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Systems with the integer rounding property in normal monomial subrings

LUIS A. DUPONT[1], **CARLOS RENTERÍA-MÁRQUEZ**[2] **and RAFAEL H. VILLARREAL**[1]

[1]Departamento de Matemáticas, Centro de Investigación y de Estudios Avanzados del IPN,
Apartado Postal 14–740, 07000 Mexico City, D.F.
[2]Departamento de Matemáticas, Escuela Superior de Física y Matemáticas, Instituto Politécnico Nacional,
Unidad Adolfo López Mateos, Edif. 9, 07738 México City, D.F.

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ABSTRACT

Let $\mathcal{C}$ be a clutter and let A be its incidence matrix. If the linear system $x \geq 0; xA \leq \mathbf{1}$ has the integ
rounding property, we give a description of the canonical module and the a-invariant of certain norm
subrings associated to $\mathcal{C}$. If the clutter is a connected graph, we describe when the aforementioned line
system has the integer rounding property in combinatorial and algebraic terms using graph theory and t
theory of Rees algebras. As a consequence we show that the extended Rees algebra of the edge ideal o
bipartite graph is Gorenstein if and only if the graph is unmixed.

Key words: canonical module, a-invariant, normal ideal, perfect graph, maximal cliques, Rees algeb
Ehrhart ring, integer rounding property.

1 INTRODUCTION

A clutter $\mathcal{C}$ with finite vertex set X is a family of subsets of X called edges, and none of which is inclu
another. The set of vertices and edges of $\mathcal{C}$ are denoted by $V(\mathcal{C})$ and $E(\mathcal{C})$, respectively. A basic ex
of a clutter is a graph.

Let $\mathcal{C}$ be a clutter with finite vertex set $X = \{x_1, \ldots, x_n\}$. We shall always assume that $\mathcal{C}$
isolated vertices, i.e., each vertex occurs in at least one edge. Let $f_1, \ldots, f_q$ be the edges of $\mathcal{C}$,
$v_k = \sum_{x_i \in f_k} e_i$ be the *characteristic vector* of f_k, where e_i is the ith unit vector in $\mathbb{R}^n$. The *incidence*
A of $\mathcal{C}$ is the $n \times q$ matrix with column vectors $v_1, \ldots, v_q$. If $a = (a_i)$ and $c = (c_i)$ are vectors
then $a \leq c$ means that $a_i \leq c_i$ for all i. Thus, $a \geq 0$ means that $a_i \geq 0$ for all i. The system $x \geq 0;$ x
has the *integer rounding property* if

$$\lceil \min\{\langle y, \mathbf{1}\rangle \mid y \geq 0;\ Ay \geq \alpha\}\rceil = \min\{\langle y, \mathbf{1}\rangle \mid Ay \geq \alpha;\ y \in \mathbb{N}^q\}$$

for each integral vector α for which $\min\{\langle y, \mathbf{1}\rangle \mid y \geq 0;\ Ay \geq \alpha\}$ is finite. Here $\mathbf{1}$ denotes the vector with all its entries equal to 1, and $\langle\ ,\ \rangle$ denotes the standard inner product. For a thorough study of this property see (Schrijver 1986).

Let $R = K[x_1, \ldots, x_n]$ be a polynomial ring over a field K, and let $w_1, \ldots, w_r$ be the set of all integral vectors α such that $0 \leq \alpha \leq v_i$ for some i. We will examine the integer rounding property using the monomial subring:

$$S = K[x^{w_1}t, \ldots, x^{w_r}t] \subset R[t],$$

where t is a new variable. As usual, we use the notation $x^a := x_1^{a_1} \cdots x_n^{a_n}$, where $a = (a_1, \ldots, a_n) \in \mathbb{N}^n$. The subring S is called *normal* if S is integrally closed, i.e., $S = \overline{S}$, where $\overline{S}$ is the integral closure of S in its field of fractions; see (Vasconcelos 2005).

