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Injective mappings and solvable vector fields

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ABSTRACT

We establish a sufficient condition for injectivity in a class of mappings defined on open connected subse of $\mathbb{R}^n$, for arbitrary n. The result relates solvability of the appropriate vector fields with injectivity of t mapping and extends a result proved by the first author for $n \leq 3$. Furthermore, we extend the result connected paracompact smooth oriented manifolds and show that the convexity condition imposes stro topological restrictions on the manifold.

Key words: fields, injectivity, mappings, solvability, vectors.

INTRODUCTION

Let Ω be an open connected subset of $\mathbb{R}^n$, and consider $\Phi(\Omega) = \{\mathrm{F} \in C^\infty(\Omega, \mathbb{R}^n) \colon \det(\mathrm{DF})(x) \neq$ We denote an element of $\Phi(\Omega)$ by $\mathrm{F} = (\mathrm{f}_1, \ldots, \mathrm{f}_n)$. A very basic problem in Mathematical Anal to determine additional conditions under which F is injective. The most famous related problem so called Jacobian Conjecture, which states that for a polynomial mapping on $\mathbb{C}^n$ no additional co is necessary, see (Bass et al. 1982). For applications and related bibliography see (Santos Filho As in Santos Filho, for each $i \in \{1, 2, \ldots, n\}$, we consider $\mathcal{V}_{\mathrm{F},i}$ the real vector field defined by

$$\mathcal{V}_{\mathrm{F},i}(\phi)(x) = \det(\mathrm{DF}_{i,\phi})(x)), \quad \text{for all } \phi \in C^\infty.$$

Here the mapping $\mathrm{F}_{i,\phi}$ is given by: Its j-component is equal to f_j if $j \neq i$ and its i-compo equal to ϕ. It follows that the non-empty connected components of $\{x \in \Omega; \mathrm{f}_j(x) = c_j, \ j$ where $c_1, \ldots, c_{i-1}, c_{i+1}, \ldots, c_n \in \mathbb{R}$, defines a smooth one-dimensional foliation of Ω. This folia precisely that one of the characteristic curves of $\mathcal{V}_{\mathrm{F},i}$, because each f_j is a first integral of on a manifold, in (Malgrange 1956), B. Malgrange introduced a notion of convexity, namely the condition (a)(2) of Theorem 2.1 below. As in (Santos Filho 2004), a similar notion is in order.

DEFINITION 1.1. Let $\mathrm{F} \in \Phi(\Omega)$. We say that Ω is F-convex if there are an open set of Ω_1 of $\mathbb{R}^n$, $\mathrm{G}_1 \in \Phi(\Omega_1)$ with $\mathrm{G}_1(\Omega_1) = \Omega$, and $\mathrm{G}_2 \in \Phi(\mathrm{F}(\Omega))$ where, for $j = 1, 2$, each G_j is a diffeomorphism over its image such that there are $n-1$ different indices $i_1, \ldots, i_{n-1} \in \{1, \ldots, n\}$ where Ω_1 is $\mathcal{V}_{F_1,i_j}$-convex for $j \in \{1, \ldots, n-1\}$ and $\mathrm{F}_1 = \mathrm{G}_2 \circ \mathrm{F} \circ \mathrm{G}_1$.

For $n = 2, 3$ it was proved in Theorems 0.1 and Theorem 0.2 of (Santos Filho 2004), that: If Ω is F-convex then F is injective. Also Ω is simply connected when $n = 2$.

Here we give a full generalization of these results. We mean not only for arbitrary dimensions, but also for the smooth category of *oriented* manifolds too. Our main goal is to generalize, for higher dimensions and arbitrary manifolds, those results. First we address the euclidean case as a introduction in Theorem 1.1 and then we furnish the needed tools for the general case in Theorem 2.2 below.

