108	0,00111	0,00071	0,00076
17	0,00004	0,00115	0,00122	0,00064	0,00065
20	0,00004	0,00153	0,00153	0,00035	0,00030
23	0,00001	0,00183	0,00169	0,00092	0,00106
26	0,00002	0,00138	0,00113	0,00060	0,00063
28	0,00000	0,00256	0,00256	0,00081	0,00096
Mean	0,00010	0,00107	0,00087	0,00077	0,00054

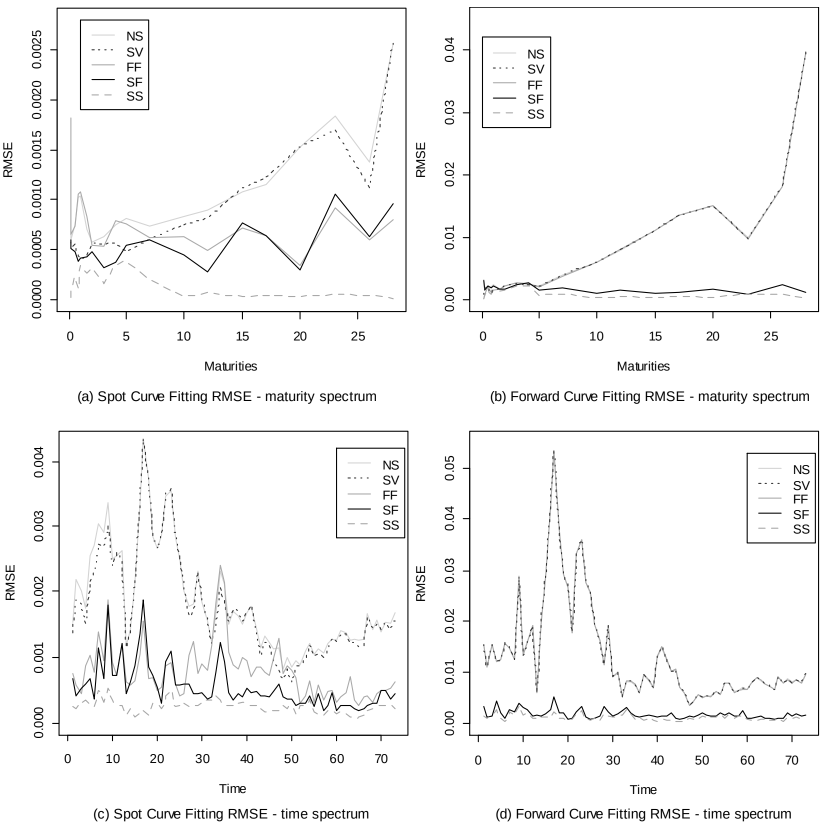
Note: this table shows the spot curves fitting RMSE (in average and by maturities) for the SS, NS, SV, FF and SF models.

The advantage of the SF over the other parametric models is, however, more notable for the forward curves, as shown by the Table 3 and Figures 4(b) and 4(d). In the time domain the superiority along the entire sample is apparent, and in the maturity domain it is more apparent for the vertices from 5 to 28 years. The fitting improvements are greater in the medium and long term maturities and they clearly influence the outcome of the average RMSE. Notice that the SF is 320% more accurate, in average, than the FF. The results also show that the difference between the SF and SS models is smaller for the forward curves than for the spot ones.

Table 3
Forward curve fitting – RMSE

Maturities in years	Models				
	SS	NS	SV	FF	SF
0,083	0,00015	0,00333	0,00081	0,00019	0,00300
0,25	0,00072	0,00189	0,00185	0,00096	0,00159
0,5	0,00156	0,00214	0,00203	0,00209	0,00215
0,75	0,00090	0,00206	0,00109	0,00129	0,00197
1	0,00151	0,00243	0,00160	0,00155	0,00218
1,5	0,00142	0,00179	0,00165	0,00162	0,00170
2	0,00155	0,00210	0,00223	0,00215	0,00166
3	0,00221	0,00262	0,00252	0,00263	0,00245
4	0,00227	0,00243	0,00240	0,00258	0,00268
5	0,00061	0,00221	0,00218	0,00199	0,00151
7	0,00091	0,00392	0,00406	0,00384	0,00196
10	0,00028	0,00595	0,00609	0,00603	0,00103
12	0,00054	0,00799	0,00807	0,00809	0,00154
15	0,00039	0,01110	0,01108	0,01115	0,00100
17	0,00057	0,01350	0,01344	0,01352	0,00114
20	0,00045	0,01514	0,01505	0,01512	0,00160
23	0,00095	0,00988	0,00981	0,00983	0,00097
26	0,00080	0,01814	0,01824	0,01815	0,00236
28	0,00029	0,03958	0,03969	0,03961	0,00124
Mean	0,00095	0,00780	0,00757	0,00749	0,00178

Note: this table shows the forward curves fitting RMSE (in average and by maturities) for the SS, NS, SV, FF and SF models.