The contents of this paper are as follows. One of the results in (Brennan et al. 2008) shows that S is normal if and only if the system $x \geq 0;\ xA \leq \mathbf{1}$ has the integer rounding property (see Theorem 2.1). As a consequence, we show that if all edges of $\mathcal{C}$ have the same number of elements and either linear system $x \geq 0; xA \leq \mathbf{1}$ or $x \geq 0; xA \geq \mathbf{1}$ has the integer rounding property, then, the subring $K[x^{v_1}t, \ldots, x^{v_q}t]$ is normal (see Corollary 2.4).

Let G be a connected graph and let A be its incidence matrix. The main results of Section 3 show that the following conditions are equivalent (see Theorems 3.2 and 3.3):

(a) $x \geq 0; xA \leq \mathbf{1}$ has the integer rounding property.

(b) $x \geq 0; xA \geq \mathbf{1}$ has the integer rounding property (see Definition 2.2).

(c) $R[x^{v_1}t, \ldots, x^{v_q}t]$ is normal, where $v_1, \ldots, v_q$ are the column vectors of A.

(d) $K[x^{v_1}t, \ldots, x^{v_q}t]$ is normal.

(e) $K[t, x_1t, \ldots, x_nt, x^{v_1}t, \ldots, x^{v_q}t]$ is normal.

(f) The induced subgraph of the vertices of any two vertex disjoint odd cycles of G is connected.

The most interesting part of this result is the equivalence between (a), which is a linear optimization property, and (f), which is a graph theoretical property. Edge ideals are defined in Section 2. We prove that the ring in (e) is isomorphic to the extended Rees algebra of the edge ideal of G (see Proposition 3.1). If G is bipartite and $I = I(G)$ is its edge ideal, we are able to prove that the extended Rees algebra of I is a Gorenstein standard K-algebra if and only if G is unmixed (see Corollary 4.3). If we work in the more general context of clutters, none of the conditions (a) to (e) are equivalent. Some of these conditions are equivalent under certain assumptions (Gitler et al. 2009).

In Section 4 we introduce the canonical module and the a-invariant of S. This invariant plays a key

methods. If S is a normal domain, we express the canonical module of S and its a-invariant in te
the vertices of the polytope

$$\{x \mid x \geq 0;\, xA \leq \mathbf{1}\}$$

(see Theorem 4.1). We are able to give an explicit description of the canonical module of S a
a-invariant when $\mathcal{C}$ is the clutter of maximal cliques of a perfect graph (Theorem 4.2).

For unexplained terminology and notation on commutative algebra and integer programming, w
to (Bruns and Herzog 1997, Vasconcelos 2005, Villarreal 2001) and (Schrijver 1986), respectively.

2 INTEGER ROUNDING AND NORMALITY

We continue using the definitions and terms from the Introduction. In what follows, $\mathbb{N}$ denotes the
non-negative integers and $\mathbb{R}_+$ denotes the set of non-negative real numbers. Let $\mathcal{A} \subset \mathbb{Z}^n$. The conv
of $\mathcal{A}$ is denoted by $\mathrm{conv}(\mathcal{A})$, and the cone generated by $\mathcal{A}$ is denoted by $\mathbb{R}_+\mathcal{A}$.

THEOREM 2.1 (Brennan et al. 2008). *Let $\mathcal{C}$ be a clutter and let $v_1, \ldots, v_q$ be the columns of the inc
matrix A of $\mathcal{C}$. If $w_1, \ldots, w_r$ is the set of all $\alpha \in \mathbb{N}^n$ such that $\alpha \leq v_i$ for some i, then the
$x \geq 0;\ xA \leq \mathbf{1}$ has the integer rounding property if and only if the subring $K[x^{w_1}t, \ldots, x^{w_r}t]$ is no*

Next we give an application of this result, but first we need to introduce some more terminolo
notation. We have already defined in the Introduction when the linear system $x \geq 0; xA \leq \mathbf{1}$ has the i
rounding property. The following is a dual notion.

DEFINITION 2.2. Let A be a matrix with entries in $\mathbb{N}$. The system $x \geq 0; xA \geq \mathbf{1}$ has the *integer ro
property* if

$$\max\{\langle y, \mathbf{1}\rangle \mid Ay \leq w;\, y \in \mathbb{N}^q\} = \lfloor \max\{\langle y, \mathbf{1}\rangle \mid y \geq 0;\, Ay \leq w\} \rfloor$$

for each integral vector w for which the right hand side is finite.

