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Nonparametric Estimation of Risk-Neutral Distribution via the Empirical Esscher Transform

(Estimação Não Paramétrica da Distribuição Neutra ao Risco através da Transformada de Esscher Empírica)

Manoel Pereira*
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Abstract

This paper introduces an empirical version of the Esscher transform for nonparametric option pricing. Traditional parametric methods require the formulation of an explicit risk-neutral model and are operational only for a few probability distributions for the returns of the underlying asset. In our proposal, we make only mild assumptions on the pricing kernel and there is no need for the formulation of the risk-neutral model. First, we simulate sample paths for the returns under the physical measure. Then, based on the empirical Esscher transform, the sample is reweighted, giving rise to a risk-neutralized sample from which derivative prices can be obtained by a weighted sum of the options' payoffs in each path. We analyze our proposal in experiments with artificial and real data.
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Keywords: Nonparametric estimation, risk-neutral distribution, option pricing, empirical Esscher transform.

JEL Codes: C1, C5, C6, G1.

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Resumo

Este artigo introduz uma versão empírica da transformada de Esscher para o apreçamento não-paramétrico de opções. Os métodos paramétricos tradicionais exigem a formulação de um modelo risco neutro explícito e são operacionais apenas para algumas distribuições de probabilidade para os retornos do ativo subjacente. Em nossa proposta, fazemos apenas suposições moderadas sobre o kernel de preços e não há necessidade de formular o modelo risco neutro. Primeiro, simulamos trajetórias para os retornos sob a medida física. Então, com base na transformação empírica de Esscher, a amostra é reponderada, dando origem a uma amostra risco-neutralizado a partir da qual os preços dos derivativos podem ser obtidos por uma soma ponderada dos payoffs das opções em cada caminho. Analisamos nossa proposta em experimentos com dados artificiais e reais.

Keywords: Estimação não-paramétrica, distribuição neutra ao risco, apreçamento de opções, transformada empírica de Esscher.

1. Introduction

In most option pricing models, the fair price is determined from the expected value of its cash flow, under a risk-neutral probability (measure Q), and discounted by a risk-free rate. Under the assumption that the market is dynamically complete, it could be shown that every derivative security can be hedged and the measure Q is unique (Bingham and Kiesel, 2004). However, incomplete markets[1] exist for many reasons and, according to the second fundamental theorem of asset pricing, we have an infinite number of measures Q under which one can get prices of derivatives. Then, how to choose a measure Q from an infinite set of possible measures?

According to Danthine and Donaldson (2015), the literature highlights two approaches to this problem: models based on the general equilibrium (Arrow, 1964, Debreu, 1959, Lucas, 1978, Rubinstein, 1976, Brennan, 1979) and the models based on absence of arbitrage (Black-Scholes, 1973, Cox and Ross, 1976, Harrison and Kreps, 1979, Harrison and Pliska, 1981). In the general equilibrium, the supply and demand

[1] Complete markets are those in which all assets can be replicated. However, in incomplete markets the asset price may be driven by additional stochastic factors (as stochastic volatility, stochastic interest rates, or stochastic jumps) that are not traded. In these cases, an investor cannot exactly replicate the payoff of an option state by state.

interact in various markets affecting the prices of many goods simultaneously. The valuation of assets occurs when the markets are balanced, that is, when the supply equals the demand. Thus, from a theoretical connection between macroeconomics (aggregate consumption) and financial markets, the marginal rate of substitution is used to determine a measure Q by solving a utility maximization problem.

In absence of arbitrage, we are appealing to the law of one price. This states that the equilibrium prices of two separate units of what is essentially the same good should be identical. If this was not the case, a riskless and costless arbitrage opportunity would open up: buy extremely large amounts at the low price and sell them at the high price, forcing the two prices to converge. The first fundamental theory of asset pricing says that, if a market model has a measure Q, then it does not admit arbitrage. The conditions that the risk-neutral probability structure must satisfy are that the discounted price process has zero drift and it must also be equivalent to the original structure. Then, a class of pricing kernels, or Radon-Nikodym derivatives, can be specified and impose restrictions that ensure the existence of a risk-neutral measure. In this case, the measure Q can be obtained without completely characterizing equilibrium in the economy (Christoffersen, Elkamhi, Feunou, and Jacobs, 2010, Christoffersen, Jacobs and Ornthanalai, 2013).

In both cases, these approaches require the formulation of an explicit risk-neutral model and are restricted to a few probability distributions for the measure Q. First, because it is difficult to characterize the general equilibrium setup underlying a Risk-Neutral Valuation Relationship (RNVR), see for example Duan (1995, 1999). Second, it is possible to investigate option valuation for a large class of conditionally heteroskedastic processes (Gaussian or non-Gaussian), provided that the conditional moment generating function exists. Christoffersen, Jacobs and Wang (2004), cite that they help explain some stylized facts (smile effect, volatility variability over time and presence of clusters in certain periods) in a qualitative sense, but the magnitude of the effects is insufficient to completely solve the biases. The resulting pricing errors have the same sign as the Black-Scholes (1973) pricing errors, but are smaller in magnitude.

According to Haley and Walker (2010), the nonparametric option pricing techniques have expanded rapidly in recent years. It offers an alternative by avoiding possibly biased parametric restrictions and reducing the misspecification risk. As the change of measure does not involve the distribution of the model's innovations, this method of risk-neutralization is applicable even when the moment generating function of the innovations' probability distribution does not exist. In these methods, the historical distribution of prices is used to predict the distribution of future asset prices. According to Stutzer (1996), by using past data to estimate the payoff distribution at expiration, it permits more accurate assessment of the likely pricing impact caused by investors' data-based beliefs about the future value distribution.

There are two ways to nonparametrically estimate risk-neutral probabilities implicit in financial instruments: the methods that seek to infer the empirical risk-neutral probability from the options market[2] (kernel, maximum entropy, and curve fitting[3]) and the methods that seek to infer the empirical risk-neutral probability from asset price (with or without option price), as canonical valuation developed by Stutzer (1996). In the case of canonical valuation, the maximum entropy principle is employed to transform the empirical distribution into its risk-neutral counterpart, by minimizing the Kullback–Leibler information criterion (KLIC).

Several papers have extended Stutzer's original work and demonstrated that the methodology is flexible and performs very well in the presence of realistic financial time series, see Gray and Newman (2005), Gray, Edwards, and Kalotay (2007), Alcock and Carmichael (2008), Haley et al. (2010) and Almeida and Azevedo (2014). Other researchers, as Haley et al. (2010) and Almeida et al. (2014), suggested the adoption of members of the Cressie-Read family of discrepancy functions as alternative ways of measuring distance in the space of probabilities.

