Anais da Academia Brasileira de Ciências
ISSN: 0001-3765
aabc@abc.org.br
Academia Brasileira de Ciências
Brasil

Pingyan, Chen
Limiting behavior of delayed sums under a non-identically distribution setup
Anais da Academia Brasileira de Ciências, vol. 80, núm. 4, diciembre, 2008, pp. 617-625
Academia Brasileira de Ciências
Rio de Janeiro, Brasil

Anais da Academia Brasileira de Ciências (2008) 80(4): 617–625
(Annals of the Brazilian Academy of Sciences)
ISSN 0001-3765
www.scielo.br/aabc

Limiting behavior of delayed sums under a non-identically distribution setu

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Manuscript received on July 28, 2007; accepted for publication on June 9, 2008;
presented by DJAIRO G. FIGUEIREDO

ABSTRACT

We present an accurate description the limiting behavior of delayed sums under a non-identically distributio
setup, and deduce Chover-type laws of the iterated logarithm for them. These complement and extend t
results of Vasudeva and Divanji (Theory of Probability and its Applications, 37 (1992), 534–542).

Key words: stable distribution, laws of iterated logarithm, delayed sum.

1 INTRODUCTION AND MAIN RESULTS

The distribution function F of a real valued random variable X is called stable law with exp
$\alpha(0 < \alpha < 2)$, if for some $\sigma > 0$, $-1 \leq \beta \leq 1$, its characteristic function is of the form

$$E\exp(itX) = \exp\left\{-\sigma|t|^\alpha(1 + i\beta\frac{t}{|t|}\omega(t,\alpha))\right\},\ t \in I\!R$$

where

$$\omega(t,\alpha) = \begin{cases} \tan\dfrac{\pi\alpha}{2}, & \text{if } \alpha \neq 1, \\ \dfrac{2}{\pi}\ln|t|, & \text{if } \alpha = 1. \end{cases}$$

If $\beta = 0$, X is a symmetric random variable. It is well-known, if F is a stable law with exp
$\alpha(0 < \alpha < 2)$, we have the following tail behavior:

$$\lim_{t\to\infty} t^\alpha(1 - F(t) + F(-t)) = c(\alpha,\sigma),$$

where $c(\alpha,\sigma) > 0$ only depends on α and σ(cf. e.g. Feller 1971). This property will play an imp
role in this paper.

Let $\{X_n, n \geq 1\}$ be a sequence of independent random variables with its partial sums $S_n = \sum$

The sum T_n is called a forward delayed sum (see Lai 1974). Suppose X_n's involve of two distributions F_1 and F_2 which are stable laws with exponents α_1 and $\alpha_2 (0 < \alpha_1 \le \alpha_2 < 2)$. For each $n \ge 1$, let $\tau_1(n)$ denote the number of random variables in the set $\{X_1, X_2, \cdots, X_n\}$ with distribution function F_1, then $\tau_2(n) = n - \tau_1(n)$ is the number of random variables with distribution function F_2 in the set $\{X_1, X_2, \cdots, X_n\}$. Then $(\tau_1(n), \tau_2(n))$ is called the sample scheme of the sequence $\{X_n, n \ge 1\}$. Assume that $\tau_1(n) = [n^{\alpha_1/\alpha_2}]$ and $B_n = n^{1/\alpha_2}$, where $[x]$ is the integer part of x. By Sreehari (1970), S_n/B_n converges weakly to a composition of the two stable laws.

Let $U_{\tau_1(n)}$ be the sum of those $\{X_1, X_2, \cdots, X_n\}$ with distribution function F_1 and $V_{\tau_2(n)}$ be the sum of those $\{X_1, X_2, \cdots, X_n\}$ with distribution function F_2. Then $S_n = U_{\tau_1(n)} + V_{\tau_2(n)}$. One can note that in T_n there are $[(n + a_n)^{\alpha_1/\alpha_2}] - [n^{\alpha_1/\alpha_2}]$ random variables with distribution function F_1 and $n + a_n - [(n + a_n)^{\alpha_1/\alpha_2}] - (n - [n^{\alpha_1/\alpha_2}])$ random variables with distribution function F_2.