Let $\mathcal{A} = \{v_1, \ldots, v_q\}$ be a set of points in $\mathbb{N}^n$, let P be the convex hull of $\mathcal{A}$, and let $R = K[x_1, .$
be a polynomial ring over a field K. The *Ehrhart ring* of the lattice polytope P is the monomial sub

$$A(P) = K[\{x^a t^i \mid a \in \mathbb{Z}^n \cap iP;\, i \in \mathbb{N}\}] \subset R[t],$$

where t is a new variable. A nice property of $A(P)$ is that it is always a normal domain (Bruns and H
1997). Let $\mathcal{C}$ be a clutter with vertex set $X = \{x_1, \ldots, x_n\}$ and edge set $E(\mathcal{C})$. For use below reca
$\mathcal{C}$ is called *uniform* if all its edges have the same number of elements. The *edge ideal* of $\mathcal{C}$, deno
$I(\mathcal{C})$, is the ideal of R generated by all monomials $\prod_{x_i \in e} x_i = x_e$ such that $e \in E(\mathcal{C})$. The *Rees alge*
$I = I(\mathcal{C})$, denoted by $R[It]$, is given by

$$R[It] := R \oplus It \oplus \cdots \oplus I^i t^i \oplus \cdots \subset R[t],$$

(Vasconcelos 2005, Chapter 1) for various presentation of Rees algebr

THEOREM 2.3 (Dupont and Villarreal 2010). *Let $I = (x^{v_1}, \dots, x^{v_q}) \subset R$ be a monomial ideal, and let A be the matrix with column vectors $v_1, \dots, v_q$. Then, the system $x \geq 0; xA \geq \mathbf{1}$ has the integer rounding property if and only if $R[It]$ is normal.*

COROLLARY 2.4. *Let $\mathcal{C}$ be a uniform clutter, let A be its incidence matrix, and let $v_1, \dots, v_q$ be the columns of A. If either system $x \geq 0; xA \leq \mathbf{1}$ or $x \geq 0; xA \geq \mathbf{1}$ has the integer rounding property and $P = \text{conv}(v_1, \dots, v_q)$, then*

$$K[x^{v_1}t, \dots, x^{v_q}t] = A(P).$$

PROOF. In general, the subring $K[x^{v_1}t, \dots, x^{v_q}t]$ is contained in $A(P)$. Assume that $x \geq 0; xA \leq \mathbf{1}$ has the integer rounding property, and that every edge of $\mathcal{C}$ has d elements. Let $w_1, \dots, w_r$ be the set of all $\alpha \in \mathbb{N}^n$ such that $\alpha \leq v_i$ for some i. Then by Theorem 2.1, the subring $K[x^{w_1}t, \dots, x^{w_r}t]$ is normal. Using that $v_1, \dots, v_q$ is the set of w_i, with $|w_i| = d$, it is not hard to see that $A(P)$ is contained in $K[x^{v_1}t, \dots, x^{v_q}t]$.

Assume that $x \geq 0; xA \geq \mathbf{1}$ has the integer rounding property. Let $I = I(\mathcal{C})$ be the edge ideal of $\mathcal{C}$, and let $R[It]$ be its Rees algebra. By Theorem 2.3, $R[It]$ is a normal domain. Since the clutter $\mathcal{C}$ is uniform, the required equality follows at once (Escobar et al. 2003, Theorem 3.15). □

The converse of Corollary 2.4 fails as the following example shows.

EXAMPLE 2.5. Let $\mathcal{C}$ be the uniform clutter with vertex set $X = \{x_1, \dots, x_8\}$ and edge set

$$E(\mathcal{C}) = \{\{x_3, x_4, x_6, x_8\},\ \{x_2, x_5, x_6, x_7\},\ \{x_1, x_4, x_5, x_8\},\ \{x_1, x_2, x_3, x_8\}\}.$$

The characteristic vectors of the edges of $\mathcal{C}$ are

$$v_1 = (0, 0, 1, 1, 0, 1, 0, 1), \quad v_2 = (0, 1, 0, 0, 1, 1, 1, 0),$$
$$v_3 = (1, 0, 0, 1, 1, 0, 0, 1), \quad v_4 = (1, 1, 1, 0, 0, 0, 0, 1).$$