(a) Spot Curve Fitting RMSE - maturity spectrum

(b) Forward Curve Fitting RMSE - maturity spectrum

(c) Spot Curve Fitting RMSE - time spectrum

(d) Forward Curve Fitting RMSE - time spectrum

Note: this figure shows the spot and forward curves fitting RMSE in the maturity and time spectrums.

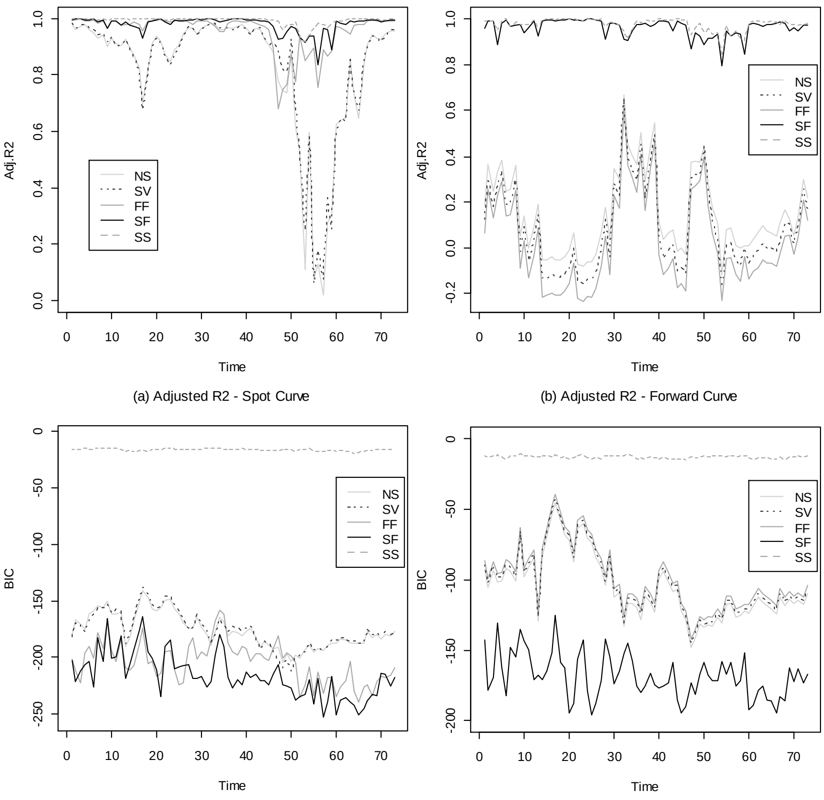
Figure 4
Fitting RMSE – spot and forward curves

Since the NS models can be considered nested,[8] a natural criticism to the results described above is that the addition of the extra curvature term necessarily leads to lower in-sample RMSE and that the superior behavior of the SF model is just a case of in-sample over fitting. To overcome such arguments and to attest the importance of the third curvature component both the Schwartz Information Criterion (SIC) and the adjusted R^2 statistic were calculated for all the cross-section regressions[9] of the sample. Figure 5(a) – 5(d) show the results. Regarding the NS models, we observe that the new component adds information in the estimation of both curves. The calculated SIC (adjusted R^2) for the SF is smaller (greater) than those calculated for the other NS models in at least 78% (83%) of the estimated yield curves. In the case of the forward curves, the results are even better. The SIC (adjusted R^2) is smaller (greater) for the SF than for the other NS models in 100% (100%) of the estimated curves. We see this result as another important contribution of the paper. Moreover, the adjusted R^2 is superior to 80% along the entire sample, and it is superior to 95% in 58 out of the 73 estimated forward curves. The SS shows the higher SIC and adjusted R^2 statistics. A reasonable explanation for the higher SIC is that it penalizes the different models for the number of degrees of freedom more harshly than the adjusted R^2.

