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Infinitesimal initial part of a singular foliation

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ABSTRACT

This work provides a necessary and sufficient condition to assure that two generalized curve singul
foliations have the same reduction of singularities and same Camacho-Sad indices at each infinitely ne
point.

Key words: singular holomorphic foliations, complex dynamics.

1 INTRODUCTION

The germs of holomorphic foliations over $(\mathbb{C}^2, 0)$ are dynamical objects much more general than
or even levels of meromorphic functions. In (Camacho et al. 1984), Camacho, Lins Neto and Sad
duce a class of foliations that share the reduction of singularities with their curve of separatrices. A
accurate approximation to a foliation (that assures coherent linear holonomies) is to compare ther
levels of multivalued functions (logarithmic foliations); this has been done in (Corral 2003) by ad
control of the Camacho-Sad indices. In this paper we show that the necessary and sufficient conditi
two foliations to share reduction of singularities, separatrices and Camacho-Sad indices is to have th
initial parts up to blow-up.

Let M be an analytic complex manifold of dimension two and consider two germs of foliati
and $\mathcal{G}$ defined in a neighbourhood of a point $p \in M$. Let $\omega^{\mathcal{F}}, \omega^{\mathcal{G}} \in \Omega^1_{M,p}$ be 1-forms defining
$\mathcal{G}$ respectively in a neighbourhood of p. It is clear that if the n-jets of $\omega^{\mathcal{F}}$ and $\omega^{\mathcal{G}}$ coincide for
enough, then the foliations $\mathcal{F}$ and $\mathcal{G}$ will share all the properties mentioned above. But this condi
not necessary as we show with examples.

2 LOCAL INVARIANTS

Let M be an analytic complex manifold of dimension two and let $\mathcal{F}$ be a singular holomorphic fo
on M. In a neighbourhood of any point $p \in M$, the foliation $\mathcal{F}$ is defined by a 1-form

with $A, B \in \mathcal{O}_{M,p}$ and $\gcd(A, B) = 1$. The multiplicity $\nu_p(\mathcal{F})$ of $\mathcal{F}$ at p is the minimum of the multiplicities $\nu_p(A)$, $\nu_p(B)$ (note that $\nu_p(\,)$ is the vanishing order at p). The point p is called a *singular point* of $\mathcal{F}$ if $\nu_p(\mathcal{F}) \geq 1$.

Let S be a germ of an irreducible analytic curve at p. We say that S is a *separatrix* of $\mathcal{F}$ through p if f divides $\omega \wedge df$, where $f = 0$ is a reduced equation of S. Denote by $\mathrm{Sep}_p(\mathcal{F})$ the set of separatrices of $\mathcal{F}$ through p.

Let $\pi_1 : M_1 \to (M, p)$ be the blow-up of M with center at p and let E_1 be the exceptional divisor $\pi_1^{-1}(p)$. The blow-up π_1 is called *non-dicritical* if E_1 is invariant by the strict transform $\pi_1^*\mathcal{F}$ of $\mathcal{F}$ by π_1; otherwise, E_1 is generically transversal to $\pi_1^*\mathcal{F}$ and we say that π_1 is *dicritical*.

Consider now a non-singular invariant curve $S = (y = 0)$ of the foliation $\mathcal{F}$ through p. The *multiplicity of $\mathcal{F}$ at p along S* is given by

$$\mu_p(\mathcal{F}, S) = \mathrm{ord}_x(B(x, 0))$$

(see (Camacho et al. 1984)). Notice that it coincides with the multiplicity of the restriction $\xi|_S = B(x, 0)\partial/\partial x$ of the vector field $\xi = B(x, y)\partial/\partial x - A(x, y)\partial/\partial y$ to the curve S. Its behaviour under blow-up is given by

$$\mu_{p_1}(\pi_1^*\mathcal{F}, S_1) = \mu_p(\mathcal{F}, S) - \nu_p(\mathcal{F}) + 1 \qquad \text{if } \pi_1 \text{ is non-dicritical,} \tag{1}$$