This paper introduces an empirical version of the Esscher transform (1932) for nonparametric option pricing. We assume that the empirical pricing kernel[4] is known and given by an empirical version of the Esscher transform (1932). This assumption is reasonable, because it is well-known in the information theory[5] that a problem of maximum entropy has its solution in the form of the Esscher transform (Buchen and Kelly, 1996, Stutzer, 1996, Duan, 2002).

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[2] There are also parametric methods such as Expansion, Distributions and Generalized Mixture. Parametric methods use a known probability distribution adjusted to observed option prices.

[3] See Aït-Sahalia and Lo (1998), Jackwerth (2000) and Shimko (1993).

[4] In many applications, the empirical pricing kernel is the object of interest because it describes risk preferences of a representative agent in an economy, and the risk aversion function estimates the investors' expectations about future return probabilities (Hansen and Jagannathan, 1991, Aït-Sahalia and Lo, 2000, and Jackwerth, 2000).

[5] Information theory is a field of mathematics that studies the information quantification. In distribution of probabilities is possible to define a quantity called entropy that has many mathematical properties. See Shannon (1948).

The empirical Esscher transform and the canonical valuation of Stutzer (1996) generate, in theory, the same risk-neutral measure: a measure that is exponential in the return of the underlying asset. The numerical differences that appear in this work are due to the different form in which the martingale condition is imposed. In the empirical Esscher transform, the martingale condition is imposed through the ratio of two empirical moment generating function's periods. In the Stutzer (1996) method, this condition is imposed via Euler's equation.

Duan (2002) also develops a nonparametric option pricing theory based on Esscher transform (1932). He uses a binary search to find the Esscher parameter and the measure Q is evaluated using the standard polynomial approximation formula. In our case, we use a consistent estimator for the moment generation function and we avoid the use of intensive computational methods. Hence, we obtain a method that does not require a set of restrictive assumptions for the formulation of a specific model; that provides a clear and easy way to obtain a risk-neutral distribution; is adaptable and flexible to respond to changes in the data generating process; and explores the whole cross-section information contained in the underlying asset's price.

The main contribution in this paper is testing the methodology proposed by Gerber and Shiu (1994) in a similar way to Stutzer (1996), Gray et al. (2005) and Haley et al. (2010). The test consists of two parts: A Monte Carlo study that assesses the method's pricing ability in the Black and Scholes (1973) model and Heston (1993) model; and an empirical assessment of the method's pricing ability for European Call options on two Brazilian Companies (Vale and Petrobras). In empirical data, we propose a methodology to construct an unknown data generation process based on bootstrap with replacement on historical returns of the underlying asset. As in previous works, the pricing ability is measured using the Mean Absolute Percentage Error of an European Call.

The paper is organized as follows. Section 2 discusses the Esscher transform. In Section 3, we introduce the empirical Esscher transform. Section 4 presents the methodology we use to compare the different pricing methods, and the results are discussed in Section 5. Finally, Section 6 concludes.

2. The Esscher transform

Let X be a random variable with probability density function $f(x)$ and let θ be a real number. Then, the Esscher transform (ET) of $f(x)$ with Esscher parameter θ is given by $f(x;\theta)$, defined as:

$$f(x;\theta) = \frac{e^{\theta x}}{\int_{-\infty}^{+\infty} e^{\theta x} f(x)dx} f(x). \quad (1)$$

Note that $f(x;\theta)$ is also a probability density function since it integrates one. Furthermore, the ET can be interpreted as a reweighted version of $f(x)$, with reweighting function given by:

$$m(x;\theta) = \frac{e^{\theta x}}{\int_{-\infty}^{+\infty} e^{\theta x} f(x)dx}. \quad (2)$$

The denominator of this expression represents the moment generating function (mgf) of $f(x)$, denoted by:

$$M(\theta) = E\left[e^{\theta X}\right] = \int_{-\infty}^{+\infty} e^{\theta x} f(x)dx. \quad (3)$$

In this case, for the Esscher transform to exist, the mgf of X must exist, which precludes some well-known density functions, like the t-student. Hence, the ET of $f(x)$ can be expressed as:

$$f(x;\theta) \quad = \quad m(x;\theta)f(x) \quad = \quad \frac{e^{\theta x}}{M(\theta)} f(x). \quad (4)$$

Consider now the ET of the density $f(x_T)$ of X_T, the log-return of an asset for a period T, given by:

$$f(x_T;\theta) = m(x_T;\theta)f(x_T) = \frac{e^{\theta x_T}}{M(\theta)} f(x_T). \quad (5)$$

Gerber and Shiu (1994) proposed to use the ET of X_T as the risk-neutral distribution (RND) for the log-return of this asset. They call it Risk-Neutral Esscher Transform (RNET). In this context, $f(x_T)$ is referred to as the physical probability measure P and $f(x_T;\theta)$, the ET of $f(x_T)$, is identified as the risk-neutral measure Q or, still, the equivalent martingale measure.

Let S_t be the price of an asset at time t. According to the fundamental theorem of asset pricing (Bingham et al, 2004), the risk-neutral value $v\left({g(S_T)}/{S_0}\right)$ of a derivative $g(S_T)$ with maturity T is given by the expected value of the payoff under the measure Q, discounted by the risk-free rate of return for period T, r_T:

$$v\left({g(S_T)}/{S_0}\right) = e^{-r_T} E^Q\left[{g(S_T)}/{S_0}\right]. \tag{6}$$

This is also true if the derivative is the asset itself so that $v\left({g(S_T)}/{S_0}\right) = S_0$ and $g(S_T) = S_T$ with $S_T = S_0 e^{X_T}$. This imposes the non-arbitrage constraint:

$$S_0 = e^{-r_T} E^Q\left[{S_0 e^{X_T}}/{S_0}\right] \rightarrow e^{r_T} = E^Q[e^{X_T}]. \tag{7}$$

Now, defining the measure Q as the ET of $f(x_T)$, we obtain the following condition for the value of θ:

$$\begin{aligned} e^{r_T} &= \int_{-\infty}^{+\infty} e^{X_T} f(x_T;\theta)\, dx_T \\ e^{r_T} &= \int_{-\infty}^{+\infty} \frac{e^{(\theta+1)x_T}}{M(\theta)} f(x_T)\, dx_T = \frac{M(\theta+1)}{M(\theta)}. \end{aligned} \tag{8}$$

Hence, the measure Q is given by $f(x_T;\theta^*)$, with $\theta^* = arg_\theta \left\{e^{r_T} = \frac{M(\theta+1)}{M(\theta)}\right\}$.