Let A be the incidence matrix of $\mathcal{C}$ with column vectors $v_1, \dots, v_4$, and let P be the convex hull of $\{v_1, \dots, v_4\}$. It is not hard to verify that the set

$$\{(v_1, 1), (v_2, 1), (v_3, 1), (v_4, 1)\}$$

is a Hilbert basis in the sense of (Schrijver 1986). Therefore, we have the equality

$$K[x^{v_1}t, x^{v_2}t, x^{v_3}t, x^{v_4}t] = A(P).$$

Using Theorem 2.3 and (Brennan et al. 2008, Theorem 2.12) it is seen that none of the two systems $x \geq 0; xA \leq \mathbf{1}$ and $x \geq 0; xA \geq \mathbf{1}$ have the integer rounding property.

rounding property of the system $x \geq 0; xA \leq \mathbf{1}$. Other equivalent algebraic conditions of this pr
will be presented.

Let $R = K[x_1, \ldots, x_n]$ be a polynomial ring over a field K, and let $I = I(G)$ be the edge idea
Recall that the *extended Rees algebra* of I is the subring

$$R[It, t^{-1}] := R[It][t^{-1}] \subset R[t, t^{-1}],$$

where $R[It]$ is the Rees algebra of I. Rees algebras of edge ideals of graphs were first studied in (S
al. 1994).

Proposition 3.1. $R[It, t^{-1}] \simeq K[t, x_1t, \ldots, x_nt, x^{v_1}t, \ldots, x^{v_q}t]$.

Proof. We set $S = K[t, x_1t, \ldots, x_nt, x^{v_1}t, \ldots, x^{v_q}t]$. Note that S and $R[It, t^{-1}]$ are both i
domains of the same Krull dimension; this follows from the dimension formula given in (Sturmfels
Lemma 4.2). Thus, it suffices to prove that there is an epimorphism $\overline{\psi}\colon S \to R[It, I^{-1}]$ of K-algeb

Let $u_0, u_1, \ldots, u_n, t_1, \ldots, t_q$ be a new set of variables, and let φ, ψ be the maps of K-al
defined by the diagram

$$\begin{tikzcd} K[u_0, u_1, \ldots, u_n, t_1, \ldots, t_q] \arrow[r, "\psi"] \arrow[d, "\varphi"] & R[It, t^{-1}] \\ S \arrow[ur, dashed, "\overline{\psi}"'] & \end{tikzcd} \qquad \begin{array}{ll} u_0 \overset{\varphi}{\longmapsto} t, & u_0 \overset{\psi}{\longmapsto} t^{-1}, \\ u_i \overset{\varphi}{\longmapsto} x_it, & u_i \overset{\psi}{\longmapsto} x_i, \\ t_i \overset{\varphi}{\longmapsto} x^{v_i}t, & t_i \overset{\psi}{\longmapsto} x^{v_i}t. \end{array}$$

To complete the proof, we will show that there is an epimorphism $\overline{\psi}$ of K-algebras that makes this di
commutative, i.e., $\psi = \overline{\psi}\varphi$. To show the existence of $\overline{\psi}$, we need only to show the inclusion ker
$\ker(\psi)$. As $\ker(\varphi)$ being a toric ideal is generated by binomials (Sturmfels 1996), it suffices to pro
any binomial of $\ker(\varphi)$ belongs to $\ker(\psi)$. Let

$$f = u_0^{a_0}u_1^{a_1}\cdots u_n^{a_n}t_1^{b_1}\cdots t_q^{b_q} - u_0^{c_0}u_1^{c_1}\cdots u_n^{c_n}t_1^{d_1}\cdots t_q^{d_q}$$

be a binomial in $\ker(\varphi)$. Then,

$$t^{a_0}(x_1t)^{a_1}\cdots(x_nt)^{a_n}(x^{v_1}t)^{b_1}\cdots(x^{v_q}t)^{b_q} = t^{c_0}(x_1t)^{c_1}\cdots(x_nt)^{c_n}(x^{v_1}t)^{d_1}\cdots(x^{v_q}t)^{d_q}$$