$$\mu_{p_1}(\pi_1^*\mathcal{F}, S_1) = \mu_p(\mathcal{F}, S) - \nu_p(\mathcal{F}) \qquad \text{if } \pi_1 \text{ is dicritical,} \tag{2}$$

where S_1 is the strict transform of S by π_1 and $p_1 = S_1 \cap E_1$. Moreover, we have that

$$\sum_{q \in E_1} \mu_q(\pi_1^*\mathcal{F}, E_1) = \nu_p(\mathcal{F}) + 1.$$

The *Camacho-Sad index* $I_p(\mathcal{F}, S)$ of $\mathcal{F}$ relative to S at p is given by

$$I_p(\mathcal{F}, S) = -\mathrm{Res}_{x=0}\frac{a(x, 0)}{B(x, 0)}$$

where $A(x, y) = ya(x, y)$ (see (Camacho and Sad 1982)).

It is important to notice that, if $\omega, \omega' \in \Omega^1_{M,p}$ are 1-forms defining $\mathcal{F}$ in a neighbourhood of p, then $\omega = h \cdot \omega'$ where $h \in \mathcal{O}_{M,p}$ is a unit. In particular, this implies that the jets of order $\nu_p = \nu_p(\mathcal{F})$ of ω and ω' coincide up to multiplication by a constant. We can write $\omega = \sum_{j \geq \nu_p} \omega_j$, where the coefficients of ω_j are homogeneous polynomials of degree j. Then the projective class $[\omega_{\nu_p}]$ of the ν_p-jet of ω is well defined. We call $J_p(\mathcal{F}) = [\omega_{\nu_p}]$ the *initial part of* $\mathcal{F}$ at p.

Let us write $\omega_j = A_j dx + B_j dy$ with $A_j, B_j \in \mathcal{O}_{M,p}$ and consider the homogeneous polynomial of degree $\nu_p + 1$ given by $P_{\nu_p+1} = xA_{\nu_p} + yB_{\nu_p}$. The *tangent cone* $TC_p(\mathcal{F})$ of $\mathcal{F}$ at p is given by

$$P_{\nu_p+1}(x, y) = 0.$$

Let us now recall the construction of the Newton polygon of a foliation. Taking local coord (x, y) in a neighbourhood of $p \in M$, a germ $f \in \mathcal{O}_{M,p}$ of a function at p can be written as a conv power series $f = \sum_{i+j \geq \nu_p(f)} f_{ij} x^i y^j$. We denote $\Delta(f; x, y) = \{(i, j) \;:\; f_{ij} \neq 0\}$. The *Newton p* $\mathcal{N}(f; x, y)$ is the convex hull of $\Delta(f; x, y) + (\mathbb{R}_{\geq 0})^2$. If C is the germ of curve at p defined by then $\mathcal{N}(C; x, y) = \mathcal{N}(f; x, y)$. Consider now a germ of foliation $\mathcal{F}$ given by $\omega = A(x, y)dx + B(x$ in a neighbourhood of p. The *Newton polygon* $\mathcal{N}(\mathcal{F}; x, y)$ of $\mathcal{F}$ is the convex hull of $\Delta(\omega; x, y) +$ where $\Delta(\omega; x, y) = \Delta(xA; x, y) \cup \Delta(yB; x, y)$.

Finally, let us recall the desingularization process of a foliation. We say that p is a *simple sing* of $\mathcal{F}$ if there are coordinates (x, y) centered at p so that $\mathcal{F}$ is defined by a 1-form of the type

$$\lambda y dx - \mu x dy + h.o.t.$$

with $\mu \neq 0$ and $\lambda/\mu \notin \mathbb{Q}_{>0}$. If $\lambda = 0$, the singularity is called a *saddle-node*. A *reduction of si rities* of $\mathcal{F}$ is a morphism

$$\pi : M' \to (M, p)$$

composition of a finite number of blow-ups of points such that the strict transform $\pi^*\mathcal{F}$ of $\mathcal{F}$ by isfies that

- each irreducible component of the exceptional divisor $D = \pi^{-1}(p)$ is either invariant by π transversal to $\pi^*\mathcal{F}$;
- all the singular points of $\pi^*\mathcal{F}$ are simple and do not belong to a dicritical component of the tional divisor D.