Gerber et al (1994) explores several different distributional assumptions to X_T, price dynamics and log-returns of an asset. They show that the RNET encompasses the classical option pricing formula of Black-

Scholes (1973) for Wiener processes, and the Binomial Model (Cox, Ross and Rubinstein, 1979).[6]

Duan (2002) explored empirical distribution of X_T and develops a nonparametric option pricing theory based on the first equation in (2.8). In this case, he used a binary search to find θ^*, the integral was evaluated numerically and $f(x_T;\theta)$ was evaluated using the standard polynomial approximation formula.

The RNET can also be applied to incomplete markets, which admit infinite measure Q. It provides an economic justification for selecting this particular transform, since it emerges as the solution for the problem of pricing a derivative under a power utility function (see Gerber et al, 1996).

Moreover, using the relative entropy principle, the risk-neutral density can be obtained from the following problem:

$$\min_{g(x)} \int_{-\infty}^{\infty} g(x) ln \frac{g(x)}{f(x)} dx \qquad (9)$$

where $g(x)$ is the model known and the discrepancy between it and another model $f(x)$ can be obtained by minimization of an information criterion. It is well-known in the information theory that the programming problem in (9) has its solution in the form of the ET (Buchen and Kelly, 1996, Stutzer, 1996, Duan, 2002).

3. The empirical Esscher transform

Consider a random sample of size n from X_T, denoted by $\{X_{T,i}\}_{i=1,n}$. Then, we define the Empirical Esscher Transform (EET) as:

$$q_{i,\theta} = \frac{e^{\theta X_{T,i}}}{\sum_{j=1}^{n} e^{\theta X_{T,j}}}. \qquad (10)$$

6 The Esscher transform can be used in both cases, discrete time (Bühlmann, 1996 and Siu, Tong and Yang, 2004) and continuous time (Gerber et al, 1994). Chan (1999) showed the relationship between minimal entropy equivalent measure and the Esscher transform when asset prices follow the Lévy process (see Chorro, Guégan and Ielpo, 2008 and Ornthanalai, 2011). For more studies see Monfort and Pegoraro (2011), Li and Badescu (2012) and Guégan, Ielpo and Lalaharison (2013).

Note that $\{q_{i,\theta}\}_{i=1,n}$ constitutes a probability mass function since $\sum_{i=1}^{n} q_{i,\theta} = 1$ and $q_{i,\theta} > 0\ \forall i$. Furthermore, in analogy to the ET, it can be interpreted as a reweighted version of the original sample $\{X_{T,i}\}_{i=1,n}$. Then, one can write:

$$q_{i,\theta} = m_q\left(X_{T,i};\theta\right)p_i \tag{11}$$

with $p_i = 1/n$ being the original weight and $m_q\left(X_{T,i};\theta\right)$ the reweighting function, given by:

$$m_q\left(X_{T,i};\theta\right) = \frac{e^{\theta X_{T,i}}}{\frac{1}{n}\sum_{j=1}^{n} e^{\theta X_{T,j}}}. \tag{12}$$

The denominator of this expression represents an estimator of the moment generating function (mgf) of $f_{X_T}(x)$:

$$\widehat{M}(\theta) = \frac{1}{n}\sum_{j=1}^{n} e^{\theta X_{T,j}}. \tag{13}$$

Then, if $\{X_{T,i}\}_{i=1,n}$ is a i.i.d sample, the weak law of large numbers assures that, if $E\left[e^{\theta X}\right]$ exists for all $\theta \in \Re$, then $\widehat{M}(\theta)$ is a consistent estimator of $M(\theta)$, i.e.:

$$\widehat{M}(\theta) \xrightarrow{P} M(\theta). \tag{14}$$

Now, take the sample version of the fundamental theorem of asset pricing stated in the preceding section. An estimate of the risk-neutral value $\hat{v}\left({g(S_T)}/{S_0}\right)$ is then given by the estimated expected value, denoted by $\hat{E}^{Q}[\cdot]$ of the payoff under the measure Q, discounted by the risk-free rate of return for period T, r_T:

$$\hat{v}\left({g(S_T)}\Big/_{S_0}\right) = e^{-r_T}\hat{E}^Q\left[{g(S_T)}\Big/_{S_0}\right] \tag{15}$$

with $S_{T,i} = S_0 e^{X_{T,i}}$ and $\hat{E}^Q\left[{g(S_T)}\Big/_{S_0}\right] = e^{-r_T}\sum_{i=1}^{n} g(S_0 e^{X_{T,i}})q_{i,\theta}$. Using the above expressions, it is easy to check that the sample version of the no-arbitrage condition is given by:

$$e^{r_T} = \frac{\widehat{M}(\theta+1)}{\widehat{M}(\theta)}. \tag{16}$$

Then, the empirical risk-neutral measure Q is given by $\{q_{i,\hat{\theta}^*}\}_{i=1,n}$, with $\hat{\theta}^* = arg_\theta\left\{e^{r_T} = \frac{\widehat{M}(\theta+1)}{\widehat{M}(\theta)}\right\}$. Again, if $E[e^{\theta X}]$ exists for all $\theta \in \mathfrak{R}$ then $\widehat{M}(\theta+1)$ and $\widehat{M}(\theta)$ will converge in probability to their respective population values and, by consequence, the solution of the non-arbitrage constraint will also converge, i.e., $\hat{\theta}^* \xrightarrow{P} \theta^*$.

The price of a European call on a non-dividend-paying stock is obtained under the risk-neutral distribution $q(S_T)$ and the payoff is discounted at the deterministic risk-free rate r:

$$C(K,T) = e^{-rT}\int_{-\infty}^{\infty}(S_T - K)^+ q(S_T)dS_T \tag{17}$$

or, alternatively,

$$C(K,T) = e^{-rT}\int_{-\infty}^{\infty}(S_T - K)^+ m(S_T)f(S_T)dS_T \tag{18}$$

where T is the time to maturity, S_T is the underlying asset price, K is the strike price, $f(S_T)$ is the physical distribution of the asset price at the option's expiration and $m(S_T) = q(S_T)/f(S_T)$ is the pricing kernel, characterizing the change of measure $f(S_T)$ to $q(S_T)$.