Figures 6 and 7 show some examples of estimated yield and forward curves in specific months. As pointed out above, the SS model presents lower interpolation errors. Note also the advantage of the SF over the other NS class models, especially for the forward curves and for the medium and long term vertices of both curves. The superiority of the SS, however, comes with a cost: its instability in interpolating some maturities of the term structure. Figures 6(c), 7(e) and 7(f) and Figures 7(c), 7(d), 7(e) and 7(f) clearly show its weakness. The SS is very unstable in interpolating the beginning of both curves. On the other hand, the SF seems to interpolate the spot and forward rates with a high smoothness and good flexibility.

[8]They may differ only accordingly to the parameters λ_1 and λ_2.

[9]The SIC for the smoothing spline method was calculated using a variant of the formula proposed by Koenker *et al.* (1994), that is, $SIC(p_\omega) = \log\left[n^{-1}\sum_{i=1}^{n}(y_i - f(x_i))^2\right] + \frac{1}{2n}p_\omega \log(n)$, where n is the number of observations and p_ω is the number of “active” knots.



(a) Adjusted R2 - Spot Curve

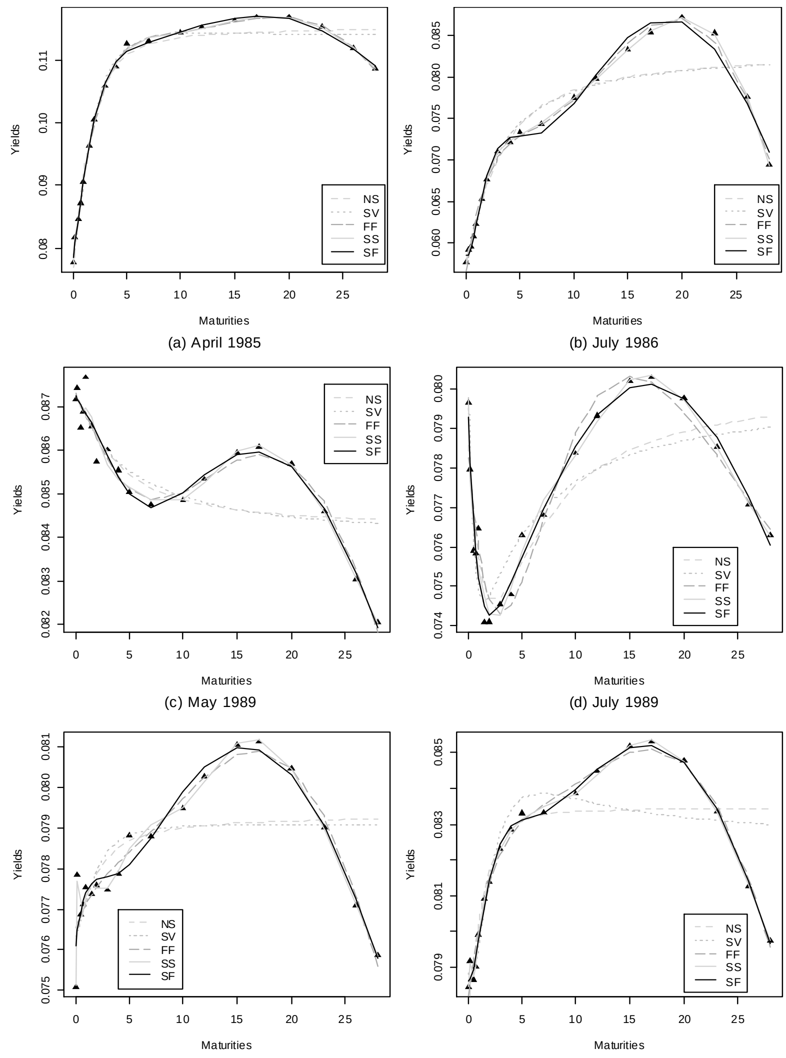
(b) Adjusted R2 - Forward Curve

(c) Schwartz Criterion (BIC) - Spot Curve

(d) Schwartz Criterion (BIC) - Forward Curve

Notes: this figure shows the computed adjusted R^2 and the Schwartz Information Criterion statistics for the cross-section regressions of the NS models.