The motivation of this paper is to extend and complement the results of Vasudeva and Divanji (1992). They obtained the following theorem in the special case that F_1 and F_2 are positive stable laws with exponents $0 < \alpha_1 \le \alpha_2 < 1$.

THEOREM A. Let $\{a_n, n \ge 1\}$ be a nondecreasing sequence with $0 < a_n \le n$ and a_n/n non-increasing. Let F_1 and F_2 are positive stable law and $0 < \alpha_1 \le \alpha_2 < 1$.

(i) If $\lim_{n\to\infty} \log(n/a_n)/\log\log n = +\infty$, then

$$\limsup_{n\to\infty} \left(\frac{T_n}{B_{a_n}}\right)^{1/\gamma_n} = e^{1/\alpha_2} \quad a.s.$$

(ii) If $\lim_{n\to\infty} \log(n/a_n)/\log\log n = 0$, then

$$\limsup_{n\to\infty} \left(\frac{T_n}{B_{a_n}}\right)^{1/\gamma_n} = e^{1/\alpha_1} \quad a.s.$$

(iii) If $\lim_{n\to\infty} \log(n/a_n)/\log\log n = s \in (0, +\infty)$, then

$$\limsup_{n\to\infty} \left(\frac{T_n}{B_{a_n}}\right)^{1/\gamma_n} = \exp\left\{\frac{\alpha_1 s + \alpha_2}{(s+1)\alpha_1\alpha_2}\right\} \quad a.s.$$

They only discuss the case that F_1 and F_2 are positive stable law with exponents $0 < \alpha_1 \le \alpha_2 < 1$. But by their method, it is impossible to discuss the rest case. In this paper, by a new method, we will complement and extend Theorem A in three directions, namely:

(i) We will obtain more exact results.

(ii) We will discuss not only that the distributions is the positive stable laws, but also that the distributions is not necessary positive stable laws and the exponents of the stable laws in (0, 2), not only in (0, 1).

Recall that the kind of type law of the iterated logarithm (LIL) was first obtained by Chover (19
symmetric stable law, and is called Chover-type LIL. By far, some papers concern with the Chove
LIL, for example, Chen (2002) for the weighted sums of symmetric stable law, Chen and Yu (2003)
weighted sums of stable law without symmetric assumption, Peng and Qi (2003) for the weighted s
law in the domain of attraction of stable law, and Chen (2004) for geometric weighted sums and
weighted sums of stable law, etc.

First we give an accurate description of the limiting behavior of S_n.

THEOREM 1.1. Let $f > 0$ be a nondecreasing function. Then with probability one

$$\limsup_{n\to\infty} \frac{|S_n|}{B_n(f(n))^{1/\alpha_1}} = \begin{cases} 0, \\ +\infty, \end{cases} \Leftrightarrow \int_1^{+\infty} \frac{dx}{xf(x)} \begin{cases} < +\infty, \\ = +\infty. \end{cases}$$

By Theorem 1.1, we have the following Corollary at once.

COROLLARY 1.1. For every $\delta > 0$, we have

$$\limsup_{n\to\infty} \frac{|S_n|}{B_n(\log n)^{(1+\delta)/\alpha_1}} = 0 \quad a.s.$$

and

$$\limsup_{n\to\infty} \frac{|S_n|}{B_n(\log n)^{1/\alpha_1}} = +\infty \quad a.s.$$

In particular

$$\limsup_{n\to\infty} \left|\frac{S_n}{B_n}\right|^{1/\log\log n} = e^{1/\alpha_1} \quad a.s.$$

REMARK 1.1. If $\alpha_1 = \alpha_2$, Corollary 1.1 extends the result of Chover (1966).