The minimal morphism $\pi_{\mathcal{F}}$, in the sense that it cannot be factorized by another morphism with the properties, is called the *minimal reduction of singularities* of $\mathcal{F}$. The centers of the blow-ups reduction of singularities of $\mathcal{F}$ are called *infinitely near points* of $\mathcal{F}$. In particular, all the centers blow-ups to obtain $\pi_{\mathcal{F}}$ and the final singularities in $\pi_{\mathcal{F}}^{-1}(p)$ are infinitely near points of $\mathcal{F}$. This extends the well known notion of infinitely near points of a curve.

3 GENERALIZED CURVE FOLIATIONS

Let $\mathbb{F} = \mathbb{F}(M, p)$ be the space of singular holomorphic foliations of (M, p). We denote by $\mathbb{G}$ the su of $\mathbb{F}$ composed by *generalized curve foliations*, that is, foliations without saddle-node singularities i reduction of singularities (see (Camacho et al. 1984)). Given a germ $C = \cup_{i=1}^{r} C_i$ of analytic cu (M, p), we denote by $\mathbb{F}_C$ the subspace of $\mathbb{F}$ composed by foliations whose curve of separatrices is we put $\mathbb{G}_C = \mathbb{G} \cap \mathbb{F}_C$. Notice that foliations in $\mathbb{F}_C$ are non-dicritical. Particular elements of $\mathbb{G}_C$ foliations given by $df = 0$, where $f = 0$ is a reduced equation of C; let $\mathcal{G}_C$ be one of such foli It is known (see (Camacho et al. 1984, Rouillé 1999)) that, for an element $\mathcal{F} \in \mathbb{G}_C$, we have that:

(iii) For any local coordinates (x, y) in a neighbourhood of p, the Newton polygons $\mathcal{N}(\mathcal{F}; x, y)$ and $\mathcal{N}(C; x, y)$ coincide.

Moreover, we have the following result

LEMMA 1. *Let $\pi : (\widetilde{M}, \tilde{p}) \to (M, p)$ be a morphism composition of a finite number of punctual blow-ups. Let $D = \pi^{-1}(p)$ be the exceptional divisor and take $E \subset D$ an irreducible component with $\tilde{p} \in E$. Then, for any foliation $\mathcal{F} \in \mathbb{G}_C$, the strict transforms $\pi^*\mathcal{F}$ and $\pi^*\mathcal{G}_C$ satisfy the following*

1. $\nu_{\tilde{p}}(\pi^*\mathcal{F}) = \nu_{\tilde{p}}(\pi^*\mathcal{G}_C)$;
2. *If $(\tilde{x}, \tilde{y})$ are local coordinates at $\tilde{p}$, then* $\mathcal{N}(\pi^*\mathcal{F}; \tilde{x}, \tilde{y}) = \mathcal{N}(\pi^*\mathcal{G}_C; \tilde{x}, \tilde{y})$;
3. $\mu_{\tilde{p}}(\pi^*\mathcal{F}, E) = \mu_{\tilde{p}}(\pi^*\mathcal{G}_C, E)$.

REMARK 1. If $\pi_1 : M_1 \to (M, p)$ is the blow-up of p with $E_1 = \pi_1^{-1}(p)$ and π_1^*C denotes the strict transform of C by π_1, then $\mu_q(\pi^*\mathcal{G}_C, E) = (\pi^*C \cdot E)_q$ for any $q \in E_1$, where $(\,\cdot\,)_q$ denotes the intersection multiplicity at q.

PROOF OF LEMMA 1. Assertions 1 and 2 are a direct consequence of the above properties (ii) and (iii) applied at each infinitely near point of $\mathcal{F}$. Let us prove assertion 3.