THEOREM 1.1. *Let Ω be an open connected subset of $\mathbb{R}^n$ and $\mathrm{F} \in \Phi(\Omega)$. If Ω is* F*-convex then* F *is injective.*

EXAMPLE. Consider the smooth mapping of the plane given by $\mathrm{F}(x, y) = (x, y \exp x^{-2})$ for $x \neq 0$ and $\mathrm{F}(0, y) = (0, 0)$ if $x = 0$. Then $\mathcal{V}_{\mathrm{F},2} = \partial/\partial y$, so $\mathbb{R}^2$ is F-convex. Moreover $\det(\mathrm{D\,F}(x)) = \exp(x^{-2}) > 0$ if $x \neq 0$, so is positive except at a closed set of null Lebesgue measure, in the other hand F is not injective. So the condition of $\mathrm{F} \in \Phi(\mathbb{R}^2)$ in Theorem 1.1 can not be in this fashion.

This paper is organized in the following way: First we prove Theorem 1.1, then extend it for connected paracompact smooth oriented n-dimensional manifold M and finally show that M must be contractible if M is F-convex for some $\mathrm{F} \in \Phi(M, \mathbb{R}^n)$. We conclude by making some remarks regarding the results.

PROOFS OF THE RESULTS AND REMARKS

Before we prove our theorems we recall part of Theorem 6.4.2 of Duistermaat and Hörmander, in (Duistermaat and Hörmander 1972), this result characterizes global solvability of vector fields considered as partial differential operators:

THEOREM 2.1. *Let L be a smooth real vector field of a C^∞ manifold M, the following conditions* (a) *and* (b) *below are equivalent:*

(a) (1) *No complete integral curve of L is contained in compact subset of M.*

(2) *For every compact subset K of M there exists a compact subset K' of M such that every compact interval of an integral curve with end points in K is contained in K'.*

(b) *There exists a manifold M_0 and an open neighborhood M_1 of $M_0 \times \{0\}$ in $M_0 \times \mathbb{R}$ which is convex in the $\mathbb{R}$ direction, and a diffeomorphism from $M \to M_1$ which carries L into the operator $\partial/\partial t$ if points of $M_0 \times \mathbb{R}$ are denoted by (y_0, t).*

PROOF OF THEOREM 1.1

Without loss of generality we can assume that $\Omega_1 = \Omega$ and $i_j = j$ for $j = 1, \dots, n-1$. Our follows by a finite induction argument once we prove injectivity for appropriate restrictions of F j an integer, $1 \leq j \leq n-1$, and $(c_1, \dots, c_n) \in Image(\mathrm{F})$, consider M_j be a connected compon $\cap_{i=1}^{j} \mathbf{f}_i^{-1}(\{c_i\})$. From the hypotheses, it follows that for each j M_j is a C^∞ submanifold of dim $n-j$ of Ω, which is invariant by the flow of of $\mathcal{V}_{\mathrm{F},i}$, for $j+1 \leq i \leq n$.

Let $j \in \{1, 2, \dots, n\}$, since $\mathcal{V}_{\mathrm{F},j}(f_i) = \det(\mathrm{D}\,\mathrm{F}(x))\delta_{ij}$ where δ_{ij} is the Kronecker symbol, w that both ω-limit and α-limit of characteristic curves of $\mathcal{V}_{\mathrm{F},j}$ are empty. Hence condition (a)(1) of rem 2.1 is true for $L = \mathcal{V}_{\mathrm{F},j}$, if $j \in \{1, \dots, n-1\}$. Therefore, in our case, condition (a) of Theorem equivalent (a)(2), which it is exactly the meaning of Ω be $\mathcal{V}_{F,i_j}$-convex. So from Theorem 1.1 we ha (b) holds, that is:

For each $1 \leq j \leq n-1$, the following holds

$(*)_j$ There exist a manifold $M_{0,j}$, an open neighborhood $M_{1,j}$ of $M_{0,i} \times \{0\}$ in $M_{0,j} \times \mathbb{R}$ which is c in the $\mathbb{R}$ direction, and a diffeomorphism $\Omega \to M_{1,j}$ which carries each $\mathcal{V}_{\mathrm{F},j}$ into the operator points in $M_{0,j} \times \mathbb{R}$ are denoted by $(y_{0,j}, t)$.