Figure 5
Adjusted R^2 and Schwartz criterion – spot and forward curves



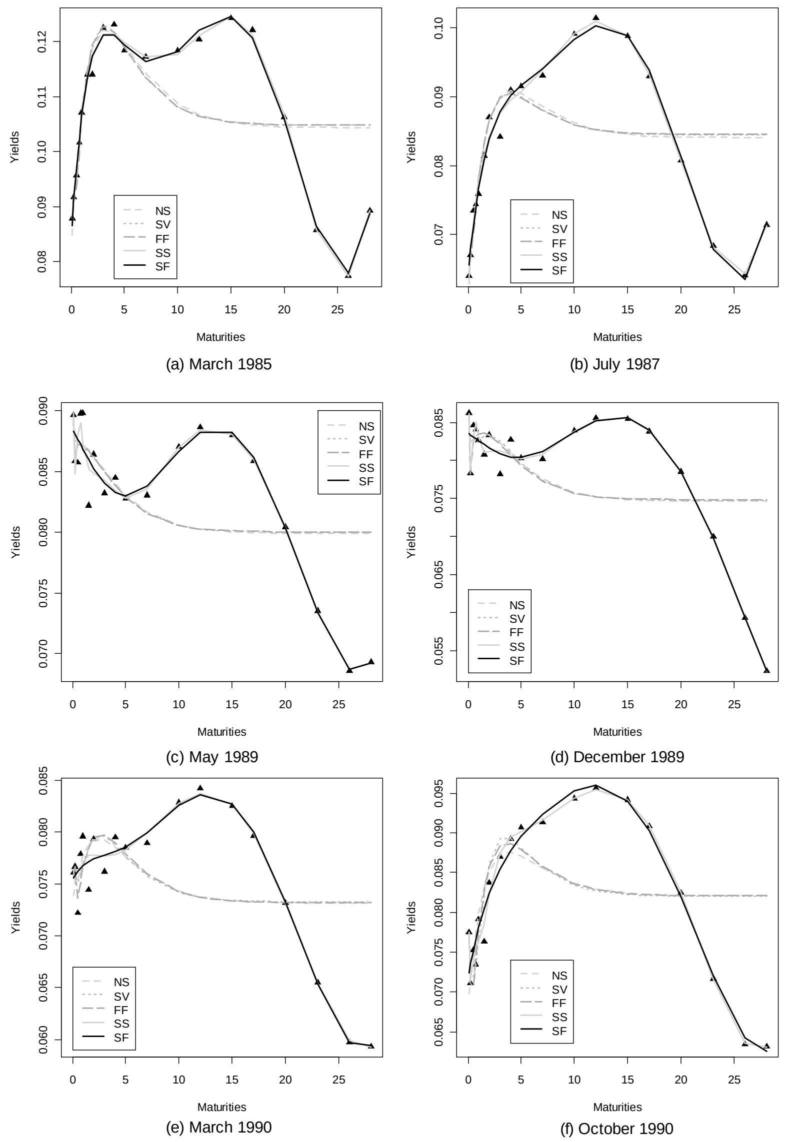
(a) April 1985 (b) July 1986

(c) May 1989 (d) July 1989

(e) December 1989 (f) June 1990

Note: figure 6 shows the McCulloch spot curves observed in six specific months of the sample and exhibits the fitting of the SS, NS, SV, FF and SF models to these curves.

Figure 6
Fitted yield curves in specific months



(a) March 1985 (b) July 1987

(c) May 1989 (d) December 1989

(e) March 1990 (f) October 1990

Note: figure 7 shows the McCulloch forward curves observed in six specific months of the sample and exhibits the fitting of the SS, NS, SV, FF and SF models to these curves.

Figure 7
Fitted forward curves in specific months

6. Conclusions

This paper compares the interpolation abilities of the most widely nonparametric and parametric term structure models used by the main Central Banks of the world. Seeking both smoothness and flexibility a new NS class parametric model is introduced. It emerges as a natural extension of the SV and FF models, proposed by Svensson (1994), De Rezende & Ferreira (2008) and Christensen *et al.* (2008).

The results show the superiority of the SS model over the other ones in interpolating the spot and forward rates, and also an advantage of the proposed SF model over the other NS models. It is also shown that the superiority of the SS, however, comes with a cost: its instability in fitting the initial vertices of the term structure. The SF, on the other hand, exhibits the desirable property of smoothness and also a high flexibility, especially for the forward curves and for the medium and long term maturities of both curves.

We also calculate along the time spectrum statistics which penalize for extra covariates and show that the third curvature term indeed adds information to the NS models, especially when the forward curve is estimated. We believe that it is an important contribution of the paper that also serves as an argument against the possible criticism of over fitting.

Despite the smoothness is important for macroeconomics purposes, the flexibility is also a desirable property. The poor construction of both the yield and forward curves can imply in the wrong understanding and measurement of important economic information carried by the term structure, especially those used for monetary policy purposes. Hence, the insertion of flexibility in a class of models largely used by Central Banks around the world, like the NS class, can improve the practice of the monetary policy. This flexibility gain can also make the NS models more useful in industry.

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