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On persistently positively expansive maps

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ABSTRACT

In this paper, we prove that any C^1-persistently positively expansive map is expanding. This improves result due to Sakai (Sakai 2004).

Key words: positively expansive maps, expanding maps.

INTRODUCTION

Let M be a compact Riemannian manifold. We say that a continuous map $f : M \to M$ is c-pos expansive for some $c > 0$ if, for any $x \neq y$, there exists $n \geq 0$, such that $d(f^n(x), f^n(y)) \geq c$. Thes were studied by many authors, see for instance (Coven and Reddy 1980), (Ruelle 1978), (Sakai 198 (Sakai 2003). An important class of positively expansive maps are the expanding ones defined as fo we say that a map is *expanding* if there are constants $C > 0$ and $\sigma > 1$, such that $\|Df^n(x)v\| \geq C$ for every $x \in M$.

Definition 1.1. *We say that a C^1 map $f : M \to M$ is C^1-persistently positively expansive there exists a C^1-neighborhood $\mathcal{U}$, such that for any $g \in \mathcal{U}$, there exists $c(g) > 0$, such that g is positively expansive.*

In this note, we shall prove the following result:

Theorem 1.2. *Let M be a compact manifold. Any C^1-persistently positively expansive map $f : M$ is expanding.*

The same type of result appears in (Sakai 2004). However, Sakai assumes that the separation pr in the definition of positive expansivity holds for any composition of maps in the neighborhood $\mathcal{U}$, his paper, $c(g)$ is also a constant in the neighborhood. Since we are not assuming these hypothes result is slightly stronger. Also, our arguments rely on a theorem by Cao (Cao 2003), which simplif

In Sakai's paper, the following definition is introduced: a map f is a C^1-stably positively expansive map with constants $\nu > 0$ and $c > 0$ if, for any sequence, $\{g_i\}_{i=1}^{\infty} \subset C^1(M)$, with $d_{C^1}(f, g_i) \leq \nu$ for all $i \in \mathbb{N}$ and for any sequences $\{x_i, y_i\}_{i=0}^{\infty}$, such that $x_i = g_i \circ \cdots \circ g_1(x_0)$, $y_i = g_i \circ \cdots \circ g_1(y_0)$ and $d(x_i, y_i) \leq c$; for all $i \geq 0$, we have that $x_0 = y_0$. Then, he proves that any C^1-stably positively expansive map is expanding.

PROOF OF THE THEOREM

First, we recall a lemma due to Franks (Franks 1971):

LEMMA 2.1. *Let f be a C^1 map and $\mathcal{U}$ a C^1-neighborhood of f. Then, there exists a neighborhood $\mathcal{U}_0 \subset \mathcal{U}$ of f and a $\delta > 0$ such that, if $g \in \mathcal{U}_0(f)$, $S = \{p_1, \ldots, p_m\} \subset M$ is a finite set, and $\{L_i : T_{p_i}M \to T_{g(p_i)}M\}_{i=1}^{m}$ are linear maps satisfying $\|L_i - Dg(p_i)\| \leq \delta$ for $i = 1, \ldots m$; then, there exists $h \in \mathcal{U}_0(f)$ satisfying $h(p_i) = g(p_i)$ and $Dh(p_i) = L_i$. Moreover, if U is a neighborhood of S, then, h can be taken such that $h(x) = g(x)$ for every $x \in S \cup (M - U)$, and locally*

$$h(x) = \exp_{g(p_i)} \circ Dg(p_i) \circ \exp_{p_i}^{-1}(x), \ \ for \ x \in B_{\varepsilon}(p_i),$$

where exp *is the exponential map of the Riemannian manifold M, and ε is sufficiently small.*

Now we fix f a C^1-persistently positively expansive map.

The following corollary is an adaptation of an argument of (Sakai 2004):

COROLLARY 2.2. *Let $\mathcal{U}$ be the neighborhood of f in the definition of C^1-persistently positively expansive map. For any $g \in \mathcal{U}$, and p as a periodic point for g with period n, if λ is an eigenvalue of $Dg^n(p)$, then $|\lambda| > 1$.*

PROOF. If not, there exists g with p as a periodic point with a contracting eigenvalue, i.e., an eigenvalue λ with $|\lambda| \leq 1$. Using Franks' lemma, we can find $h \in \mathcal{U}$, such that:

$$h(x) = \exp_{g^i(p)} \circ Dg(g^{i-1}(p)) \circ \exp_{g^{i-1}(p)}^{-1}(x), \ \text{for } x \in B_{\varepsilon}(g^{i-1}(p)),$$

where ε is small. Now, since we linearized near the periodic orbit, we obtain two distinct, but close enough, orbits z and w inside the stable local manifold of this linear map, such that $d(h^n(x), h^n(y)) \leq c(h)$ for $n \geq 0$, which is a contradiction. □

The same argument shows that any $g \in \mathcal{U}$ is a local diffeomorphism. Since there exists $p \in M$ and a zero eigenvalue of $Dg(p)$, by Franks' lemma, we make a perturbation h with a set of points S (in the direction of the eigenvalue) such that, for every $z \in S$, we have $h(z) = h(p)$, and this contradicts the positive expansivity of h.