Let (x, y) be coordinates in a neighbourhood of p and take a 1-form $\omega = A(x, y)dx + B(x, y)dy$ defining $\mathcal{F}$ in a neighbourhood of p. Consider a reduced equation $f = 0$ of C and let $\mathcal{G}_C$ be the foliation defined by $df = 0$. If we denote $\nu = \nu_0(\mathcal{F})$, then the multiplicity $\nu_p(C)$ of C at p is equal to $\nu + 1$. Therefore, we can write $f = \sum_{i \geq \nu+1} f_i$, $A = \sum_{i \geq \nu} A_i$ and $B = \sum_{i \geq \nu} B_i$ with f_i, A_i and B_i being homogeneous polynomials of degree i.

Consider $\pi_1 : M_1 \to (M, p)$ the blow-up of p and denote by $E_1 = \pi_1^{-1}(p)$ the exceptional divisor. Let us prove that

$$\mu_q\big(\pi_1^*\mathcal{F}, E_1\big) = \mu_q\big(\pi_1^*\mathcal{G}_C, E_1\big) \tag{3}$$

at each point $q \in E_1$. Then the result follows using similar arguments.

If $q \in E_1$ is a non-singular point of $\pi_1^*\mathcal{F}$, then $\mu_q(\pi_1^*\mathcal{F}, E_1) = 0$. Since the singular points of $\pi_1^*\mathcal{F}$ and $\pi_1^*\mathcal{G}_C$ in E_1 coincide, we also have that $\mu_q(\pi_1^*\mathcal{G}_C, E_1) = 0$. Consequently, we only have to prove equality (3) at the singular points $q_1, q_2, \ldots, q_t$ of $\pi_1^*\mathcal{F}$ in E_1. We can assume that all the singular points belong to the first chart of E_1 and we take coordinates (x', y') with $E_1 = (x' = 0)$ and $\pi_1(x', y') = (x', x'y')$. In these coordinates, the foliation $\pi_1^*\mathcal{G}_C$ is given by

$$\big((\nu+1) f_{\nu+1}(1, y') + x'(\ldots)\big)dx' + x'\left(\frac{\partial f_{\nu+1}}{\partial y'}(1, y') + x'(\ldots)\right)dy' = 0. \tag{4}$$

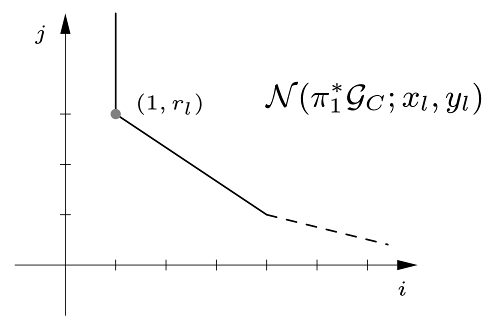
Let us write $f_{\nu+1}(1, y) = k \cdot \prod_{l=1}^{t} (y - a_l)^{r_l}$ with $k \in \mathbb{C}^*$. We can assume that the point q_l is given by $(0, a_l)$ in the coordinates (x', y'), for $l = 1, \ldots, t$. In particular, we have that $\mu_{q_l}(\pi^*\mathcal{G}_C, E_1) = r_l$

By definition, we have that $\mu_{q_l}(\pi_1^*\mathcal{F}, E_1) = \mathrm{ord}_{y=a_l}(P_{\nu+1}(1, y))$.

Take (x_l, y_l) coordinates centered at q_l with $x_l = x'$ and $y_l = y' - a_l$. From expression (
deduce that

$$\mathcal{N}(\pi_1^*\mathcal{G}_C; x_l, y_l) \subset \{(i, j) \,:\, i \geq 1\}$$

and $\mathcal{N}(\pi_1^*\mathcal{G}_C; x_l, y_l) \cap \{i = 1\} = \{(1, j) \,:\, j \geq r_l\}$.