The next step of the proof is to show that we can take $M_{0,j}$ so that its image on Ω, by the morphism of $(*)_j$, is equal to a connected component of a level set of f_j. In order to prove this, co $X_i = (\det(\mathrm{F}'(x))^{-1}\mathcal{V}_{\mathrm{F},i}$, for $i = 1, \dots, n$, so $[X_i, X_j](\mathrm{f}_k) = 0$ for all i, j and k, therefore by the I Function Theorem we have that $[X_i, X_j] = 0$ for all i and j. Moreover, the orbits of X_i, for $1 \leq$ are equal to the corresponding $\mathcal{V}_{\mathrm{F},i}$.

So for $1 \leq j \leq n-1$ $(*)_j$ holds for X_j as well. From Frobenius Theorem, since the X_j's con the image on Ω of $M_{0,j}$ by the diffeomorphism must be orthogonal to X_j. Therefore it is a con component of a pre-image of f_j, and the same holds for $\mathcal{V}_{\mathrm{F},j}$.

Assume that there are two points p_1 and p_2 of M such that $\mathrm{F}(p_1) = \mathrm{F}(p_2)$, in particular we ha $\mathrm{f}_1(p_1) = \mathrm{f}_1(p_2)$. Clearly if the characteristic curves of $\mathcal{V}_{\mathrm{F},1}$ passing through each p_i are the same th have that $p_1 = p_2$, because $\mathcal{V}_{\mathrm{F},1}(\mathrm{f}_1) \neq 0$. So we assume that the above curves are different con components of $\{x; \mathrm{f}_j(x) = \mathrm{f}_j(p_1),\ \text{for } j \neq 1\}$.

Identifying $M_{0,1}$ with its image by the diffeomorphism given in $(*)_j$, by the above it must be e a connected component of $\mathrm{f}_1^{-1}(c_1)(= M_{0,1})$, for some real c_1. Furthermore, since $\{\mathcal{V}_{\mathrm{F},i}; 2 \leq i \leq$ tangent to $\mathrm{f}_1^{-1}(c_)$, we reduce to prove the theorem for $\mathrm{F}|_{M_{0,1}} \in \Phi(M_{0,1}, \mathbb{R}^{n-1})$.

We apply the argument above until we get the restriction of F to a component connec $\cap_{1 \leq j \leq n-1} \mathrm{f}_j^{-1}(c_j)$, but on this curve the restriction of $\mathbf{F}$ is equal to f_n. So $\mathbf{F}$ must be injective th using that $\mathcal{V}_{\mathrm{F},n}\mathrm{f}_n \neq 0$. Concluding the proof of Theorem 0.1.

EXTENSION OF THEOREM 1.1

Suppose that $\mathrm{F} = (\mathrm{f}_1, \ldots, \mathrm{f}_n) \colon M \to \mathbb{R}^n$ is a smooth mapping with injective derivative at any point of M. As before such a mapping is said to belong to $\Phi(M) = \Phi(M, \mathbb{R}^n)$. From the hypothesis on M we have that there is $(U_m, \varphi_m)_{m \in \mathbb{Z}_+}$ a coordinate system for M so that U_m is pre-compact for each m. Take $(\psi_m)_{m \in \mathbb{Z}_+}$ to be a partition of unity associated to (U_m). Furthermore, we can assume that the restriction of F on U_m is a diffeomorphism over its image.

Let $1 \leq j \leq n$ consider the vector field $\mathcal{V}_j$ defined as follows, let $g \in C^\infty(M)$, take:

$$\mathcal{V}_j(g) = \mathrm{coeff}\left[(-1)^{n-j} d\mathrm{f}_1 \wedge \ldots \wedge (d\mathrm{f}_j)^v \wedge \ldots \wedge d\mathrm{f}_n \wedge d(g) \text{ in terms of } \omega\right].$$

It is a routine computation to show that $\mathcal{V}_j$ is a derivation, then it defines a vector field on M.