THEOREM 1.2. Let $\{a_n, n \geq 1\}$ be a subsequence of positive integers with $\limsup_{n\to\infty} a_n/n <$
Let $f > 0$ be a nondecreasing function. Then with probability one

$$\limsup_{n\to\infty} \frac{|T_n|}{B_n(f(n))^{1/\alpha 1}} = \begin{cases} 0, \\ +\infty, \end{cases} \Leftrightarrow \int_1^{+\infty} \frac{dx}{xf(x)} \begin{cases} < +\infty, \\ = +\infty. \end{cases}$$

COROLLARY 1.2. Let $\{a_n, n \geq 1\}$ as Theorem 1.2. Then for every $\delta > 0$, we have

$$\limsup_{n\to\infty} \frac{|T_n|}{B_n(\log n)^{(1+\delta)/\alpha_1}} = 0 \quad a.s.$$

and

$$\limsup_{n\to\infty} \frac{|T_n|}{B_n(\log n)^{1/\alpha_1}} = +\infty \quad a.s.$$

In particular

(i) If $\lim_{n\to\infty} \log(n/a_n)/\log\log n = +\infty$, then

$$\limsup_{n\to\infty} \left|\frac{T_n}{B_{a_n}}\right|^{1/\gamma_n} = e^{1/\alpha_2} \quad a.s. \tag{1.7}$$

(ii) If $\lim_{n\to\infty} \log(n/a_n)/\log\log n = 0$, then

$$\limsup_{n\to\infty} \left|\frac{T_n}{B_{a_n}}\right|^{1/\gamma_n} = e^{1/\alpha_1} \quad a.s. \tag{1.8}$$

(iii) If $\lim_{n\to\infty} \log(n/a_n)/\log\log n = s \in (0, +\infty)$, then

$$\limsup_{n\to\infty} \left|\frac{T_n}{B_{a_n}}\right|^{1/\gamma_n} = \exp\left\{\frac{\alpha_1 s + \alpha_2}{(s+1)\alpha_1\alpha_2}\right\} \quad a.s. \tag{1.9}$$

COROLLARY 1.4. Let $\{a_n, n \geq 1\}$ as Theorem 1.2. If $\alpha_1 = \alpha_2 = \alpha$, then

$$\limsup_{n\to\infty} \left|\frac{T_n}{B_{a_n}}\right|^{1/\gamma_n} = e^{1/\alpha} \quad a.s. \tag{1.10}$$

REMARK 1.2. Corollary 1.4 extends the result of Zinchenko (1994).

2 PROOFS OF THE MAIN RESULTS

We need the following lemmas.

LEMMA 2.1 (see Lemma 2.1 of Chen 2004). Let $f > 0$ be a non-decreasing function with

$$\int_1^\infty \frac{dx}{xf(x)} < +\infty,$$

then there exists a non-decreasing function $g > 0$ such that

$$g(x) \leq f(x), \quad \limsup_{x\to+\infty} g(2x)/g(x) < +\infty \quad \text{and} \quad \int_1^\infty \frac{dx}{xg(x)} < +\infty.$$

LEMMA 2.2 (see Lemma 2.2 of Chen 2002). Let $f > 0$ be a non-decreasing function satisfying

$$\int_1^\infty \frac{dx}{xf(x)} = +\infty.$$

Then there exists a non-decreasing function $h > 0$ such that

$$h(x) \to +\infty \text{ as } x \to +\infty \quad \text{and} \quad \int_1^\infty \frac{dx}{xf(x)h(x)} = +\infty.$$

In the rest of this paper, we denote C as a generic positive number which may be different at di places, and $a(n) \sim b(n)$ means $\lim_{n\to\infty} a(n)/b(n) = 1$. For the sake of simplicity, we denote r variable Y_1 with distribution function F_1 and random variable Y_2 with distribution function F_2.

PROOF OF THEOREM 1.1. Assume that $\int_1^\infty \frac{dx}{xf(x)} < \infty$. First of all, we show that

$$\frac{S_n}{B_n(f(n))^{1/\alpha_1}} \to 0 \ \text{ in probability.}$$

Note that by (1.1), $(\tau_1(n))^{-1/\alpha_1}(U_{\tau_1(n)} - b_{\tau_1(n)})$ has the same distribution as Y_1 and $(\tau_2(n))^{-1/\alpha_2}$ $- d_{\tau_2(n)})$ has the same distribution as Y_2, where $b_n = 0$ if $\alpha_1 \neq 1$ and $b_n = bn\log n$ for som $(-\infty, +\infty)$ if $\alpha_1 = 1$, and $d_n = 0$ if $\alpha_1 \neq 1$ and $d_n = dn\log n$ for some $d \in (-\infty, +\infty)$ if α $\int_1^\infty \frac{dx}{xf(x)} < \infty$ implies that $f(n) \to \infty$ and $\log n/f(n) \to 0$ as $n \to \infty$. Hence for all $\varepsilon > 0$