Then the equality of the Newton polygons $\mathcal{N}(\pi_1^*\mathcal{G}_C; x_l, y_l)$ and $\mathcal{N}(\pi_1^*\mathcal{F}; x_l, y_l)$ implies th
point $(1, r_l)$ belongs to $\Delta(\omega_1; x_l, y_l)$. Taking into account that

$$\Delta\big(\omega_1; x_l, y_l\big) \cap \{i = 1\} = \Delta\big(x_l P_{\nu+1}(1, y_l + a_l); x_l, y_l\big) \cup \Delta\big(x_l y_l B_\nu(1, y_l + a_l); x_l, y_l\big)$$

we deduce that

$$\mathrm{ord}_{y=a_l}\big(P_{\nu+1}(1, y)\big) \geq r_l; \qquad \mathrm{ord}_{y=a_l}\big(B_\nu(1, y)\big) \geq r_l - 1.$$

Thus $\mu_{q_l}(\pi_1^*\mathcal{F}, E_1) \geq \mu_{q_l}(\pi_1^*\mathcal{G}_C, E_1)$ for each $l = 1, \ldots, t$, and the following equality

$$\nu + 1 = \sum_{l=1}^{t} \mu_{q_l}\big(\pi_1^*\mathcal{F}, E_1\big) = \sum_{l=1}^{t} \mu_{q_l}\big(\pi_1^*\mathcal{G}_C, E_1\big).$$

gives the result.

COROLLARY 1. *Given two foliations $\mathcal{F}$ and $\mathcal{G}$ of $\mathbb{G}_C$, the polynomials $\mathfrak{p}_p(\mathcal{F})$ and $\mathfrak{p}_p(\mathcal{G})$ coinc
each infinitely near point p of C.*

4 LOGARITHMIC FOLIATIONS

Given a germ of plane curve $C = \cup_{i=1}^{r} C_i$ in (M, p), a logarithmic foliation $\mathcal{L}$ in $\mathbb{G}_C$ is given by a
meromorphic 1-form $\eta = 0$ where

$$\eta = \sum_{i=1}^{r} \lambda_i \frac{df_i}{f_i},$$

with $f_i = 0$ being a reduced equation of C_i and $\lambda_i \neq 0$. Notice that if η as above defines a logar

We denote by $\mathcal{L}_{\underline{\lambda}}$ one of these foliations. Observe that not all the foliations defined by a 1-form of the type $(*)$ belong to $\mathbb{G}_C$ since they could be dicritical foliations.

Recall that a logarithmic foliation $\mathcal{L}$ is a *logarithmic model* of a foliation $\mathcal{F}$ if they have the same separatrices, the same reduction of singularities and the same Camacho-Sad indices at the final singularities after desingularization. The existence of logarithmic models for non-dicritical generalized curves was proved in (Corral 2003) and is unique once a reduced equation of the separatrices is fixed. Then, for each foliation $\mathcal{F} \in \mathbb{G}_C$, we denote by $\underline{\lambda}(\mathcal{F})$ the exponent vector of the logarithmic model of $\mathcal{F}$. We denote by $\mathbb{G}_{C,\underline{\lambda}}$ the set of foliations $\mathcal{F} \in \mathbb{G}_C$ such that $\underline{\lambda}(\mathcal{F}) = \underline{\lambda}$. Thus, the set $\mathbb{G}_C$ is equal to

$$\mathbb{G}_C = \cup_{\underline{\lambda}\in\mathbb{P}^{r-1}_{\mathbb{C}}} \mathbb{G}_{C,\underline{\lambda}}.$$

Notice that $\mathbb{G}_{C,\underline{\lambda}} \neq \emptyset$ if and only if the foliation given by $\sum_{i=1}^r \lambda_i df_i/f_i = 0$ is non-dicritical. The goal of this article is to study the properties which characterize the foliations in one of such sets $\mathbb{G}_{C,\underline{\lambda}}$.

Consider now a meromorphic 1-form η of the type $(*)$ and assume that the foliation given by $\eta = 0$ belongs to $\mathbb{G}_C$. Thus all the foliations defined by $\eta + \alpha = 0$, with α holomorphic, belong to $\mathbb{G}_{C,\underline{\lambda}}$ but in general they are not logarithmic. However, it is not true that all foliations of $\mathbb{G}_{C,\underline{\lambda}}$ are defined by a 1-form of the type $\eta + \alpha = 0$ with α holomorphic.

