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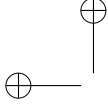
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## Injectivity of the Dirichlet-to-Neumann Functional and the Schwarzian Derivative

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### ABSTRACT

In this article, we show the relation between the Schwartz kernels of the Dirichlet-to-Neumann operator associated to the metrics  $g_0$  and  $h = F^*(e^{2\varphi}g_0)$  on the circular annulus  $A_R$ , and the Schwarzian Derivative of the argument function  $f$  of the restriction of the diffeomorphism  $F$  to the boundary of  $A_R$ .

**Key words:** annulus, Dirichlet-to-Neumann Functional, Schwarzian Derivative.

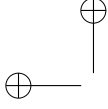
### 1 INTRODUCTION

Let  $\mathcal{M}(\overline{\Omega})$  denote the space of all Riemannian metrics on a compact manifold  $\overline{\Omega}$ , with  $C^\infty$  boundary  $\partial\Omega$ , and denote by  $\mathcal{O}_p(\partial\Omega)$  the space of continuous linear operators acting on  $C^\infty(\partial\Omega)$ .

The Dirichlet-to-Neumann functional  $\Lambda$  is a mapping from  $\mathcal{M}(\overline{\Omega})$  into  $\mathcal{O}_p(\partial\Omega)$  such that, for  $g \in \mathcal{M}(\overline{\Omega})$ ,  $\Lambda_g$  takes Dirichlet boundary values to Neumann boundary values. More precisely,  $\Lambda_g(\varphi)$  is the unique solution of the Dirichlet problem  $\Delta_g u = 0$  in  $\Omega$ ,  $u|_{\partial\Omega} = \varphi \in C^\infty(\partial\Omega)$ ,  $\Lambda_g(\varphi) = du(v_g) \in C^\infty(\partial\Omega)$ , where  $\Delta_g$  (resp.  $v_g$ ) is the Laplace-Beltrami operator (resp. unit normal vector field) associated to the metric  $g$ . The study of this functional goes back to the seminal work of (Calderón 1980).

It is known (Lee and Uhlmann 1989) that  $\Lambda_g$  is in fact an elliptic self-adjoint pseudo-differential operator of order one, whose principal symbol is  $|\xi|_{h_0}$ ,  $\xi \in T^*\partial\Omega$ , and  $h_0 := g|_{\partial\Omega}$ .

Let  $\mathcal{D}(\overline{\Omega})$  be the group of diffeomorphism of  $\overline{\Omega}$ . The semi-direct product  $\mathcal{D}(\overline{\Omega}) \ltimes C^\infty(\overline{\Omega})$  (Poisson 1987) of the groups  $\mathcal{D}(\overline{\Omega})$  and  $C^\infty(\overline{\Omega})$  defined by



provides a natural right action on  $\mathcal{M}(\overline{\Omega})$ , given by

$$g \bullet (F, \varphi) = F^* e^{2\varphi} g,$$

where  $F^*$  denotes the pull-back of  $F$ .

The main obstruction to injectivity, in the two-dimensional case, is the semidirect product of the groups of diffeomorphisms that restricts to the identity on the boundary, and the Abelian group of real-valued functions that equals zero on it. In fact, as formula (2.1) shows, the Dirichlet-to-Neumann Functional is constant on the orbits determined by  $\mathcal{D}_0(\overline{\Omega}) \ltimes \mathcal{C}_0^\infty(\overline{\Omega})$ ; this is a normal subgroup of  $\mathcal{D}(\overline{\Omega}) \ltimes \mathcal{C}^\infty(\overline{\Omega})$ .

With respect to the determination of the metric  $g$  from the Dirichlet-to-Neumann Operator, we recommend the papers (Lee and Uhlmann 1989), (Lassas and Uhlmann 2001) and (Lassas et al. 2003). In these papers, they solve, in a more general setting, the problem of recovering the manifold and the metric.