Consider the discretization of integral in (18):

$$C(K,T) = e^{-rT}\left[\sum_{j=1}^{n}\left(S_{T,j} - K\right)^{+} m(S_T)p(S_T)\right] \qquad (19)$$

where $q(S_T) = m(S_T)p(S_T)$ is the risk-neutral probability mass function.

4. Methodology

This section presents the methodology that is used to compare the proposed method to artificial and real data. To investigate its applicability in some settings, the empirical Esscher transform is applied to price European call options across a range of moneyness and maturities. The algorithm for our method is:

1. Simulate the physical distribution for $S_{T,i}, i = 1, \dots, n$;
2. Compute the empirical Esscher parameter, $\hat{\theta}^*$, using the equation (16);
3. Compute the option price with the equation (19).

In experiment 1, the asset price (step 1) follows a Geometric Brownian Model (GBM), under the physical measure:

$$dS_t = \mu S_t dt + \sigma S_t dz_t \qquad (20)$$

where S_t is the underlying asset's price at time t, μ is the expected rate of return, σ is the volatility and dz_t follows a Wiener process. We compare the empirical Esscher transform simulated option prices to the true price, the Black-Scholes model.

Based on the works of Hutchinson and Poggio (1994), Stutzer (1996) and Gray et al (2005), we use an annualized volatility of 20%, a drift of 10% and the riskless rate of interest is assumed to be a constant of 5%.

In experiment 2, the asset price (step 1) follows the Heston (1993) model, which assumes a diffusion process for the asset price and another stochastic process for the volatility. The asset price S_t follows the diffusion, under the physical measure:

$$dS_t = \mu S_t dt + \sqrt{v_t} S_t dz_{1,t} \qquad (21)$$

where $z_{1,t}$ is a Wiener process. The volatility $\sqrt{v_t}$ follows the diffusion:

$$dv_t = \kappa(\theta - v_t)dt + \sigma\sqrt{v_t}dz_{2,t} \quad (22)$$

$$dz_{1,t}dz_{2,t} = pdt \quad (23)$$

where $z_{2,t}$ is a Wiener process that has correlation ρ with $z_{1,t}$, θ is the long-run mean of the variance, κ is a mean reversion parameter, σ is the volatility of variance. We compare the empirical Esscher transform simulated option prices to the true price, the Heston models.

We use the Euler discretizations for the stochastic process of price and volatility. The first step in a simulation scheme is usually approximate a continuous-time process with a discrete time process. A standard method such as the Euler scheme has the advantage of facilitate understanding and their convergence properties are well-known. However, the Euler scheme must generate bias on the final price. In general, this bias decreases as the number of steps increases. In order to reduce the bias of discretization in our final prices, we use the number of time steps equal the 100, as in Rouah (2013) and Kienitz and Wetterau (2012).[7]

Based on the works of Lin, Strong and Xu (2001), Zhang and Shu (2003) and Gray et al (2005), we use the following values: $\kappa = 3.00$, $\theta = 0.04$, $\sigma = 0.40$ and $\rho = -0.50$. The initial value of the volatility equals to its long-term average. For consistency with the Black-Scholes world simulations, the drift of 10% and the riskless rate of interest is assumed to be a constant 5% continuously compounded.

In these artificial experiments, the moneyness (S/K) is equal to 0.90, 0.97, 1.00, 1.03, 1.125 and the maturities are equal to $1/12$, $1/4$, $1/2$ and 1 years. For each time to maturity T, 200 returns are drawn to generate the distribution of T-year forward. We obtain the risk-neutral measure (step 2) and we calculate the option price (step 3). This procedure is repeated 10.000 times and we calculate the Mean Absolute Percentage Error (MAPE). We repeat the artificial experiments with 5×10^4 returns, and 200 repetitions, to analyze if the accuracy improves with an increase in the sample size.

In experiment 3, we evaluate nonparametric opting pricing with real data. We compare the prices of the proposed method (EET) with the Stutzer prices (STZ) and the Black-Scholes prices (BSM). The comparison

[7] Broadie and Kaya (2006) find that this bias may be very large in some cases even if a large number of time steps are used.

with Stutzer (1996) is necessary due to the similarity between the pricing equations. To analyze the performance of nonparametric methods, we compared to the simplest parametric case: Black-Scholes (1973).

In order to construct the prices of nonparametric methods, in each time to maturity T, we perform bootstrap with replacement on historical returns of the underlying asset. We follow the sequence: (a) we construct a single trajectory for the asset price by drawing a certain quantity of historical log returns. For example, if the option has 17 days to maturity, then we draw the same quantity; (b) we accumulate the log returns of this trajectory and we obtain one price; (c) we repeat the process (a) and (b) 252 times to construct the physical distribution for the price at maturity (step 1). We obtain the risk-neutral measure (step 2) and we calculate the option price (step 3). We repeat this procedure 15.000 times and we calculate the MAPE.

The Stutzer (1996) method begins with the asset's historical distribution of T-year gross returns R_i, $i = 1, \dots, n$, which are expressed as price relatives. By the maximum entropy principle, it shows that the risk-neutral probabilities are:

$$\hat{\pi}_i^* = \frac{exp\left(\gamma^* \frac{R_i}{(1+r)^T}\right)}{\sum_{i=1}^n exp\left(\gamma^* \frac{R_i}{(1+r)^T}\right)} \tag{24}$$

where γ^* is the Lagrange multiplier, given by the following minimization problem:

$$\gamma^* = arg \min_{\gamma} \sum_{i=1}^{n} exp\left[\gamma\left(\frac{R_i}{(1+r)^T} - 1\right)\right]. \tag{25}$$

Note that the equation (24) is similar to risk-neutral probabilities of the empirical version of the Esscher transform in the equation (10).[8] Canonical option prices follow from the equation:

[8] Stutzer (1996) reports that the performance of canonical valuation improves when a small amount of option data is used.

$$C(K,T) = \frac{1}{(1+r)^T}\left[\sum_{i=1}^{n}\left(P_{T,i} - K\right)^+ \hat{\pi}_i^*\right], \qquad P_{T,i} = P_0 R_i, \qquad (26)$$
$$i = 1, \ldots, n.$$

where $P_{T,i}$ are the prices of underlying asset.

