



Anais da Academia Brasileira de Ciências

ISSN: 0001-3765

aabc@abc.org.br

Academia Brasileira de Ciências

Brasil

Arbieto, Alexander

On persistently positively expansive maps

Anais da Academia Brasileira de Ciências, vol. 82, núm. 2, junio, 2010, pp. 263-266

Academia Brasileira de Ciências

Rio de Janeiro, Brasil

Available in: <http://www.redalyc.org/articulo.oa?id=32713482002>

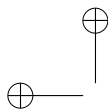
- How to cite
- Complete issue
- More information about this article
- Journal's homepage in redalyc.org

redalyc.org

Scientific Information System

Network of Scientific Journals from Latin America, the Caribbean, Spain and Portugal

Non-profit academic project, developed under the open access initiative



Anais da Academia Brasileira de Ciências (2010) 82(2): 263-266
(Annals of the Brazilian Academy of Sciences)
ISSN 0001-3765
www.scielo.br/aabc

On persistently positively expansive maps

ALEXANDER ARBIETO

Instituto de Matemática, Universidade Federal do Rio de Janeiro, Caixa Postal 68530, 21945-970 Rio de Janeiro, RJ, Brazil

Manuscript received on May 19, 2009; accepted for publication on September 30, 2009

ABSTRACT

In this paper, we prove that any C^1 -persistently positively expansive map is expanding. This improves a result due to Sakai (Sakai 2004).

Key words: positively expansive maps, expanding maps.

INTRODUCTION

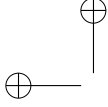
Let M be a compact Riemannian manifold. We say that a continuous map $f : M \rightarrow M$ is c -positively expansive for some $c > 0$ if, for any $x \neq y$, there exists $n \geq 0$, such that $d(f^n(x), f^n(y)) \geq c$. These maps were studied by many authors, see for instance (Coven and Reddy 1980), (Ruelle 1978), (Sakai 1980), (Sakai 2003). An important class of positively expansive maps are the expanding ones defined as follows: we say that a map is *expanding* if there are constants $C > 0$ and $\sigma > 1$, such that $\|Df^n(x)v\| \geq C\sigma^n\|v\|$ for every $x \in M$.

DEFINITION 1.1. *We say that a C^1 map $f : M \rightarrow M$ is C^1 -persistently positively expansive if there exists a C^1 -neighborhood \mathcal{U} , such that for any $g \in \mathcal{U}$, there exists $c(g) > 0$, such that g is $c(g)$ -positively expansive.*

In this note, we shall prove the following result:

THEOREM 1.2. *Let M be a compact manifold. Any C^1 -persistently positively expansive map $f : M \rightarrow M$ is expanding.*

The same type of result appears in (Sakai 2004). However, Sakai assumes that the separation property in the definition of positive expansivity holds for any composition of maps in the neighborhood \mathcal{U} , and, in this paper, $c(g)$ is also a constant in the neighborhood. Since we are not assuming these hypotheses, our result is slightly stronger. Also, our arguments rely on a theorem by Cao (Cao 2003), which simplifies the proof.



In Sakai’s paper, the following definition is introduced: a map f is a C^1 -stably positively expansive map with constants $\nu > 0$ and $c > 0$ if, for any sequence, $\{g_i\}_{i=1}^\infty \subset C^1(M)$, with $d_{C^1}(f, g_i) \leq \nu$ for all $i \in \mathbb{N}$ and for any sequences $\{x_i, y_i\}_{i=0}^\infty$, such that $x_i = g_i \circ \cdots \circ g_1(x_0)$, $y_i = g_i \circ \cdots \circ g_1(y_0)$ and $d(x_i, y_i) \leq c$; for all $i \geq 0$, we have that $x_0 = y_0$. Then, he proves that any C^1 -stably positively expansive map is expanding.

PROOF OF THE THEOREM

First, we recall a lemma due to Franks (Franks 1971):

LEMMA 2.1. *Let f be a C^1 map and \mathcal{U} a C^1 -neighborhood of f . Then, there exists a neighborhood $\mathcal{U}_0 \subset \mathcal{U}$ of f and a $\delta > 0$ such that, if $g \in \mathcal{U}_0(f)$, $S = \{p_1, \dots, p_m\} \subset M$ is a finite set, and $\{L_i : T_{p_i}M \rightarrow T_{g(p_i)}M\}_{i=1}^m$ are linear maps satisfying $\|L_i - Dg(p_i)\| \leq \delta$ for $i = 1, \dots, m$; then, there exists $h \in \mathcal{U}_0(f)$ satisfying $h(p_i) = g(p_i)$ and $Dh(p_i) = L_i$. Moreover, if U is a neighborhood of S , then, h can be taken such that $h(x) = g(x)$ for every $x \in S \cup (M - U)$, and locally*

$$h(x) = \exp_{g(p_i)} \circ Dg(p_i) \circ \exp_{p_i}^{-1}(x), \text{ for } x \in B_\varepsilon(p_i),$$

where \exp is the exponential map of the Riemannian manifold M , and ε is sufficiently small.

Now we fix f a C^1 -persistently positively expansive map.