In the case of a fixed annulus, all metrics can be written as  $h = F^* e^{2\varphi} g_0$ , for  $g_0$  coming from the pull-back of the euclidean metric in the annulus of radius 1 and  $R^2$ ,  $R > 1$ . We prove, in this special case, that the equality of the Dirichlet-to-Neumann Operators associated to both metrics  $h$  and  $g_0$  gives us a relation involving the Schwarzian derivative of  $f$  ( $f$  the lifting to  $\mathbb{R}$  of the restriction to the boundary of the diffeomorphism  $F$ ).

Furthermore, we also show that the conformal factor restricted to the boundary of the annulus is determined by  $f$ .

More precisely, we shall prove in Section 2 that, if  $\Omega$  is the annulus

$$A_R = \left\{ z \in \mathbb{C} : \frac{1}{R} < |z| < R \right\},$$

$g_0 \in \mathcal{M}(\overline{A_R})$  is conformal to the euclidean metric,  $h = F^*(e^{2\varphi} g_0)$ , where  $F \in \mathcal{D}(\overline{A_R})$  and  $\varphi \in \mathcal{C}^\infty(\overline{A_R})$ ; the equality of the Schwartz kernels of  $\Lambda_{g_0}$  and of  $\Lambda_h$  implies that the argument function  $f$ , of the restriction of  $F$  to  $\partial A_R$ , satisfies the differential equation

$$S(f) = \lambda(R)([f']^2 - 1) \quad \text{and} \quad e^{-\varphi \circ F} = f',$$

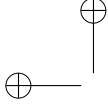
and  $S(f)$  denotes the Schwarzian Derivative of  $f$ . It follows that, if  $\lambda(R) \geq 0$ , then  $f' = 1$ ,  $f(\theta) = \theta + c$  and  $\varphi$  equals zero on the boundary.

## 2 GEOMETRIC FORMULATION

Here on we will denote by  $\mathcal{N}_g$  the Schwartz kernel of  $\Lambda_g$ . We start with two lemmas.

LEMMA 2.1. *Given a two-dimensional compact manifold  $\overline{\Omega}$  with  $\mathcal{C}^\infty$  boundary and  $F \in \mathcal{D}(\overline{\Omega})$ ,  $\varphi \in \mathcal{C}^\infty(\overline{\Omega})$  and  $g \in \mathcal{M}(\overline{\Omega})$ , we have*

$$\Lambda_{F^* e^{2\varphi} g} = F^* \circ \Lambda_g \circ F^{-1*} \quad (2.1)$$



LEMMA 2.2. *Let  $\overline{\Omega}$  be a two-dimensional compact manifold with  $\mathcal{C}^\infty$  boundary,  $h = F^*(e^{2\phi}g)$ ,  $F \in \mathcal{D}(\overline{\Omega})$ ,  $\varphi \in \mathcal{C}^\infty(\overline{\Omega})$ ,  $g \in \mathcal{M}(\overline{\Omega})$  and  $E$  the unitary vector field to  $\partial\Omega$ , with respect to the metric  $g$ . Then,*

$$\begin{cases} \mathcal{N}_{F^*g}(x, y) = \mathcal{N}_g(F(x), F(y))F'(y), \\ \mathcal{N}_{e^{2\varphi}g}(x, y) = e^{-\varphi(x)}\mathcal{N}_g(x, y). \end{cases}$$

where  $F'$  denotes the real, valued function on  $\partial\Omega$  such that  $F_*E = F'E \circ F$ .

PROOF. Let  $x \in \partial\Omega$ ,

$$\begin{aligned} \Lambda_{F^*g}(\psi)(x) &= \int_{\partial\Omega} \mathcal{N}_{F^*g}(x, y)\psi(y)v_h(y) \\ &= F^* \circ \Lambda_g \circ (F^{-1})^* \circ \psi(x) \\ &= F^* \circ \Lambda_g(\psi \circ F^{-1})(x) \\ &= \Lambda_g(\psi \circ F^{-1})(F(x)) \\ &= \int_{\partial\Omega} \mathcal{N}_g(F(x), z)(\psi \circ F^{-1})(z)v_h(z) \end{aligned}$$

changing variables  $F(y) = z$  we get:

$$\begin{aligned} \Lambda_{F^*g}(\psi)(x) &= \int_{\partial\Omega} \mathcal{N}_g(F(x), F(y))\psi(y)F^*(v_h)(y) \\ \Lambda_{F^*g}(\psi)(x) &= \int_{\partial\Omega} \mathcal{N}_g(F(x), F(y))\psi(y)F'(y)(v_h)(y) \end{aligned}$$