Taking degrees in t and $\underline{x} = \{x_1, \ldots, x_n\}$, we obtain

$$\begin{aligned} a_0 + (a_1 + \cdots + a_n) + (b_1 + \cdots + b_q) &= c_0 + (c_1 + \cdots + c_n) + (d_1 + \cdots + d_q), \\ a_1 + \cdots + a_n + 2(b_1 + \cdots + b_q) &= c_1 + \cdots + c_n + 2(d_1 + \cdots + d_q). \end{aligned}$$

Thus $-a_0 + b_1 + \cdots + b_q = -c_0 + d_1 + \cdots + d_q$, and we obtain the equality

We come to the main result of this section.

THEOREM 3.2. *Let G be a connected graph and let A be its incidence matrix. Then, the system*

$$x \geq 0; xA \leq \mathbf{1}$$

has the integer rounding property if and only if the induced subgraph of the vertices of any two vertex disjoint odd cycles of G is connected.

PROOF. Let $v_1, \ldots, v_q$ be the column vectors of A. According to (Simis et al. 1998, Theorem 1.1, cf. Villarreal 2005, Corollary 3.10), the subring $K[Gt] := K[x^{v_1}t, \ldots, x^{v_q}t]$ is normal if and only if any two vertex disjoint odd cycles of G can be connected by at least one edge of G. Thus, we need only to show that $K[Gt]$ is normal if and only the system $x \geq 0; xA \leq \mathbf{1}$ has the integer rounding property. Let $I = I(G)$ be the edge ideal of G. Since G is connected, the subring $K[Gt]$ is normal if and only if the Rees algebra $R[It]$ of I is normal (Simis et al. 1998, Corollary 2.8). By a result of (Herzog et al. 1991), $R[It]$ is normal if and only if $R[It, t^{-1}]$ is normal. By Proposition 3.1, $R[It, t^{-1}]$ is normal if and only if the subring

$$S = K[t, x_1t, \ldots, x_nt, x^{v_1}t, \ldots, x^{v_q}t]$$

is normal. Thus, we can apply Theorem 2.1 to conclude that S is normal if and only if the system $x \geq 0; xA \leq \mathbf{1}$ has the integer rounding property. □

THEOREM 3.3. *Let G be a connected graph and let A be its incidence matrix. Then, the system $x \geq 0; xA \leq \mathbf{1}$ has the integer rounding property if and only if any of the following equivalent conditions hold*

(a) $x \geq 0; xA \geq \mathbf{1}$ *is a system with the integer rounding property.*

(b) $R[It]$ *is a normal domain, where $I = I(G)$ is the edge ideal of G.*

(c) $K[x^{v_1}t, \ldots, x^{v_q}t]$ *is normal, where $v_1, \ldots, v_q$ are the column vectors of A.*

(d) $K[t, x_1t, \ldots, x_nt, x^{v_1}t, \ldots, x^{v_q}t]$ *is normal.*

PROOF. According to Theorem 2.3, the system $x \geq 0; xA \geq \mathbf{1}$ has the integer rounding property if and only if the Rees algebra $R[It]$ is normal. Thus, the result follows from the proof of Theorem 3.2. □

4 THE CANONICAL MODULE AND THE a-INVARIANT

In this section, we give a description of the canonical module and the a-invariant for subrings arising from systems with the integer rounding property.

Let $\mathcal{C}$ be a clutter with vertex set $X = \{x_1, \ldots, x_n\}$, and let $v_1, \ldots, v_q$ be the columns of the incidence matrix A of $\mathcal{C}$. For use below, consider the set $w_1, \ldots, w_r$ of all $\alpha \in \mathbb{N}^n$ such that $\alpha \leq v_i$ for some i. Let

be the subring of $R[t]$ generated by $x^{w_1}t, \ldots, x^{w_r}t$, where t is a new variable. As $(w_i, 1)$ lies
hyperplane $x_{n+1} = 1$ for all i, S is a standard K-algebra. Thus, a monomial $x^a t^b$ in S has degree
what follows, we assume that S has this grading. Recall that the *a-invariant* of S, denoted $a(S)$,
degree as a rational function of the Hilbert series of S, see for instance (Villarreal 2001, p. 99).
Cohen-Macaulay and ω_S is the canonical module of S, then