5 INFINITESIMAL INITIAL PART

Given a non-dicritical foliation $\mathcal{F} \in \mathbb{F}$, we define the *infinitesimal initial part* $\mathcal{J}(\mathcal{F})$ of $\mathcal{F}$ to be the family $\{J_q(\mathcal{F})\}$ where q varies among the infinitely near points of $\mathcal{F}$. We wonder under what conditions two foliations $\mathcal{F}$ and $\mathcal{G}$ have the same infinitesimal initial part. It is clear that having the same curve of separatrices and the same reduction of singularities are necessary conditions. But these conditions are not enough even if we work with generalized curve foliations. Notice also that a sufficient condition is that of $\mathcal{F}$ and $\mathcal{G}$ having the same n-jet at the point p, with n being big enough. However, this condition is not necessary as shown by $d\big(y^3 - x^{11}\big) = 0$ and $11\big(-x^{10} + yx^7\big)dx + 3\big(y^2 - x^8\big)dy = 0$. Then, the result is

THEOREM 1. *Let $\mathcal{F}$ and $\mathcal{G}$ be two foliations in $\mathbb{G}_C$. The foliations $\mathcal{F}$ and $\mathcal{G}$ have the same infinitesimal initial part if and only if $\underline{\lambda}(\mathcal{F}) = \underline{\lambda}(\mathcal{G})$.*

PROOF. Take $\mathcal{F}$ and $\mathcal{G}$ two foliations in $\mathbb{G}_C$ and assume that $\mathcal{J}(\mathcal{F}) = \mathcal{J}(\mathcal{G})$. Consider the minimal reduction of singularities $\pi_C : \widetilde{M} \to (M, p)$ of C (notice that $\pi_C = \pi_{\mathcal{F}} = \pi_{\mathcal{G}}$). By hypothesis, the initial parts $J_q(\mathcal{F})$ and $J_q(\mathcal{G})$ coincide at each point $q \in \pi_C^{-1}(p)$. In particular, this implies that the Camacho-Sad indices $I_q(\pi^*\mathcal{F}, F)$ and $I_q(\pi^*\mathcal{G}, F)$ coincide for a component $F \subset \pi_C^{-1}(p)$ through q and consequently $\underline{\lambda}(\mathcal{F}) = \underline{\lambda}(\mathcal{G})$.

Reciprocally, take a foliation $\mathcal{F} \in \mathbb{G}_{C,\underline{\lambda}}$ and let us show that $J_p(\mathcal{F}) = J_p(\mathcal{L})$, where $\mathcal{L} = \mathcal{L}_{\underline{\lambda}}$ is a logarithmic foliation in $\mathbb{G}_{C,\underline{\lambda}}$. Put $\nu = \nu_p(\mathcal{F})$ and assume that $\mathcal{F}$ and $\mathcal{L}$ are given by the holomorphic 1-forms $\omega^{\mathcal{F}} = 0$ and $\omega^{\mathcal{L}} = 0$ respectively, where with A_i^-, B_i^- being homogeneous polynomials of degree i. Consider the blow-up $\pi_1 \colon M_1 \to (M$ p and let $q_1, q_2, \ldots, q_t$ be the singular points of $\pi_1^*\mathcal{F}$ in $E_1 = \pi_1^{-1}(p)$. We can assume, without generality, that all the points q_i belong to the first chart of E_1. Take (x', y') coordinates in the firs of E_1 with $\pi_1(x', y') = (x', x'y')$ and $E_1 = (x' = 0)$. Then, the foliations $\pi_1^*\mathcal{F}$ and $\pi_1^*\mathcal{L}$ are gi $\omega_1^{\mathcal{F}} = 0$ and $\omega_1^{\mathcal{L}} = 0$ respectively, where

$$\begin{aligned}
\omega_1^{\mathcal{F}} &= \big(P_{\nu+1}^{\mathcal{F}}(1, y') + x'(\cdots)\big)dx' + x'(B_\nu^{\mathcal{F}}(1, y') + x'(\cdots))dy';\\
\omega_1^{\mathcal{L}} &= \big(P_{\nu+1}^{\mathcal{L}}(1, y') + x'(\cdots)\big)dx' + x'(B_\nu^{\mathcal{L}}(1, y') + x'(\cdots))dy'.
\end{aligned}$$