$$\begin{aligned} P\big(|U_{\tau_1(n)} - b_{\tau_1(n)}| \geq \varepsilon B_n(f(n))^{1/\alpha_1}\big) &= P\big(|Y_1| \geq \varepsilon B_n(f(n))^{1/\alpha_1}/(\tau_1(n))^{1/\alpha_1}\big) \\ &\sim Cn^{-\alpha_1/\alpha_2}(f(n))^{-1}\tau_1(n) \\ &\sim C(f(n))^{-1} \to 0, \quad n \to \infty \end{aligned}$$

and

$$\begin{aligned} P\big(|V_{\tau_2(n)} - d_{\tau_2(n)}| \geq \varepsilon B_n(f(n))^{1/\alpha_1}\big) &= P\big(|Y_2| \geq \varepsilon B_n(f(n))^{1/\alpha_1}/(\tau_2(n))^{1/\alpha_2}\big) \\ &\sim Cn^{-1}(f(n))^{-\alpha_2/\alpha_1}\tau_2(n) \\ &\sim C(f(n))^{-\alpha_2/\alpha_1} \to 0, \quad n \to \infty. \end{aligned}$$

Hence (2.1) holds. So by standard symmetric argument (see Lemma 3.2.1 of Stout 1974), we need o prove the result for $\{X_n, n \geq 1\}$ symmetric.

By Lemma 2.1 of Chen (2002),

$$\frac{U_{\tau_1(n)}}{(\tau_1(n)f(\tau_1(n)))^{1/\alpha_1}} \to 0 \ \ a.s. \qquad \text{and} \qquad \frac{V_{\tau_2(n)}}{(\tau_2(n)f(\tau_2(n)))^{1/\alpha_2}} \to 0 \ \ a.s.$$

Note that

$$\limsup_{n\to\infty} \frac{(\tau_1(n)f(\tau_1(n)))^{1/\alpha_1}}{B_n(f(n))^{1/\alpha_1}} < \infty \qquad \text{and} \qquad \limsup_{n\to\infty} \frac{(\tau_2(n)f(\tau_2(n)))^{1/\alpha_2}}{B_n(f(n))^{1/\alpha_1}} < \infty.$$

Hence

$$\begin{aligned} \limsup_{n\to\infty} \frac{|S_n|}{B_n(f(n))^{1/\alpha_1}} &\leq \limsup_{n\to\infty} \frac{|U_{\tau_1(n)}|}{B_n(f(n))^{1/\alpha_1}} + \limsup_{n\to\infty} \frac{|V_{\tau_2(n)}|}{B_n(f(n))^{1/\alpha_1}} \\ &\leq \limsup_{n\to\infty} \frac{(\tau_1(n)f(\tau_1(n)))^{1/\alpha_1}}{B_n(f(n))^{1/\alpha_1}} \times \frac{|U_{\tau_1(n)}|}{(\tau_1(n)f(\tau_1(n)))^{1/\alpha_1}} \end{aligned}$$

So we complete the proof of the convergent part.

Now we assume that $\int_1^\infty \frac{dx}{xf(x)} = +\infty$. If

$$\sum_{n=1}^{\infty} P\big(|X_n| \geq MB_n(f(n))^{1/\alpha_1}\big) = +\infty, \quad \forall M > 0 \tag{2.2}$$

holds, then by the Borel-Cantelli lemma, we have

$$\limsup_{n\to\infty} \frac{|X_n|}{B_n(f(n))^{1/\alpha_1}} = +\infty \ \ a.s.$$

and note that

$$\begin{aligned}
\limsup_{n\to\infty} \frac{|X_n|}{B_n(f(n))^{1/\alpha_1}} &\leq \limsup_{n\to\infty} \frac{|S_n|}{B_n(f(n))^{1/\alpha_1}} + \limsup_{n\to\infty} \frac{B_{n-1}(f(n-1))^{1/\alpha_1}}{B_n(f(n))^{1/\alpha_1}} \times \frac{|S_{n-1}|}{B_{n-1}(f(n-1))^{1/\alpha_1}} \\
&\leq 2 \limsup_{n\to\infty} \frac{|S_n|}{B_n(f(n))^{1/\alpha_1}},
\end{aligned}$$