$$a(S) = -\min\{\, i \mid (\omega_S)_i \neq 0\},$$

see (Bruns and Herzog 1997, p. 141) and (Villarreal 2001, Proposition 4.2.3). This formula appli
is normal because normal monomial subrings are Cohen-Macaulay (Hochster 1972). If S is norma
by a formula of Danilov-Stanley, see (Bruns and Herzog 1997, Theorem 6.3.5) and (Danilov 197
canonical module of S is the ideal given by

$$\omega_S = (\{x^a t^b \mid (a, b) \in \mathbb{N}\mathcal{B} \cap (\mathbb{R}_+\mathcal{B})^{\mathrm{o}}\}),$$

where $\mathcal{B} = \{(w_1, 1), \ldots, (w_r, 1)\}$ and $(\mathbb{R}_+\mathcal{B})^{\mathrm{o}}$ is the interior of $\mathbb{R}_+\mathcal{B}$ relative to $aff(\mathbb{R}_+\mathcal{B})$, the
hull of $\mathbb{R}_+\mathcal{B}$. In our case, $aff(\mathbb{R}_+\mathcal{B}) = \mathbb{R}^{n+1}$.

The next theorem complements a result of (Brennan et al. 2008). In loc. cit. a somewhat di
expression for the canonical module and a-invariant are shown. Our expressions are simpler becaus
only involve the vertices of a certain polytope, while in (Brennan et al. 2008) some other paramet
involved.

THEOREM 4.1. *Let $\mathcal{C}$ be a clutter with incidence matrix A, let $v_1, \ldots, v_q$ be the columns of A,*
$w_1, \ldots, w_r$ be the set of all $\alpha \in \mathbb{N}^n$ such that $0 \leq \alpha \leq v_i$ for some i. If the system $x \geq 0; xA \leq \mathbf{1}$
integer rounding property and $\ell_1, \ldots, \ell_m$ are the non-zero vertices of $P = \{x \mid x \geq 0; xA \leq \mathbf{1}\}$, th
subring $S = K[x^{w_1}t, \ldots, x^{w_r}t]$ is normal, the canonical module of S is given by

$$\omega_S = \left(\left\{ x^a t^b \,\middle|\, (a, b) \begin{pmatrix} -\ell_1 & \cdots & -\ell_m & e_1 & \cdots & e_n \\ 1 & \cdots & 1 & 0 & \cdots & 0 \end{pmatrix} > 0 \right\}\right),$$

and the a-invariant of S is equal to $-(\max_i\{\lfloor|\ell_i|\rfloor\} + 1)$. Here $|\ell_i| = \langle \ell_i, \mathbf{1}\rangle$.

PROOF. Note that in Eq. (3) we regard the ℓ_i's and $e_i's$ as column vectors. The normality of S f
from Theorem 2.1. Let $P = \{x \mid x \geq 0;\ xA \leq \mathbf{1}\}$ and let $T(P)$ be its antiblocking polyhedron

$$T(P) := \{z \mid z \geq 0; \langle z, x\rangle \leq 1 \ \text{ for all } \ x \in P\}.$$

By the finite basis theorem (Schrijver 1986), we can write

$$P = \{z \mid z \geq 0; \langle z, w_i\rangle \leq 1\, \forall i\} = \mathrm{conv}(\ell_0, \ell_1, \ldots, \ell_m),$$

where $\ell_1, \ldots, \ell_m, \ell_0$ are the vertices of P and $\ell_0 = 0$. Notice that the vertices of P are in $\mathbb{Q}^n$

Using Eq. (4) again and noticing that $\langle \ell_i, w_j \rangle \leq 1$ for all i, j, we get

$$\mathbb{R}_+^n \cap (\mathrm{conv}(\ell_0, \ldots, \ell_m) + \mathbb{R}_+\{-e_1, \ldots, -e_n\}) = \{z \mid z \geq 0; \langle z, w_i \rangle \leq 1 \, \forall i\}.$$