The following corollary is an adaptation of an argument of (Sakai 2004):

COROLLARY 2.2. *Let \mathcal{U} be the neighborhood of f in the definition of C^1 -persistently positively expansive map. For any $g \in \mathcal{U}$, and p as a periodic point for g with period n , if λ is an eigenvalue of $Dg^n(p)$, then $|\lambda| > 1$.*

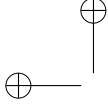
PROOF. If not, there exists g with p as a periodic point with a contracting eigenvalue, i.e., an eigenvalue λ with $|\lambda| \leq 1$. Using Franks’ lemma, we can find $h \in \mathcal{U}$, such that:

$$h(x) = \exp_{g^i(p)} \circ Dg(g^{i-1}(p)) \circ \exp_{g^{i-1}(p)}^{-1}(x), \text{ for } x \in B_\varepsilon(g^{i-1}(p)),$$

where ε is small. Now, since we linearized near the periodic orbit, we obtain two distinct, but close enough, orbits z and w inside the stable local manifold of this linear map, such that $d(h^n(x), h^n(y)) \leq c(h)$ for $n \geq 0$, which is a contradiction. \square

The same argument shows that any $g \in \mathcal{U}$ is a local diffeomorphism. Since there exists $p \in M$ and a zero eigenvalue of $Dg(p)$, by Franks’ lemma, we make a perturbation h with a set of points S (in the direction of the eigenvalue) such that, for every $z \in S$, we have $h(z) = h(p)$, and this contradicts the positive expansivity of h .

A δ -pseudo orbit $\{x_i\}$ for f is a sequence of points in M such that, for every i , we have $d(f(x_i), x_{i+1}) <$



Since f is a local diffeomorphism, in particular it is open. Now, (Ruelle 1978) and (Sakai 2003) show that any positively expansive map that is open has the shadowing property. Also, since the map is positively expansive, the shadow is unique. In particular, if the pseudo-orbit is periodic, the shadow is a periodic point.

We recall that, for any $x \in M$ and $v \in T_x M$, the latter being a nonzero vector, the Lyapunov exponent associated is:

$$\lambda(x, v) = \lim_{k \rightarrow \infty} \frac{1}{k} \log \|Df^k(x)v\|$$

whenever this limit exists. By Oseledets' theorem (Oseledets 1968), this limit exists for μ -almost every point x and any nonzero vector $v \in T_x M$, if μ is a finite invariant measure.

LEMMA 2.3. *Let μ be a finite invariant measure of f . Then, for μ -almost every x , the Lyapunov exponents $\lambda_i(f, x)$ are positive.*

PROOF. Fix $\delta > 0$ as the constant given by Franks' lemma. Let $\nu > 0$, such that, if $d(x, y) < \nu$, then $d(Df(x), Df(y)) < \delta$. Finally, fix $\varepsilon > 0$ as the constant given by the shadowing property associated to f for ν -pseudo orbits.

If the lemma is false, there exists a measure μ with a non-positive Lyapunov exponent. By Poincaré recurrence, there exists $x \in M$, $v \in T_x M$, $0 < \eta \ll \delta^1$ and N large enough, such that:

$$\|Df^N(x)v\| \leq (1 + \eta)^N \|v\|, \quad d(f^N(x), x) < \nu,$$

and there exists i much smaller than N , such that $d(f^i(x), x) < \nu$ and the directions

$$\frac{Df^i(x)v}{\|Df^i(x)v\|} \quad \text{and} \quad \frac{Df^N(x)v}{\|Df^N(x)v\|}$$

are close enough.

The shadowing lemma implies that there exists a periodic orbit $f^{N-i}(p) = p$, such that $d(f^j(p), f^i(p)) < \nu$ for $j = 0, \dots, N - i - 1$; moreover, there exists some vector $w \in T_p M - \{0\}$ close to $Df^{N-i}(p)w$. Since η is small, we can use Frank's lemma to perturb f , replacing the derivative in the shadow of p by the derivative of the segment of orbit of x , dividing by $1 + 2\eta$ on each iterate, and composing with the last derivative (i.e., the last iterate of the periodic orbit) with a small rotation. It was obtained $g \in \mathcal{G}$ with a periodic point p of period $N - i$ with a contractive eigen-vector w for the derivative $Df^{N-i}(p)$, which contradicts the previous lemma.

Now we invoke a theorem due to Cao (Cao 2003):

THEOREM 2.4 (Cao). *Let $f : M \rightarrow M$ be a C^1 -local diffeomorphism. If the Lyapunov exponents of f -invariant probability measure are positive, then f is uniformly expanding.*



ACKNOWLEDGMENTS

The author would like to thank Professor C. Morales by useful conversations. The author would also like to thank the useful comments of an anonymous referee that clarified the proof of lemma 2.3. This work was partially supported by Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq) and Programa de Apoio a Projetos Institucionais com a Participação de Recém-Doutores – Coordenação de Aperfeiçoamento de Pessoal de Nível Superior (PRODOC–CAPES).

RESUMO

Neste artigo, mostramos que todo mapa C^1 -persistentemente positivamente expansivo é expansor. Isto melhora um resultado devido a Sakai (Sakai 2004).

Palavras-chave: mapas positivamente expansivos, mapas expansores.

REFERENCES

- CAO Y. 2003. Non-zero Lyapunov exponents and uniform hyperbolicity. *Nonlinearity* 16: 1473–1479.
- COVEN E AND REDDY W. 1980. Positively Expansive maps of compact manifolds. *Lecture Notes in Math* 819: 96–110.
- FRANKS J. 1971. Necessary conditions for stability of diffeomorphisms. *Trans Amer Math Soc* 158: 301–308.
- RUELLE D. 1978. Thermodynamical Formalism. *Encyclopedia Math Appl* n. 5.
- SAKAI K. 1985. Periodic Points of Positively Expansive maps. *Proc Amer Math Soc* 94: 531–534.
- SAKAI K. 2003. Various shadowing properties for positively expansive maps. *Topology Appl* 131: 15–31.
- SAKAI K. 2004. C^1 -Stably Positively Expansive Maps. *Bull of the Polish Acad Sci Math* 52: 197–209.
- OSELEDETS V. 1968. A multiplicative ergodic theorem: Lyapunov characteristic numbers for dynamical systems. *Trans Moscow Math Soc* 19: 197–231.