We consider the closing values of two daily databases of Vale's and Petrobras' prices from January 17, 2011 to January 17, 2012, containing 251 observations for each database.[9] We set time 0 to January 17, 2012 (the end point of the data sample period). The closing value of Vale on that day was $S_0 = 41.13$ and of Petrobras was $S_0 = 24.37$. This date presents the largest set of data with different maturities in that year, which allowed evaluate the methods in several maturities. We use the corresponding true market price on the valuation date as a benchmark.[10] The true market prices of the options with the strikes and maturities under consideration are shown in Table 2. The maturities are equal to $17/252$, $40/252$, $59/252$ and $121/252$ years for Petrobras (PETR4) and only the three first for Vale (VALE5). To the risk-free rate, were used the database of future markets of swap contracts DI X PRE[11], with maturities of 30, 60, 90 and 120 days. The interest risk-free rate was 10.3499% (17/252), 10.2485% (40/252), 10.1721% (59/252) and 10.032% (121/252) obtained by linear interpolations.[12] Table 1 presents the main descriptive statistics of the log returns. In Black-Scholes prices, we use the annualized historical volatility.

Table 1: Descriptive Statistics of the Log Returns

	Mean	Standard Deviation	Skewness	Kurtosis	Maximum	Minimum
Petrobras	-0.0007	0.0180	-0.5621	5.2172	0.0480	-0.0788
Vale	-0.0011	0.0174	-0.6538	7.1367	0.0577	-0.0962

[9] These data are specifics to the Brazilian market.

[10] All required data are obtained from Bovespa (http://www.bmfbovespa.com.br). Accessed in 12/20/2017.

[11] The DI spot rate is the overnight interbank deposit rate. It corresponds to the interest rate in which a depository institution lends immediately available funds to another depository institution. It provides an efficient method whereby banks can access short-term financing from central bank depositories.

[12] http://www.bcb.gov.br. Accessed in 12/20/2017.

Figure 1: Vale prices from January 17, 2011 to January 17, 2012.

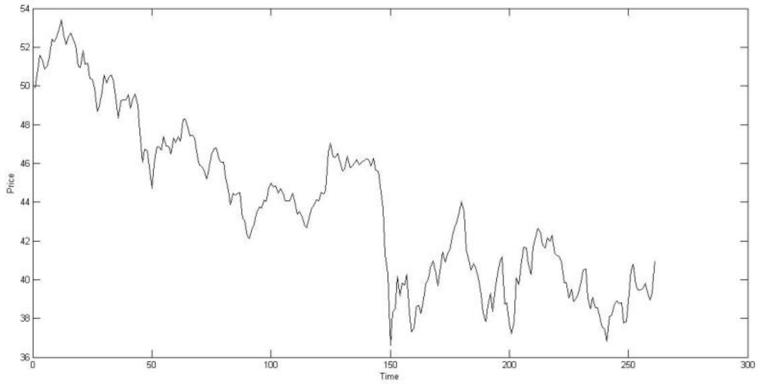
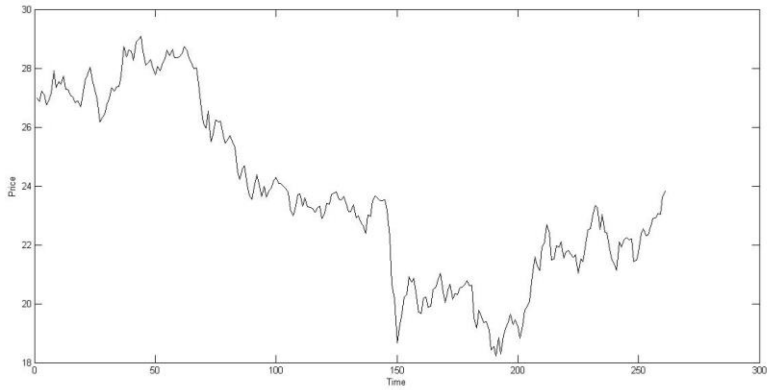


Figure 2: Petrobras prices from January 17, 2011 to January 17, 2012.



Figures 1 and 2 present the prices' behavior during the studied period. Since the 2008 crisis, the prices of these companies have presented a downward trend. The main factors that have contributed to the prices' fall were the Arab spring, the downgrade in the US credit rating from Standard and Poor's and the crisis in the Eurozone (Greece, Italy, Ireland and Portugal declared inability to pay their debts).

Table 2: Market prices of options from Vale and Petrobras

	Vale (VALE5)		Petrobras (PETR4)	
Maturity (days)	**Strike**	**Option price**	**Strike**	**Option price**
17	44.00	0.11	25.66	0.18
	43.14	0.21	25.16	0.27
	42.00	0.54	24.83	0.43
	41.14	0.99	23.66	1.12
	41.00	1.03	22.83	1.80
	40.14	1.72	21.66	2.88
	39.07	2.50	20.83	3.63
	38.57	2.75	19.66	4.94
	37.14	4.24	18.66	5.78
	37.00	4.41	17.66	6.78
	36.14	5.32	15.16	9.19
	36.00	5.34		
	35.00	6.05		
	34.00	7.31		
	30.14	11.41		
	30.00	11.65		
	28.00	13.59		
40	46.07	0.15	27.83	0.09
	46.00	0.20	27.00	0.17
	45.57	0.19	25.83	0.40
	44.07	0.42	25.33	0.61
	43.07	0.86	25.00	0.77
	42.07	1.25	23.83	1.38
	42.00	1.32	22.83	2.11
	41.00	1.83	21.66	3.08
	40.57	2.01	21.00	3.70
	40.00	2.51	19.66	4.95
	38.00	4.00	18.66	5.79
	37.00	4.75	17.83	6.60
	36.57	5.21		
	35.07	6.50		
	32.00	9.30		
59	48.00	0.21	26.00	0.58
	44.14	0.87	24.00	1.50
	44.00	0.90	21.83	3.21
	41.07	2.30		
	40.00	3.00		
121			25.50	1.85

5. Results

Experiment 1 compares the Empirical Esscher transform (EET) method performance when the conditions are the same as in the Black-Scholes (1973) model through the MAPE. Results in the table 3 show that the EET prices reproduced BSM prices when the sample size increases. This is expected because Gerber and Shiu (1994) proved the consistency of their method with the Black-Scholes (1973) model.

Experiment 2 compares the EET method performance when the conditions are the same as in the Heston (1993) model through the MAPE. Results in the table 4, as in the table 3, show that the EET prices improves with sample size.