hence we have

$$\limsup_{n\to\infty} \frac{|S_n|}{B_n(f(n))^{1/\alpha_1}} = +\infty \ \ a.s.$$

Now we prove (2.2). Note that

$$\begin{aligned}
\sum_{n=1}^{\infty} P\big(|X_n| \geq MB_n(f(n))^{1/\alpha_1}\big) &= \sum_{k=0}^{\infty} \sum_{n=2^k}^{2^{k+1}-1} P\big(|X_n| \geq MB_n(f(n))^{1/\alpha_1}\big) \\
&\geq \sum_{k=0}^{\infty} \sum_{n=2^k}^{2^{k+1}-1} P\big(|X_n| \geq MB_{2^{k+1}}(f(2^{k+1}))^{1/\alpha_1}\big) \\
&\geq \sum_{k=0}^{\infty} (\tau_1(2^{k+1}-1) - \tau_1(2^k-1)) P\big(|Y_1| \geq MB_{2^{k+1}}(f(2^{k+1}))^{1/\alpha_1}\big) \\
&\geq C\sum_{k=0}^{\infty} (\tau_1(2^{k+1}-1) - \tau_1(2^k-1))(2^{k+1})^{-\alpha_1/\alpha_2}(f(2^{k+1}))^{-1} \\
&\geq C\sum_{k=0}^{\infty} (f(2^{k+1}))^{-1}
\end{aligned}$$

and $\int_1^\infty \frac{dx}{xf(x)} = +\infty$ implies $\sum_{k=0}^\infty (f(2^{k+1}))^{-1} = +\infty$, so (2.2) holds.

Proof of Theorem 1.2. Assume that $\int_1^\infty \frac{dx}{xf(x)} < \infty$, by Lemma 2.1, without loss of generality, we

Note that $\limsup_{n\to\infty} \frac{B_{n+a_n}(f(n+a_n))^{1/\alpha_1}}{B_n(f(n))^{1/\alpha_1}} < \infty$, hence

$$\begin{aligned}\limsup_{n\to\infty} \frac{|T_n|}{B_n(f(n))^{1/\alpha_1}} &\le \limsup_{n\to\infty} \frac{|S_{n+a_n}|}{B_n(f(n))^{1/\alpha_1}} + \limsup_{n\to\infty} \frac{|S_n|}{B_n(f(n))^{1/\alpha_1}} \\ &= \limsup_{n\to\infty} \frac{B_{n+a_n}(f(n+a_n))^{1/\alpha_1}}{B_n(f(n))^{1/\alpha_1}} \times \frac{|S_{n+a_n}|}{B_{n+a_n}(f(n+a_n))^{1/\alpha_1}} \\ &= 0 \quad a.s.\end{aligned}$$

Now we assume that $\int_1^\infty \frac{dx}{xf(x)} = +\infty$. Suppose

$$\limsup_{n\to\infty} \frac{|T_n|}{B_n(f(n))^{1/\alpha_1}} = +\infty \quad a.s.$$

does not hold, then by Kolmogorov 0-1 law, there exists a constant $c_0 \in [0, \infty)$ such that

$$\limsup_{n\to\infty} \frac{|T_n|}{B_n(f(n))^{1/\alpha_1}} = c_0 \quad a.s.$$

Hence

$$\lim_{n\to\infty} \frac{T_n}{B_n(f(n)h(n))^{1/\alpha_1}} = 0 \quad a.s.$$

where $h(x)$ is given by Lemma 2.2. It is easy to show that

$$\frac{X_{n+1}}{B_n(f(n)h(n))^{1/\alpha_1}} \to 0 \text{ in probability,}$$

i.e.

$$\frac{T_n - X_{n+1}}{B_n(f(n)h(n))^{1/\alpha_1}} \to 0 \text{ in probability.}$$

By Lemma 2.3, we have

$$\frac{X_{n+1}}{B_n(f(n)h(n))^{1/\alpha_1}} \to 0 \quad a.s.$$

By the Borel-Cantelli lemma

$$\sum_{n=1}^{\infty} P\left(|X_n| \ge B_n(f(n)h(n))^{1/\alpha_1}\right) < \infty.$$

But by the same argument in the proof of Theorem 1.1, we have

$$\sum_{n=1}^{\infty} P\left(|X_n| \ge B_n(f(n)h(n))^{1/\alpha_1}\right) = \infty.$$

This leads to a contradiction, so we complete the proof.