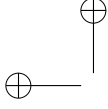
where  $F'$  denotes the unique real, valued function defined on  $\partial\Omega$  such that

$$F_*E = F'E \circ F$$

and  $E$  is the tangent unitary vector field on  $\partial\Omega$  such that  $h(E, E) = 1$  and  $v_h(E) = +1$ . The equation means at every point  $p \in \partial\Omega$  the following:  $F_*E(p)$  and  $E \circ F(p)$  belong to the same one-dimensional tangent space  $T_{F(p)}(\partial\Omega)$ ; consequently, the first one is a real multiple of the second. This multiple is unique and it is equal to  $F'(p)$ .

For the second equality,

$$\begin{aligned} \Lambda_{e^{2\varphi}g}(\psi)(x) &= \int_{\partial\Omega} \mathcal{N}_{e^{2\varphi}g}(x, y)\psi(y)v_h(y) \\ &= e^{-\varphi(x)}\Lambda_g(\psi)(x) \end{aligned}$$



The next Lemma establish, a relation between  $\mathcal{N}_g$  and the Green function  $G(z, z')$  of the Laplacian  $\Delta_g$  with Dirichlet condition on  $\partial\Omega$  (Guillarmou and Sá Barreto 2009).

LEMMA 2.3. *The Schwartz kernel  $\mathcal{N}_g(y, y')$  of  $\Lambda_g$  is given for  $y, y' \in \partial\Omega$ ,  $y \neq y'$ , by*

$$\mathcal{N}_g(y, y') = \partial_n \partial_{n'} G(z, z') \big|_{z=y, z'=y'}$$

where  $\partial_n, \partial_{n'}$  are, respectively, the inward pointing vector fields to the boundary in variable  $z$  and  $z'$ .

PROOF. Let  $x$  be the distance function to the boundary in  $\overline{\Omega}$ ; it is smooth in a neighborhood of  $\partial\Omega$  and the normal vector field to the boundary is the gradient  $\partial_n = \nabla^g x$  of  $x$ . The flow  $e^{t\partial_n}$  of  $\nabla^g x$  induces a diffeomorphism  $\phi : [0, \epsilon)_t \times \partial\Omega \rightarrow \phi([0, \epsilon)_t \times \partial\Omega)$  defined by  $\phi(t, y) := e^{t\partial_n}(y)$ , and we have  $x(\phi(t, y)) = t$ . This induces natural coordinates  $z = (x, y)$  near the boundary, these are normal geodesic coordinates. The function  $u$  is the unique solution of the Dirichlet problem  $\Delta_g u = 0$  in  $\Omega$ , and  $u|_{\partial\Omega} = \varphi \in C^\infty(\partial\Omega)$  can be obtained by taking

$$u(z) := \chi(z) - \int_{\overline{\Omega}} G(z, z') (\Delta_g \chi)(z') dz'$$

where  $\chi$  is any smooth function on  $\overline{\Omega}$  such that  $\chi = \varphi + O(x^2)$ . Now, using Green's formula and  $\Delta_g(z)G(z, z') = \delta(z - z') = \Delta_g(z')G(z, z')$ , where  $\delta(z - z')$  is the Dirac mass on the diagonal, we obtain for  $z \in \Omega$

$$\begin{aligned} u(z) &= \int_{\partial\Omega} (\partial_{n'} G(z, z') \chi(z')) \big|_{z'=y'} dy' - \int_{\partial\Omega} (G(z, z') (\partial_n \chi)(z')) \big|_{z'=y'} dy' \\ u(z) &= \int_{\partial\Omega} (\partial_{n'} G(z, z')) \big|_{z'=y'} \varphi(y') dy'. \end{aligned}$$