Hence, using this equality and (Schrijver 1986, Theorem 9.4), we obtain

$$\mathbb{R}_+^n \cap (\mathrm{conv}(w_1, \ldots, w_r) + \mathbb{R}_+\{-e_1, \ldots, -e_n\}) = \{z \mid z \geq 0; \langle z, \ell_i \rangle \leq 1 \, \forall i\}. \tag{6}$$

By (Fulkerson 1971, Theorem 8), we have the equality

$$\mathrm{conv}(w_1, \ldots, w_r) = \mathbb{R}_+^n \cap (\mathrm{conv}(w_1, \ldots, w_r) + \mathbb{R}_+\{-e_1, \ldots, -e_n\}).$$

Therefore, using Eqs. (5) and (6), we conclude the following duality:

$$\begin{aligned} P = \{x \mid x \geq 0; \langle x, w_i \rangle \leq 1 \, \forall i\} &= \mathrm{conv}(\ell_0, \ell_1, \ldots, \ell_m), \\ \mathrm{conv}(w_1, \ldots, w_r) &= \{x \mid x \geq 0; \langle x, \ell_i \rangle \leq 1 \forall i\} = T(P). \end{aligned} \tag{7}$$

We set $\mathcal{B} = \{(w_1, 1), \ldots, (w_r, 1)\}$. Note that $\mathbb{Z}\mathcal{B} = \mathbb{Z}^{n+1}$. From Eq. (7), it is seen that

$$\mathbb{R}_+\mathcal{B} = H_{e_1}^+ \cap \cdots \cap H_{e_n}^+ \cap H_{(-\ell_1,1)}^+ \cap \cdots \cap H_{(-\ell_m,1)}^+. \tag{8}$$

Here H_a^+ denotes the closed halfspace $H_a^+ = \{x \mid \langle x, a \rangle \geq 0\}$ and H_a stands for the hyperplane through the origin with normal vector a. Notice that

$$H_{e_1} \cap \mathbb{R}_+\mathcal{B}, \ldots, H_{e_n} \cap \mathbb{R}_+\mathcal{B}, H_{(-\ell_1,1)} \cap \mathbb{R}_+\mathcal{B}, \ldots, H_{(-\ell_m,1)} \cap \mathbb{R}_+\mathcal{B}$$

are proper faces of $\mathbb{R}_+\mathcal{B}$. Hence, from Eq. (8), we get that a vector (a, b), with $a \in \mathbb{R}^n$, $b \in \mathbb{R}$, is in the relative interior of $\mathbb{R}_+\mathcal{B}$ if and only if the entries of a are positive and $\langle (a, b), (-\ell_i, 1) \rangle > 0$ for all i. Thus, the required expression for ω_S, i.e., Eq. (3), follows using the normality of S and the Danilov-Stanley formula given in Eq. (2).

It remains to prove the formula for $a(S)$, the a-invariant of S. Consider the vector $(\mathbf{1}, b_0)$, where $b_0 = \max_i\{\lfloor |\ell_i| \rfloor\} + 1$. Using Eq. (3), it is not hard to see (by direct substitution of $(\mathbf{1}, b_0)$) that the monomial $x^{\mathbf{1}} t^{b_0}$ is in ω_S. Thus, from Eq. (1), we get $a(S) \geq -b_0$. Conversely, if the monomial $x^a t^b$ is in ω_s, then again from Eq. (3) we get $\langle (-\ell_i, 1), (a, b) \rangle > 0$ for all i and $a_i \geq 1$ for all i, where $a = (a_i)$. Hence,

$$b > \langle a, \ell_i \rangle \geq \langle \mathbf{1}, \ell_i \rangle = |\ell_i| \geq \lfloor |\ell_i| \rfloor.$$

Since b is an integer, we obtain $b \geq \lfloor |\ell_i| \rfloor + 1$ for all i. Therefore, $b \geq b_0$, i.e., $\deg(x^a t^b) = b \geq b_0$. As

MONOMIAL SUBRINGS OF CLIQUES OF PERFECT GRAPHS

Let S be a set of vertices of a graph G, the *induced subgraph* $\langle S\rangle$ is the maximal subgraph of (vertex set S. A *clique* of a graph G is a subset of the set of vertices that induces a complete subgraph. be a graph with vertex set $X=\{x_1,\ldots,x_n\}$. A *colouring* of the vertices of G is an assignment of c to the vertices of G in such a way that adjacent vertices have distinct colours. The *chromatic numbe* is the minimal number of colours in a colouring of G. A graph is *perfect* if, for every induced sul H, the chromatic number of H equals the size of the largest complete subgraph of H. Let S be a sul the vertices of G. The set S is called *independent* if no two vertices of S are adjacent.