Hence we have

$$P\big(|T_n| \geq B_n(\log n)^{(1+\delta)/\alpha_1},\ \text{i.o.}\big) = 0,\ \ \forall\delta > 0 \qquad \text{and} \qquad P\big(|T_n| \geq B_n(\log n)^{1/\alpha_1},\ \text{i.o.}\big) = 1,$$

where $P(A_n,\ \text{i.o.}) = P(\limsup_{n\to\infty} A_n)$ and A_n is a sequence of events. So we have

$$P\left(\log\left|\frac{T_n}{B_{a_n}}\right| \geq (1/\alpha_2)\log(n/a_n) + ((1+\delta)/\alpha_1)\log\log n,\ \ \text{i.o.}\right) = 0,\ \ \forall\delta > 0,$$

and

$$P\left(\log\left|\frac{T_n}{B_{a_n}}\right| \geq (1/\alpha_2)\log(n/a_n) + (1/\alpha_1)\log\log n,\ \text{i.o.}\right) = 1.$$

(i) If $\lim_{n\to\infty}\log(n/a_n)/\log\log n = \infty$, then

$$P\left(\log\left|\frac{T_n}{B_{a_n}}\right| \geq (1+\delta_1)\gamma_n/\alpha_2,\ \text{i.o.}\right) = 0,\ \ \forall\delta_1 > 0$$

and

$$P\left(\log\left|\frac{T_n}{B_{a_n}}\right| \geq (1-\delta_2)\gamma_n/\alpha_2,\ \ \text{i.o.}\right) = 1,\ \ \forall\delta_2 > 0,$$

hence we have

$$\limsup_{n\to\infty}\left|\frac{T_n}{B_{a_n}}\right|^{1/\gamma_n} = e^{1/\alpha_2}\ \ a.s.$$

(ii) If $\lim_{n\to\infty}\log(n/a_n)/\log\log n = 0$, then

$$P\left(\log\left|\frac{T_n}{B_{a_n}}\right| \geq (1+\delta_3)\gamma_n/\alpha_1,\ \ \text{i.o.}\right) = 0,\ \ \forall\delta_3 > 0$$

and

$$P\left(\log\left|\frac{T_n}{B_{a_n}}\right| \geq (1-\delta_4)\gamma_n/\alpha_1,\ \ \text{i.o.}\right) = 1,\ \ \forall\delta_4 > 0,$$

hence we have

$$\limsup_{n\to\infty}\left|\frac{T_n}{B_{a_n}}\right|^{1/\gamma_n} = e^{1/\alpha_1}\ \ a.s.$$

(iii) If $\lim_{n\to\infty}\log(n/a_n)/\log\log n = s \in (0,\infty)$, then

$$P\left(\log\left|\frac{T_n}{B_{a_n}}\right| \geq \left(\frac{\alpha_1 s + \alpha_2}{\alpha_1\alpha_2(s+1)} + \delta_5\right)\gamma_n,\ \ \text{i.o.}\right) = 0,\ \ \forall\delta_5 > 0$$

and

$$P\left(\log\left|\frac{T_n}{B_{a_n}}\right| \geq \left(\frac{\alpha_1 s + \alpha_2}{\alpha_1\alpha_2(s+1)} - \delta_6\right)\gamma_n,\ \text{i.o.}\right) = 1,\ \ \forall\delta_6 > 0,$$

ACKNOWLEDGMENTS

Research supported by the National Natural Science Foundation of China. The author is thankful referee for her/his helpful remarks which improved the presentation of the paper.

RESUMO

Apresentamos uma descrição precisa do comportamento limite de somas retardadas, e deduzimos leis do tipo de logaritmo iterado para as mesmas. Isso completa e estende os resultados de Vasudeva e Divanji (Th Probability and its Aplications, 37 (1992), 534–542).

Palavras-chave: distribuição estável, leis do logaritmo iterado, somas retardadas.

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