In both experiments, the MAPE decreases monotonically as increases moneyness, and it decreases (increases) in maturity for out-of-the-money (in-the-money) options. These findings are consistent with a similar analysis performed by Stutzer (1996) and Gray et al. (2005) and Haley et al (2010). The disparity in the performance increases as the data generating process moves further from a constant volatility model to a stochastic volatility model, the MAPE of proposed method are the biggest.

In experiment 3, we compare the EET, the Stutzer method (STZ) and the Black-Scholes model (BSM) in real data (from Petrobras and Vale). Tables 5 and 6 show that the lowest pricing errors are between the nonparametric methods. Table 5, for the maturity equal to 17, the nonparametric methods present similar results. For others maturities, the proposed method presents the lowest MAPE for moneyness equal to deep-out-of-the-money, out-of-the-money and at-the-money. Results in the table 6 are similar to the ones in table 5.

The size of the MAPE errors can be very large, especially for the real data. At least four factors are contributing to our results: the lack of synchronization of prices, the liquidity of the derivative, the data generating process proposed and the empirical Esscher transform. First, option prices are extremely sensitive to the underlying asset price, and a lack of synchronization can generate: “arbitrage” opportunities (Galai, 1979), violate lower bound constraints (Bhattacharya, 1983, Culumovic and Welsh, 1994, Stephan and Whaley, 1990, Fleming, Ostdiek and Whaley, 1995) and can generate large errors in option prices, especially for low-priced out-of-the-money options (George and Longstaff, 1993).

Second, liquidity risk is characterized by a low or even lack of demand for the asset. Hence, become difficult to sell the asset because the price may be quite different from the fair price. Third, the unknown data generating process proposed may not have been able to capture the time series' stylized facts of the underlying asset. Four, the empirical Esscher transform has deficiencies or it does not provides correct pricing when the markets are in fall (see figures 1 and 2).

We also analyze the behavior of the empirical Esscher parameter. The results are presented in Figure 3 and tables 7, 8, 9 and 10. In the Figure 3, the panels (a) and (d) present the empirical Esscher parameters obtained for 200 returns (and 10,000 repetitions) in the Black-Scholes and Heston worlds, respectively. We note a cloud of points when the size of the sample is small. Histograms in panels (b) and (e) show that the Esscher parameter is symmetric in the Black-Scholes world and negatively skewed in the Heston world, what can suggest that the probability distribution of the empirical Esscher parameter follows the behavior of data generating process. The same behavior was observed in others maturities. The panels (c) and (f) present the empirical parameters for 5x10 returns (and 200 repetitions) in the Black-Scholes and Heston worlds, respectively. These figures show that the empirical parameter converges for one specific value when the sample increases.

Tables 7 and 8 present the main descriptive statistics of the empirical parameter in the Black-Scholes and Heston world. We can highlight that the standard deviation decreases along with the maturity and with the increase in the sample size, and the statistics' values begin to converge to a constant value in larger samples.

Tables 9 and 10 present the main descriptive statistics of the empirical parameters of both methods: Esscher (θ^*) and Stutzer (γ^*). Note that the values are close. When we compare only the empirical Esscher parameter obtained for synthetic data with the one obtained for real data, the more important change is the signal. That is, the Esscher parameters obtained with synthetic data are simulated with a drift $(\mu = 10.00\%)$ greater than the risk-free rate $(r = 5.00\%)$. Thus, the negative parameter shifts the risk-neutral distribution to the left, what eliminates the risk premium and assures the average yield equal to risk-free rate. In real data, the opposite happens. The positive parameter shifts the risk-neutral distribution to the right. This is contrary to financial theory. However, this does not constitute an arbitrage opportunity, because the daily risk-free rate is between the worst and the best daily return (see Cox et al, 1979). Again, as we can see (figures 1 and 2, and in Table 2) the price time series are in fall, and in this case, applications in risk-free interest rates are paying more than these stocks.

Table 3: MAPE of Empirical Esscher Transform Estimates in a Black-Scholes World

This table contains the prices for a European call option, where the underlying prices are simulated by a geometric Brownian motion with $\mu = 0.1$ and $\sigma = 0.20$, and they are compared to the true Black-Scholes call prices. The top and bottom numbers reported for each combination are the mean absolute percentage error (MAPE) of the EET with 200 returns and with 5×10^4 estimated returns respectively over 10,000 and 200 simulations.

	Time to expiration (years)			
Moneyness (S/K)	1/12	1/4	1/2	1
Deep-out-of-the-money	25.1459	5.9172	2.9991	2.0394
(0.90)	1.6812	0.3472	0.1720	0.1325
Out-of-the-money	2.8328	1.8461	1.6741	1.6500
(0.97)	0.1870	0.1166	0.1041	0.1054
At-the-money	1.4156	1.4219	1.4480	1.5203
(1.00)	0.0932	0.0817	0.0898	0.0964
In-the-money	0.9585	1.1688	1.2757	1.3936
(1.03)	0.0612	0.0710	0.0787	0.0862
Deep-in-the-money	0.1512	0.5227	0.7681	0.9774
(1.125)	0.0092	0.0320	0.0477	0.0628

Table 4: MAPE of Empirical Esscher Transform Estimates in a Heston World

This table contains the prices for a European call option, where the underlying prices are simulated by stochastic volatility with $\mu = 0.1$, $\kappa = 3.00$, $\theta = 0.04$, $\sigma = 0.40$ and $\rho = -0.50$, and they are compared to the true Heston call prices. The top and bottom numbers reported for each combination are the mean absolute percentage error (MAPE) of the EET with 200 returns and with 5×10^4 estimated returns respectively over 10,000 and 200 simulations.

	Time to expiration (years)			
Moneyness (S/K)	1/12	1/4	1/2	1
Deep-out-of-the-money	36.7566	13.4996	9.5056	7.6861
(0.90)	4.8636	4.9842	5.8228	6.0545
Out-of-the-money	7.2624	5.6063	5.1907	5.0316
(0.97)	1.2073	2.3118	3.2218	3.9084
At-the-money	3.8561	3.9084	4.0431	4.2104
(1.00)	0.6422	1.5878	2.4599	3.2255
In-the-money	2.0482	2.7513	3.1526	3.5286
(1.03)	0.3030	1.0644	1.8609	2.6532
Deep-in-the-money	0.2332	0.8956	1.4536	2.0503
(1.125)	0.0171	0.2615	0.7414	1.4129

Table 5: MAPE of Empirical Esscher Transform Estimates in Petrobras
This table contains the prices for a European call option, where the underlying prices are simulated from bootstrap with replacement on historical returns, and they are compared to the true market price. The numbers reported for each combination are the MAPE of the EET (proposed method), STZ (Stutzer method) and BSM (Black-Scholes model) with 252 returns and the simulation is repeated 15,000 times.