For use below we consider the empty set as a clique whose vertex set is empty. The *suppo* monomial x^a is given by $\mathrm{supp}(x^a)=\{x_i\mid a_i>0\}$. Note that $\mathrm{supp}(x^a)=\emptyset$ if and only if $a=0$.

THEOREM 4.2. *Let G be a perfect graph and let $S=K[x^{\omega_1}t,\ldots,x^{\omega_r}t]$ be the subring generated square-free monomials x^at such that $supp(x^a)$ is a clique of G. Then, the canonical module of S i: by*

$$\omega_S=\left(\left\{x^at^b\Big|\ (a,b)\begin{pmatrix}-a_1&\cdots&-a_m&e_1&\cdots&e_n\\1&\cdots&1&0&\cdots&0\end{pmatrix}\geq\mathbf{1}\right\}\right),$$

where $a_1,\ldots,a_m$ are the characteristic vectors of the maximal independent sets of G, and the a-inv of S is equal to $-(\max_i\{|a_i|\}+1)$.

PROOF. Let $v_1,\ldots,v_q$ be the set of characteristic vectors of the maximal cliques of G. No $w_1,\ldots,w_r$ is the set of all $\alpha\in\mathbb{N}^n$ such that $\alpha\leq v_i$ for some i. Since G is a perfect gra (Korte and Vygen 2000, Theorem 16.14) we have the equality

$$P=\{x|x\geq 0;\,xA\leq\mathbf{1}\}=\mathrm{conv}(a_0,a_1,\ldots,a_p),$$

where $a_0=0$ and $a_1,\ldots,a_p$ are the characteristic vectors of the independent sets of G. We may a that $a_1,\ldots,a_m$ correspond to the maximal independent sets of G. Furthermore, since P has only i vertices, by a result of (Lovász 1972), the system $x\geq 0;\,xA\leq\mathbf{1}$ is totally dual integral, i.e., the mi in the LP-duality equation

$$\max\{\langle\alpha,x\rangle\,|\,x\geq 0;\,xA\leq\mathbf{1}\}=\min\{\langle y,\mathbf{1}\rangle\,|\,y\geq 0;\,Ay\geq\alpha\}$$

has an integral optimum solution y for each integral vector α with finite minimum. In particu system $x\geq 0;\,xA\leq\mathbf{1}$ satisfies the integer rounding property. Therefore, the result follows readil Theorem 4.1.

For use below recall that a graph G is called *unmixed* if all maximal independent sets of G ha same cardinality. Unmixed bipartite graphs have been nicely characterized in (Villarreal 2007).

COROLLARY 4.3. *Let G be a connected bipartite graph, and let $I=I(G)$ be its edge ideal. Th extended Rees algebra $R[It,t^{-1}]$ is a Gorenstein standard K-algebra if and only if G is unmixed.*

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RESUMO

Seja $\mathcal{C}$ uma desordem (família de Sperner) e seja A sua matriz de incidência. Se o sistema linear $x \geq 0; xA \leq \mathbf{1}$ tem a propriedade do arredondamento inteiro, fornecemos a descrição do módulo canônico e do a-invariante de certos subaneis monomiais associados a $\mathcal{C}$. Se a desordem é um grafo conexo, descreve-se quando o supra-mencionado sistema linear tem a propriedade do arredondamento inteiro em termos combinatórios e algébricos, usando a teoria dos grafos e a teoria das álgebras de Rees. Como consequência, mostra-se que a álgebra de Rees estendida do ideal de arestas de um grafo bipartido é um anel de Gorenstein se e somente se o grafo é de altura pura.

Palavras-chave: módulo canônico, a-invariante, ideal normal, grafo perfeito, cliques maximais, álgebra de Rees, anel de Erhart, propriedade do arredondamento inteiro.

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