We have Taylor expansion  $u(x, y) = \varphi(y) + x \Lambda_g \varphi(y) + O(x^2)$  near the boundary. Let  $y \in \partial\Omega$  and take  $\phi \in C^\infty(\Omega)$  supported near  $y$ . Thus, pairing with  $\phi \in C^\infty(\partial\Omega)$  gives

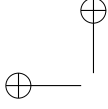
$$\int_{\partial\Omega} u(x, y) \phi(y) dy = \int_{\partial\Omega} \varphi(y) \phi(y) dy - x \int_{\partial\Omega} \phi(y) \Lambda_g \varphi(y) dy + O(x^2). \quad (2.2)$$

Now taking  $\phi$  with support disjoint to the support of  $\varphi$ , thus  $\phi\varphi = 0$ , and differentiating (2.2) in  $x$ , we see, in view of the fact that Green's function  $G(z, z')$  is smooth outside the diagonal, that

$$\int_{\partial\Omega} \phi(y) \Lambda_g \varphi(y) dy = \int_{\partial\Omega} \int_{\partial\Omega} (\partial_n \partial_{n'} G(z, z')) \big|_{z=y, z'=y'} \varphi(y') \phi(y) dy dy',$$

which proves the claim. □

Let  $(\partial\Omega, g)$  be a Riemannian manifold, and let us denote by  $d_g(x, y)$  the geodesic distance between  $x, y \in \partial\Omega$ , and we denote  $[d_g(x, y)]^2 = d_g^2(x, y)$ . If



COROLLARY 2.4. *If  $\Lambda_{F^*e^{2\psi}g} = \Lambda_g$  then  $e^{-\phi \circ F(x)} = F'(x)$  for  $x \in \partial\Omega$ .*

PROOF. Using the equalities of the Dirichlet-to- Neumann operators and Lemma 2.2 we have

$$\frac{d_g^2(x, y)}{d_g^2(F(x), F(y))} e^{-\phi \circ F(x)} d_g^2(F(x), F(y)) \mathcal{N}_g(F(x), F(y)) F'(y) = d_g^2(x, y) \mathcal{N}_g(x, y)$$

On the other hand, since

$$\lim_{y \rightarrow x} \frac{d_g(x, y)}{d_g(F(x), F(y))} = \frac{1}{F'(x)},$$

then, taking the limit when  $y \rightarrow x$  in (2.3), the demonstration follows.

REMARK 2.5. From Lemma 2.2 and Corollary 2.4 we have the following equation,

$$\mathcal{N}_g(F(x), F(y)) F'(x) F'(y) = \mathcal{N}_g(x, y).$$

The set of solutions of equation (2.4) is a group with multiplication law given by composition of functions that is, if  $F$  and  $G$  are solutions of the equation (2.4), then,  $G \circ F$  is solution of (2.4). In fact,

$$\begin{aligned} & \mathcal{N}_g((G \circ F)(x), (G \circ F)(y)) (G \circ F)'(x) (G \circ F)'(y) \\ &= \mathcal{N}_g(G(F(x)), G(F(y))) G'(F(x)) G'(F(y)) F'(x) F'(y) \\ &= \mathcal{N}_g(F(x), F(y)) F'(x) F'(y) = \mathcal{N}_g(x, y). \end{aligned}$$

In what follows, we use an explicit formula for the Green’s Function of  $\Delta_{g_0}$  on the annulus  $A_R$  and Guisa 1998). There,  $g_0$  is given in polar coordinates by:

$$g_0 = \frac{1}{2} \left( 1 + \frac{1}{\rho^2} \right) (d\rho^2 + \rho^2 d\theta^2),$$

and it is conformal to the euclidean metric, with conformal factor  $f(\rho, \theta) = \frac{1}{2} \left( 1 + \frac{1}{\rho^2} \right)$ .