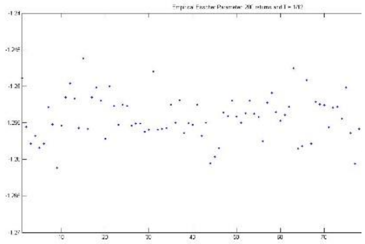
Maturity	Moneyness (spot/strike)		EET	STZ	BSM
T = 17/252	Deep-out-of-the-money	0.95	60.2770	61.5768	71.3672
	Out-of-the-money	0.97	63.0951	63.9341	70.4573
		0.98	32.2144	32.6802	36.5306
	In-the-money	1.03	8.6391	8.6142	9.3748
		1.07	2.5588	2.4217	2.5840
	Deep-in-the-money	1.13	0.4307	0.3273	0.2894
		1.17	1.6701	1.5079	1.6076
		1.24	1.9466	2.0699	1.9588
		1.31	0.9689	0.8672	0.9671
		1.38	0.7246	0.6422	0.7244
		1.61	1.3303	1.2781	1.3303
T =40/252	Deep-out-of-the-money	0.88	131.3580	140.6030	157.7360
		0.9	102.7306	108.7454	118.8495
		0.94	62.9430	65.8111	70.0874
		0.96	36.6367	38.4690	41.1048
	Out-of-the-money	0.97	26.2260	27.6094	29.5751
	At-the-money	1.02	15.0387	15.5256	16.3215
	In-the-money	1.07	7.3354	7.4032	7.7823
	Deep-in-the-money	1.13	4.0906	3.9246	4.1547
		1.16	2.4239	2.2025	2.4157
		1.24	1.9286	1.6844	1.8991
		1.31	3.8815	3.6555	3.8650
		1.37	3.4446	3.2459	3.4375
T = 59/252	Deep-out-of-the-money	0.94	52.6996	56.6112	58.6471
	At-the-money	1.02	20.8055	21.8995	22.4323
	Deep-in-the-money	1.12	3.5929	3.5943	3.8422
T = 121/252	Deep-out-of-the-money	0.96	6.4109	8.8424	6.3078

Table 6: MAPE of Empirical Esscher Transform Estimates in Vale
This table contains the prices for a European call option, where the underlying prices are simulated from bootstrap with replacement on historical returns, and they are compared to the true market price. The numbers reported for each combination are the MAPE of the EET (proposed method), STZ (Stutzer method) and BSM (Black-Scholes model) with 252 returns and the simulation is repeated 15,000 times.

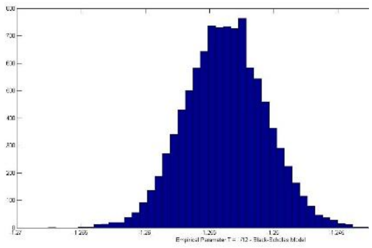
Maturity	Moneyness (spot/strike)		EET	STZ	BSM
T = 17/252	Deep-out-of-the-money	0.93	175.9523	182.4160	207.2992
		0.95	134.9038	138.7734	154.2885
	Out-of-the-money	0.98	61.8209	63.3503	69.8371
	At-the-money	1	28.7498	29.5111	32.7514
		1	31.0346	31.7410	34.7763
		1.02	8.7360	9.0209	10.4410
	In-the-money	1.05	5.9744	6.0083	6.5923
		1.07	11.1145	11.0728	11.4601
		1.11	1.9831	1.8404	1.9505
		1.11	1.0093	0.8644	0.9658
	Deep-In-the-money	1.14	0.9143	1.0715	0.9810
		1.14	1.2425	1.0812	1.1752
		1.18	5.4708	5.3079	5.4227
		1.21	0.7746	0.6326	0.7515
		1.36	1.8743	1.9600	1.8745
		1.37	2.7023	2.7859	2.7025
		1.47	1.9758	2.0428	1.9759
T = 40/252	Deep-out-of-the-money	0.89	193.5641	211.4168	228.3421
		0.89	126.1923	139.6921	152.3182
		0.9	178.4586	193.7011	207.7265
		0.93	111.0552	119.0766	125.9295
		0.95	41.1680	45.1967	48.5118
	Out-of-the-money	0.98	29.8445	32.5372	34.7016
		0.98	25.3795	27.9143	29.9489
	At-the-money	1	17.9030	19.5386	20.8631
		1.01	19.4561	20.8401	21.9774
	In-the-money	1.03	9.5245	10.4966	11.3238
		1.08	4.4847	4.7102	5.0866
		1.11	5.2491	5.2654	5.5572
	Deep-In-the-money	1.12	3.0515	3.0024	3.2671
		1.17	3.3883	3.2020	3.4340
		1.29	3.8424	3.6109	3.8307
T = 59/252	Deep-out-of-the-money	0.86	116.3855	135.1830	144.3093
		0.93	51.9814	58.8867	61.0783
		0.93	52.1273	58.8390	60.9299
	At-the-money	1	15.6564	18.0059	18.5215
	In-the-money	1.03	9.4821	11.0098	11.3472

Figure.3: Empirical Esscher Parameter T=1/12.