Now we will see that these vector fields generalize for M the vector fields considered before on the the euclidean context. In fact, let $g \in C^\infty(M)$ and write $g = \Sigma_m \psi_n g$, therefore the support of $\psi_m g$ is contained on U_m. Since the restriction of F to U_m is a diffeomorphism we can find a $g_m \in C_c^\infty(\mathrm{F}(U_m))$ such that $\psi_m g = g_m \circ \mathrm{F}$. So $g = \Sigma_m g_m \circ \mathrm{F}$. Then

$$\left[(-1)^{n-j} d\mathrm{f}_1 \wedge \cdots \wedge (d\mathrm{f}_j)^v \wedge \cdots \wedge d\mathrm{f}_n \wedge d(g)\right] = \Sigma_m \partial_n g_m d\mathrm{f}_1 \wedge \cdots \wedge d\mathrm{f}_n\,.$$

Defining $\mathrm{Det}(D\mathrm{F}(x))$ to be the coefficient of $d\mathrm{f}_1 \wedge \cdots \wedge d\mathrm{f}_n$ in terms of ω, we have that

$$\mathcal{V}_j(g) = \Sigma_m \partial_n g_m \mathrm{Det}(D\mathrm{F}(x))\,.$$

As before we have that f_j, for $j \neq i$, is a first integral of $\mathcal{V}_i$. Also the vector fields are in involution. We consider an extension of Definition 1.1:

Definition 2.1. Let $\mathrm{F} \in \Phi(M)$. We say that M is F-convex if there are an smooth manifold M_1, $\mathrm{G}_1 \in \Phi(M_1, M)$ with $\mathrm{G}_1(M_1) = M$ and $\mathrm{G}_2 \in \Phi(\mathrm{F}(\Omega))$ where, for $j = 1, 2$, each G_j is a diffeomorphism over its image such that there are $n-1$ different indices $i_1, \ldots, i_{n-1} \in \{1, \ldots, n\}$ where M_1 is $\mathcal{V}_{F_1, i_j}$-convex for $j \in \{1, \ldots, n-1\}$ and $\mathrm{F}_1 = \mathrm{G}_2 \circ \mathrm{F} \circ \mathrm{G}_1$.

From inspection on the proof of Theorem 1.1 we have:

Theorem 2.2. *Let M be a connected paracompact smooth oriented manifold and* $\mathrm{F} \in \Phi(M)$. *If M is* F-*convex then* F *is injective.*

Theorem 2.2 could be used to decide whether a local parametrization of a manifold, defined globally, is a global parametrization. The result below extends for arbitrary dimension Theorem 0.3 of (Santos Filho 2004):

Corollary 2.1. *Let M be be a connected paracompact smooth oriented manifold and* $\mathrm{F} \in \Phi(M)$ *so that* Ω *is* F-*convex. Then M is contractible.*

We observe from the proof of Theorem 1.1 that M is a diffeomorphic image of $\gamma \times \mathbb{R}^{n-1}$ where

REMARKS.

1) The proof that each pair of $\mathcal{V}_{\mathrm{F},j}$'s is in involution can also be proved without normalizing
That is without considering the $X_j's$. In fact, let $\mathcal{V}_{\mathrm{F},j_1}$ and $\mathcal{V}_{\mathrm{F},j_2}$, with $j_1 \neq j_2$, then $\mathcal{V}_{\mathrm{F},j_i}(f_j$
if $j \neq j_i$ for $i \in \{1,2\}$. Therefore $[\mathcal{V}_{\mathrm{F},j_1}, \mathcal{V}_{\mathrm{F},j_2}]$ at any point is linear combination of $\mathcal{V}_{\mathrm{F},}$
$\mathcal{V}_{\mathrm{F},j_2}$, proving the assertion.

2) From Theorem 2.1, we can not have a vector field globally solvable on a compact manifold w
boundary. Otherwise the manifold would be a diffeomorphic image of a cylinder $M_0 \times \mathbb{R}$, wh
is a $n-1$ dimensional manifold, therefore imposing a strong restriction on the topology of M.

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RESUMO

Nós estabelecemos uma condição suficiente para injetividade numa classe de aplicações definidas em subco
abertos conexos de $\mathbb{R}^n$, para n arbitrário. O resultado relaciona resolubilidade de campos de vetores aprop
com injetividade da aplicação e estende o resultado demonstrado pelo primeiro autor quando $n \leq 3$. Alén
nós estendemos o resultado para variedades suaves orientadas e para-compactas e mostramos que a condi
convexidade impõe fortes restrições topológicas na variedade.

Palavras-chave: campos, injetividade, aplicações, resolubilidade, vetores.

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