Then, the normal derivative of  $u \in C^\infty(\overline{A_R})$ , with respect to  $g_0$  on  $|z| = R$ , is:

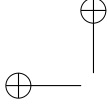
$$\left. \frac{\partial u}{\partial \nu_{g_0}} \right|_{\rho=R} = \frac{2R^2}{1+R^2} \left. \frac{\partial u}{\partial \rho} \right|_{\rho=R}$$

Analogously, the normal derivative of  $u$ , with respect to  $g_0$  on  $|z| = \frac{1}{R}$ , is:

$$\left. \frac{\partial u}{\partial \nu_{g_0}} \right|_{\rho=\frac{1}{R}} = \frac{-2}{1+R^2} \left. \frac{\partial u}{\partial \rho} \right|_{\rho=\frac{1}{R}}$$

The Green’s function of  $\overline{A_R}$  is given by

$$G(z, \zeta) = \ln(rR) + \sum_{n=1}^{\infty} \frac{1}{2n} \frac{r^n + (-r)^{-n}}{R^{2n} - (-R)^{-2n}} \frac{\rho^n + (-\rho)^{-n}}{R^{2n} - (-R)^{-2n}} \cos n(\theta - \alpha)$$



where  $z = \rho e^{i\theta}$ ,  $\frac{1}{R} \leq \rho \leq R$ ,  $\zeta = r e^{i\alpha}$ ,  $\frac{1}{R} < r < R$ ,  $0 < \theta < 2\pi$ ,  $0 < \alpha < 2\pi$ .

LEMA 2.6. *The Schwartz kernel of  $\Lambda_{g_0}$ ,  $g_0 \in \mathcal{M}(\overline{A_R})$  being of the form (2.5), is*

$$\begin{aligned} \mathcal{N}_{g_0}(R e^{i\theta}, R e^{i\alpha}) &= \frac{4R^2}{(1+R^2)^2} \sum_{n=1}^{\infty} n \frac{R^n - (-R)^{-n}}{R^n} \frac{R^n - (-R)^{-n}}{R^n + (-R)^{-n}} \cos n(\theta - \alpha) \\ &+ \frac{2R^2}{(1+R^2)^2} \frac{1}{1 - \cos(\theta - \alpha)} - \frac{4R^4}{(1+R^2)^2} \frac{R^{-4} \cos(\theta - \alpha) + \cos(\theta - \alpha) + 2R^{-2}}{(R^{-2} + 2 \cos(\theta - \alpha) + R^2)^2} \end{aligned} \quad (2.6)$$

$$\begin{aligned} \mathcal{N}_{g_0}(R^{-1} e^{i\theta}, R^{-1} e^{i\alpha}) &= \frac{4R^2}{(1+R^2)^2} \sum_{n=1}^{\infty} n \frac{\left(\frac{1}{R}\right)^n - \left(-\frac{1}{R}\right)^{-n}}{R^n} \frac{\left(\frac{1}{R}\right)^n - \left(-\frac{1}{R}\right)^{-n}}{R^n + (-R)^{-n}} \cos n(\theta - \alpha) \\ &+ \frac{2R^2}{(1+R^2)^2} \frac{1}{1 - \cos(\theta - \alpha)} - \frac{4}{(1+R^2)^2} \cdot \frac{R^4 \cos(\theta - \alpha) + \cos(\theta - \alpha) + 2R^2}{(R^2 + 2 \cos(\theta - \alpha) + R^{-2})^2} \end{aligned} \quad (2.7)$$

The equality above is in the distributions sense.

PROPOSITION 2.7. *Let  $p, q \in \partial A_R$ , then,*

$$\lim_{p \rightarrow q} d_{\bar{g}_{eucl}}^2(p, q) \mathcal{N}_{g_0}(p, q) = \frac{-4R^4}{(1+R^2)^2} \quad \text{on } |z| = R \quad (2.8)$$

and

$$\lim_{p \rightarrow q} d_{\bar{g}_{eucl}}^2(p, q) \mathcal{N}_{g_0}(p, q) = \frac{-4}{(1+R^2)^2} \quad \text{on } |z| = \frac{1}{R}, \quad (2.9)$$

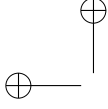
where  $d_{\bar{g}_{eucl}}$  denotes the geodesic distance between  $p$  and  $q$  with respect to the Euclidean metric in  $\partial A_R$ .