A δ-pseudo orbit $\{x_i\}$ for f is a sequence of points in M such that, for every i, we have $d(f(x_i), x_{i+1}) <$

Since f is a local diffeomorphism, in particular it is open. Now, (Ruelle 1978) and (Sakai 2003) that any positively expansive map that is open has the shadowing property. Also, since the map is pos expansive, the shadow is unique. In particular, if the pseudo-orbit is periodic, the shadow is a pe point.

We recall that, for any $x \in M$ and $v \in T_x M$, the latter being a nonzero vector, the Lyapunov exp associated is:

$$\lambda(x, v) = \lim_{k \to \infty} \frac{1}{k} \log \| Df^k(x) v \|$$

whenever this limit exists. By Oseledets' theorem (Oseledets 1968), this limit exists for μ-almost point x and any nonzero vector $v \in T_x M$, if μ is a finite invariant measure.

LEMMA 2.3. *Let μ be a finite invariant measure of f. Then, for μ-almost every x, the Lyapunov nents $\lambda_i(f, x)$ are positive.*

PROOF. Fix $\delta > 0$ as the constant given by Franks' lemma. Let $\nu > 0$, such that, if $d(x, y) <$ $d(Df(x), Df(y)) < \delta$. Finally, fix $\varepsilon > 0$ as the constant given by the shadowing property associ ν-pseudo orbits.

If the lemma is false, there exists a measure μ with a non-positive Lyapunov exponent. By Po recurrence, there exists $x \in M$, $v \in T_x M$, $0 < \eta \ll \delta^1$ and N large enough, such that:

$$\| Df^N(x) v \| \leq (1 + \eta)^N \| v \|, \quad d(f^N(x), x) < \nu,$$

and there exists i much smaller than N, such that $d(f^i(x), x) < \nu$ and the directions

$$\frac{Df^i(x) v}{\| Df^i(x) v \|} \quad \text{and} \quad \frac{Df^N(x) v}{\| Df^N(x) v \|}$$

are close enough.

The shadowing lemma implies that there exists a periodic orbit $f^{N-i}(p) = p$, such that $d($ $f^j(p)) < \nu$ for $j = 0, \ldots N - i - 1$; moreover, there exists some vector $w \in T_p M - \{0\}$ cl $Df^{N-i}(p) w$. Since η is small, we can use Frank's lemma to perturb f, replacing the derivative in th of p by the derivative of the segment of orbit of x, dividing by $1 + 2\eta$ on each iterate, and compo last derivative (i.e., the last iterate of the periodic orbit) with a small rotation. It was obtained $g \in$ a periodic point p of period $N - i$ with a contractive eigen- vector w for the derivative $Df^{N-i}(p)$ contradicts the previous lemma.

Now we invoke a theorem due to Cao (Cao 2003):

THEOREM 2.4 (Cao). *Let $f : M \to M$ be a C^1-local diffeomorphism. If the Lyapunov exponents o f-invariant probability measure are positive, then f is uniformly expanding.*

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RESUMO

Neste artigo, mostramos que todo mapa C^1-persistentemente positivamente expansivo é expansor. Isto melhora um resultado devido a Sakai (Sakai 2004).

Palavras-chave: mapas positivamente expansivos, mapas expansores.

REFERENCES

CAO Y. 2003. Non-zero Lyapunov exponents and uniform hyperbolicity. Nonlinearity 16: 1473–1479.

COVEN E AND REDDY W. 1980. Positively Expansive maps of compact manifolds. Lecture Notes in Math 819: 96–110.

FRANKS J. 1971. Necessary conditions for stability of diffeomorphisms. Trans Amer Math Soc 158: 301–308.

RUELLE D. 1978. Thermodynamical Formalism. Encyclopedia Math Appl n. 5.

SAKAI K. 1985. Periodic Points of Positively Expansive maps. Proc Amer Math Soc 94: 531–534.

SAKAI K. 2003. Various shadowing properties for positively expansive maps. Topology Appl 131: 15–31.

SAKAI K. 2004. C^1-Stably Positively Expansive Maps. Bull of the Polish Acad Sci Math 52: 197–209.

OSELEDETS V. 1968. A multiplicative ergodic theorem: Lyapunov characteristic numbers for dynamical systems. Trans Moscow Math Soc 19: 197–231.