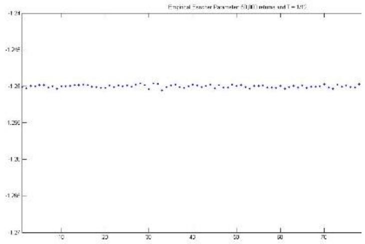
Black-Scholes World



(a) 200 returns

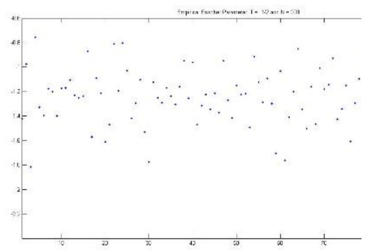


(b) Histogram for 200 returns

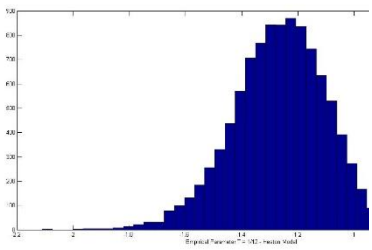


(c) 5x10^4 returns

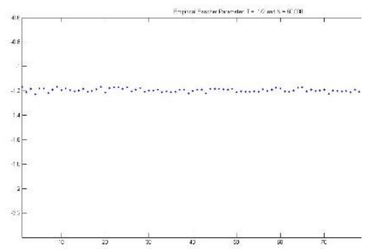
Heston World



(d) 200 returns



(e) Histogram for 200 returns



(f) 5x10^4 returns

Table 7: Empirical Escher Parameter in Black-Scholes World

	200 returns and 10.000 repetitions				$5x10^4$ returns and 200 repetitions			
	$T = 1/12$	$T = 1/4$	$T = 1/2$	$T = 1$	$T = 1/12$	$T = 1/4$	$T = 1/2$	$T = 1$
Mean	-1.2538	-1.2539	-1.2539	-1.2543	-1.2500	-1.2500	-1.2500	-1.2500
Std Deviation	0.0032	0.0056	0.0079	0.0110	0.0002	0.0003	0.0005	0.0008
Maximum	-1.2379	-1.2255	-1.2194	-1.2020	-1.2495	-1.2492	-1.2487	-1.2482
Minimum	-1.2676	-1.2792	-1.2836	-1.3007	-1.2506	-1.2509	-1.2518	-1.2520

Table 8: Empirical Escher Parameter in Heston World

	200 returns and 10.000 repetitions				$5x10^4$ returns and 200 repetitions			
	T = 21	T = 63	T = 126	T = 252	T = 21	T = 63	T = 126	T = 252
Mean	-1.2665	-1.2545	-1.2434	-1.2318	-1.2484	-1.2366	-1.2264	-1.2139
Std Deviation	0.1665	0.1548	0.1406	0.1239	0.0116	0.0098	0.0088	0.0078
Maximum	-0.6649	-0.6557	-0.7943	-0.8156	-1.2189	-1.2126	-1.2073	-1.1982
Minimum	-2.1098	-2.0004	-1.8774	-1.7747	-1.2758	-1.2568	-1.2509	-1.2378

Table 9: Empirical Parameters - Petrobras

	Esscher (θ^*)				Stutzer (γ^*)			
	T = 17	T = 40	T = 59	T = 121	T = 17	T = 40	T = 59	T = 121
Mean	2.8557	2.8891	2.9031	2.9309	2.8002	2.8253	2.8327	2.8460
Std Deviation	0.9265	0.6626	0.5826	0.4915	0.9276	0.6652	0.5881	0.5127
Maximum	7.0618	6.4443	5.7783	5.4746	6.9985	6.3773	5.6778	5.5471
Minimum	-0.5597	0.6241	0.7970	1.3742	-0.6260	0.5585	0.7238	1.1808

Table 10: Empirical Parameters - Vale

	Esscher (θ^*)			Stutzer (γ^*)		
	T = 17	T = 40	T = 59	T = 17	T = 40	T = 59
Mean	4.8156	4.8404	4.8830	4.7550	4.7724	4.8129
Std Deviation	1.0412	0.8152	0.7700	1.0451	0.8298	0.7974
Maximum	9.7780	8.4302	8.8304	9.7615	8.4394	8.8084
Minimum	0.9284	2.2304	2.7088	0.8686	2.1183	2.3691

6. Conclusion

In this paper, we propose an empirical version of the Esscher transform for nonparametric option pricing. We conduct artificial experiments in Black-Scholes and Heston worlds and real experiments to explore the potential usefulness of the proposed method.

Artificial results show that the EET prices improve alongside sample size. EET also provides higher prices for all maturities, and the MAPE decreases as moneyness does.

Real data results show that, when the stochastic process of underlying asset is unknown, the lowest pricing errors are between the nonparametric methods. The empirical Esscher transform and the canonical valuation of Stutzer (1996) generate, in theory, the same risk-neutral measure: a measure that is exponential in the return of the underlying asset. The numerical differences that appear in this work are due to the different form in which the martingale condition is imposed. In the empirical Esscher transform, the martingale condition is imposed through the ratio of two periods empirical moment generating function's periods. In the Stutzer (1996) method, this condition is imposed via Euler's equation.

The size of the MAPE errors can be very large, especially for the real data. At least four factors are contributing to our results: the lack of synchronization of prices, the liquidity of the derivative, the data generating process proposed and the empirical Esscher transform. First, option prices are extremely sensitive to the underlying asset price, and a lack of synchronization can generate: "arbitrage" opportunities (Galai, 1979), violate lower bound constraints (Bhattacharya, 1983, Culumovic and Welsh, 1994, Stephan and Whaley, 1990, Fleming, Ostdiek and Whaley, 1995) and can generate large errors in option prices, especially for low-priced out-of-the-money options (George and Longstaff, 1993).

Second, liquidity risk is characterized by a low or even lack of demand for the asset. Hence, become difficult to sell the asset because the price may be quite different from the fair price. Third, the unknown data generating process proposed may not have been able to capture the time series' stylized facts of the underlying asset. Four, the empirical Esscher transform has deficiencies or it does not provides correct pricing when the markets are in fall (see figures 1 and 2).

We also analyze the behavior of the empirical Esscher parameter. We can highlight that the standard deviation decreases with the maturity and with the increase in the sample size, and the values of the descriptive statistics begin to converge to a constant value in larger samples. When we compare only the empirical Esscher parameter obtained for synthetic and real data, the more important change is the signal. That is, the Esscher parameters obtained with synthetic data are simulated with a drift $(\mu = 10.00\%)$ greater than the risk-free rate $(r = 5.00\%)$. Thus, the negative parameter shifts the risk-neutral distribution to the left, what eliminates the risk premium and assures the average yield equal to risk-free rate. With real data, the opposite happens. The positive parameter shifts the risk-neutral distribution to the right. This is contrary to financial theory. However, this does not constitute an arbitrage opportunity, because the daily risk-free rate is between the worst and the best daily return (see Cox et al, 1979). Price time series are have been falling and in this case, applications in risk-free interest rates are paying more than in these stocks.

Further research can be done comparing the proposed method to other nonparametric pricing methodologies, verifying Monte Carlo simulation techniques, conducting an extensive empirical study on the performance of our proposed pricing method, considering asset returns of different frequencies, multiple cross-sections of market option prices and long-dated options, and verify the results with other data generating processes and/or include real option prices to help with pricing accuracy.

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