PROOF. In order to prove equation (2.8), we write

$$a_n = \frac{(R^n - (-R)^{-n})^2}{R^n (R^n + (-R)^{-n})}.$$

Then, the sequence  $b_n = n(a_n - 1)$  has the following property:  $|b_n| < \frac{4n}{R^n} < C(k, R)n^{-k}$  for all  $n \in \mathbb{N}$  where  $C(k, R)$  is a constant that depends only on  $k$  and  $R$ . In fact,  $\frac{n^{k+1}}{R^n} < \frac{(k+1)!}{(\ln R)^{k+1}}$ . Hence, the series  $\sum_{n=1}^{\infty} b_n \cos n(\theta - \alpha)$  represents a  $\mathcal{C}^\infty$  function. On the other hand, using the Fourier series of the function  $f(x) = \ln(|\sin(\frac{x}{2})|)$ , with  $0 < x < \pi$ , we have that

$$\ln\left(|\sin\left(\frac{x}{2}\right)|\right) = -\left\{\ln 2 + \sum_{n=1}^{\infty} \frac{\cos nx}{n}\right\},$$



which implies:

$$\sum_{n=1}^{\infty} n \cos nx = -\frac{1}{1 - \cos x},$$

the equality being in the distributions sense.

Then, multiplying (2.6) by  $d_{\tilde{g}_{eucl}}^2(p, q)$  and taking the limit as  $q \rightarrow p$ , we get the following:

$$\lim_{\theta \rightarrow \alpha} \frac{-2R^2}{(1 + R^2)^2} \cdot \frac{R^2(\theta - \alpha)^2}{1 - \cos(\theta - \alpha)} = \frac{-4R^4}{(1 + R^2)^2}.$$

Analogously, we get (2.9).

REMARK 2.8. It follows from the proof of the Proposition (2.6) that the Schwartz kernel of  $\Lambda_{g_0}$  written as:

$$\mathcal{N}_{g_0}(Re^{i\theta}, Re^{i\alpha}) = H(Re^{i\theta}, Re^{i\alpha}) - \frac{2R^2}{(1 + R^2)^2} \cdot \frac{1}{1 - \cos(\theta - \alpha)} \quad \text{on } |z| = R$$

$$\mathcal{N}_{g_0}(R^{-1}e^{i\theta}, R^{-1}e^{i\alpha}) = H(R^{-1}e^{i\theta}, R^{-1}e^{i\alpha}) - \frac{2R^2}{(1 + R^2)^2} \cdot \frac{1}{1 - \cos(\theta - \alpha)} \quad \text{on } |z| = \frac{1}{R}$$

where  $H$  is a  $\mathcal{C}^\infty$  function given by

$$H(Re^{i\theta}, Re^{i\alpha}) = \frac{4R^2}{(1 + R^2)^2} \left\{ \sum_{n=1}^{\infty} b_n \cos n(\theta - \alpha) - \frac{R^{-2} \cos(\theta - \alpha) + R^2 \cos(\theta - \alpha) + 2}{(R^{-2} + 2 \cos(\theta - \alpha) + R^2)^2} \right\}.$$

TEOREMA 2.9. Let  $g_0$  be a metric as in (2.5),  $h = F^*(e^{2\varphi}g_0)$  where  $F \in \mathcal{D}(\overline{A_R})$ ,  $\varphi \in \mathcal{C}^\infty(\overline{A_R})$ ,  $F(Re^{i\theta}) = Re^{if(\theta)}$ . If  $\Lambda_h = \Lambda_{g_0}$ , then,

$$\begin{cases} S(f) = \lambda([f']^2 - 1) \\ e^{-\varphi \circ F} = f' \end{cases}$$

where  $S(f)$  denotes the Schwarzian Derivative of  $f$  (see (2.18) and the line right after it).