Assume that the point q_i is given by $(0, a_i)$ in the coordinates (x', y') and put $r_i = \mu_{q_i}(\pi_1^*\mathcal{F}, E_1)$. by Lemma 1 and equation (5), we have that

$$P_{\nu+1}^{\mathcal{F}}(1, y) = k_1 \prod_{i=1}^{t} (y - a_i)^{r_i}; \quad P_{\nu+1}^{\mathcal{L}}(1, y) = k_2 \prod_{i=1}^{t} (y - a_i)^{r_i},$$

where k_1 and k_2 are non-zero constants. The second inequality of (5) implies that

$$B_\nu^{\mathcal{F}}(1, y) = H^{\mathcal{F}}(y) \cdot \prod_{i=1}^{t} (y - a_i)^{r_i - 1}; \quad B_\nu^{\mathcal{L}}(1, y) = H^{\mathcal{L}}(y) \cdot \prod_{i=1}^{t} (y - a_i)^{r_i - 1},$$

with $H^{\mathcal{F}}(y)$ and $H^{\mathcal{L}}(y)$ being polynomials of degree $t - 1$. Now, since $\mathcal{L}$ is a logarithmic model we have the equality of the Camacho-Sad indices

$$I_{q_i}\big(\pi_1^*\mathcal{F}, E_1\big) = I_{q_i}\big(\pi_1^*\mathcal{L}, E_1\big) \quad \text{for } i = 1, \ldots, t.$$

Computation of the indices gives that

$$\begin{aligned}
I_{q_i}(\pi_1^*\mathcal{F}, E_1) &= -\mathrm{Res}_{y=a_i} \frac{B_{\nu+1}^{\mathcal{F}}(1, y)}{P_{\nu+1}^{\mathcal{F}}(1, y)} = -\frac{H^{\mathcal{F}}(a_i)}{k_1 \cdot \prod_{\substack{j=1\\ j\neq i}}^{t} (a_i - a_j)}\\
I_{q_i}(\pi_1^*\mathcal{L}, E_1) &= -\mathrm{Res}_{y=a_i} \frac{B_{\nu+1}^{\mathcal{L}}(1, y)}{P_{\nu+1}^{\mathcal{L}}(1, y)} = -\frac{H^{\mathcal{L}}(a_i)}{k_2 \cdot \prod_{\substack{j=1\\ j\neq i}}^{t} (a_i - a_j)}
\end{aligned}$$

The equalities of the Camacho-Sad indices imply that $H^{\mathcal{F}}(y) = kH^{\mathcal{L}}(y)$ with $k = k_1/k_2$. The have that $B_\nu^{\mathcal{F}}(1, y) = kB_\nu^{\mathcal{L}}(1, y)$. Moreover, since $P_{\nu+1}^{\mathcal{F}}(1, y) = kP_{\nu+1}^{\mathcal{L}}(1, y)$, we get that $A_\nu^{\mathcal{F}}(1$ $kA_\nu^{\mathcal{L}}(1, y)$. It follows that $\mathcal{F}$ and $\mathcal{L}$ have the same initial parts $J_p(\mathcal{F})$ and $J_p(\mathcal{L})$ at p.

The same arguments prove the equality $J_q(\mathcal{F}) = J_q(\mathcal{G})$ at each infinitely near point q of C, a result is straightforward.

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RESUMO

Este trabalho fornece uma condição necessária e suficiente a fim de que duas folheações singulares curva generalizada admitam mesma redução de singularidades e mesmo índice de Camacho-Sad em cada ponto infinitamente vizinho.

Palavras-chave: folheações holomorfas singulares, dinâmica complexa.

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