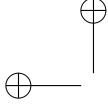
PROOF. Using the equality of the Dirichlet-to-Neumann operators, it follows from Lemma 2.2 that

$$\mathcal{N}_{F^*e^{2\varphi}g_0}(x, y) = e^{-\varphi \circ F(x)} \mathcal{N}_{g_0}(F(x), F(y)) F'(y) = \mathcal{N}_{g_0}(x, y).$$

Writing  $x = Re^{i\theta}$ ,  $y = Re^{i\alpha}$  and using (2.8), we have that

$$e^{-\varphi \circ F(Re^{i\theta})} \left\{ H(Re^{if(\theta)}, Re^{if(\alpha)}) - \frac{2R^2}{(1 + R^2)^2} \frac{1}{1 - \cos(f(\theta) - f(\alpha))} \right\} f'(\alpha)$$





On the other hand, we have from Corollary 2.4 and (2.8) that  $e^{-\varphi \circ F} = f'$  on the boundary. Hence,

$$\begin{aligned} & \left\{ H(Re^{if(\theta)}, Re^{if(\alpha)}) - \frac{2R^2}{(1+R^2)^2} \frac{1}{1 - \cos(f(\theta) - f(\alpha))} \right\} f'(\alpha) f'(\theta) \\ &= H(Re^{i\theta}, Re^{i\alpha}) - \frac{2R^2}{(1+R^2)^2} \frac{1}{1 - \cos(\theta - \alpha)}. \end{aligned}$$

We obtain, then,

$$\begin{aligned} & H(Re^{if(\theta)}, Re^{if(\alpha)}) f'(\alpha) f'(\theta) - H(Re^{i\theta}, Re^{i\alpha}) \\ &= \frac{2R^2}{(1+R^2)^2} \left\{ \frac{f'(\alpha) f'(\theta)}{1 - \cos(f(\theta) - f(\alpha))} - \frac{1}{1 - \cos(\theta - \alpha)} \right\}. \end{aligned} \quad (2.12)$$

Since the left hand side of the equation (2.12) is the  $C^\infty$  component of the Schwartz kernel, then if we take  $\alpha \rightarrow \theta$ , we get

$$H(R, R) \left\{ [f'(\theta)]^2 - 1 \right\}.$$

In what concerns the right hand side of the equation (2.12), we use Taylor expansion of order 4 of the expression in brackets, for  $\alpha$  near  $\theta$ ; we get, with  $\delta = \alpha - \theta$ ,

$$\frac{f'(\theta) \left\{ f'(\theta) + f''(\theta)\delta + \frac{f'''(\theta)\delta^2}{2!} \right\}}{[f'(\theta)]^2 \frac{\delta^2}{2!} + 3f'(\theta)f''(\theta)\frac{\delta^3}{3!} + \left\{ -[f'(\theta)]^4 + 3[f''(\theta)]^2 + 4f'(\theta)f'''(\theta) \right\} \frac{\delta^4}{4!}} - \frac{1}{\frac{\delta^2}{2!} - \frac{\delta^4}{4!}},$$

which can be written as follows,

$$\frac{\left\{ -[f'(\theta)]^2 - 3[f''(\theta)]^2 + 2f'(\theta)f'''(\theta) + [f'(\theta)]^4 \right\} \frac{\delta^4}{4!} + \mathcal{O}(\delta^5)}{[f'(\theta)]^2 \frac{\delta^4}{4} + \mathcal{O}(\delta^5)}. \quad (2.13)$$

Since the limit exists, when  $\delta \rightarrow 0$ , we obtain from (2.12) and (2.13) that

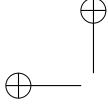
$$3! \frac{(1+R^2)^2}{2R^2} H(R, R) \left\{ [f'(\theta)]^2 - 1 \right\} = [f'(\theta)]^2 - 1 - 3 \left[ \frac{f''(\theta)}{f'(\theta)} \right]^2 + 2 \frac{f'''(\theta)}{f'(\theta)},$$

which implies:

$$\left\{ [f'(\theta)]^2 - 1 \right\} \left\{ 3! \frac{(1+R^2)^2}{2R^2} H(R, R) - 1 \right\} = -3 \left[ \frac{f''(\theta)}{f'(\theta)} \right]^2 + 2 \frac{f'''(\theta)}{f'(\theta)},$$

or, more precisely:

$$\left\{ [f'(\theta)]^2 - 1 \right\} \left\{ 3! \frac{(1+R^2)^2}{2R^2} H(R, R) - 1 \right\} = 2 \left[ \frac{f''(\theta)}{f'(\theta)} \right]' - \left[ \frac{f''(\theta)}{f'(\theta)} \right]^2. \quad (2.14)$$



Then, it, follows that

$$2\lambda(R) = \frac{4!}{2} \left\{ \sum_{n=1}^{\infty} b_n - \frac{R^2}{(1+R^2)^2} \right\} - 1,$$

where

$$b_n = n \left\{ \frac{-3(-1)^n + \frac{1}{R^{2n}}}{R^{2n} + (-1)^n} \right\}.$$

From equations (2.14) and (2.15) we have that

$$\left\{ [f'(\theta)]^2 - 1 \right\} 2\lambda(R) = 2 \left[ \frac{f''(\theta)}{f'(\theta)} \right]' - \left[ \frac{f''(\theta)}{f'(\theta)} \right]^2,$$

which implies:

$$\left\{ [f'(\theta)]^2 - 1 \right\} \lambda(R) = \left[ \frac{f''(\theta)}{f'(\theta)} \right]' - \frac{1}{2} \left[ \frac{f''(\theta)}{f'(\theta)} \right]^2 = \frac{f'''(\theta)}{f'(\theta)} - \frac{3}{2} \left[ \frac{f''(\theta)}{f'(\theta)} \right]^2.$$

The right-hand side of (2.18) is called the Schwarzian Derivative  $S(f)$  of  $f$  (Navas 2007).

REMARK 2.10. The numerical study of  $\lambda(R)$  defined in (2.16) is done in Mendoza et al. 2009.

COROLLARY 2.11. *The solution of the equation (2.18) for  $\lambda(R) \geq 0$  is  $f'(\theta) = 1$ .*

PROOF. Making the change of variables:  $y(\theta) = \ln(f'(\theta))$ , the equation (2.18) becomes

$$\{e^{2y(\theta)} - 1\} \lambda(R) = y''(\theta) - \frac{1}{2} [y'(\theta)]^2,$$

that is,

$$y'' = \frac{1}{2} [y']^2 + \lambda \{e^{2y(\theta)} - 1\}.$$

Since  $f(\theta + 2\pi) = f(\theta) + 2\pi$ , we have that  $f''$  and  $f'$  are periodic of period  $2\pi$ . Then, integrating between 0 and  $2\pi$  we obtain

$$0 = \frac{1}{2} \int_0^{2\pi} [y']^2 d\theta + \lambda \left\{ \int_0^{2\pi} e^{2y} d\theta - 2\pi \right\}.$$

On the other hand,

$$0 \leq \int_0^{2\pi} 1 \cdot f' d\theta \leq \left( \int_0^{2\pi} 1 d\theta \right)^{\frac{1}{2}} \cdot \left( \int_0^{2\pi} [f']^2 d\theta \right)^{\frac{1}{2}}$$

that is,

$$2\pi \leq \int_0^{2\pi} [f']^2 d\theta = \int_0^{2\pi} e^{2y} d\theta,$$

which implies that  $y' = 0$ . Because there is  $0 \leq \theta_0 \leq 2\pi$  such that  $f'(\theta_0) = 1$ , we get  $y = 0$ . The  $f' = 1$ .

It follows that  $F$  restricted to the exterior boundary is a rotation and  $\varphi$  equals zero there. The



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#### RESUMO

Neste artigo mostramos a relação entre os núcleos de Schwartz dos operadores Dirichlet-to-Neumann associados à métrica  $g_0$  e  $h = F^*(e^{2\phi} g_0)$ , no anel circular  $A_R$ , e a Derivada Schwarziana da função argumento  $f$ , da restrição do difeomorfismo  $F$  à fronteira de  $A_R$ .

**Palavras-chave:** anel, Funcional Dirichlet-Neumann, Derivada Schwarziana.

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