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# Flags of holomorphic foliations

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**ABSTRACT**

A flag of holomorphic foliations on a complex manifold $M$ is an object consisting of a finite number of singular holomorphic foliations on $M$ of growing dimensions such that the tangent sheaf of a fixed foliation is a subsheaf of the tangent sheaf of any of the foliations of higher dimension. We study some basic properties of these objects and, in $\mathbb{P}^n_{\mathbb{C}}$, $n \geq 3$, we establish some necessary conditions for a foliation of lower dimension to leave invariant foliations of codimension one. Finally, still in $\mathbb{P}^n_{\mathbb{C}}$, we find bounds involving the degrees of polar classes of foliations in a flag.

**Key words:** holomorphic foliations, polar varieties, invariant varieties.

## INTRODUCTION

Let $M$ be a complex manifold of dimension $m$ with tangent bundle $TM$. Let us denote by $\Theta = \mathcal{O}(TM)$ its tangent sheaf. A *singular holomorphic foliation*, or shortly *foliation*, is a coherent analytic subsheaf $\mathcal{T}$ of $\Theta$ that is involutive, which means that its stalks are invariant by the Lie bracket:

$$\left[\mathcal{T}_x, \mathcal{T}_x\right] \subset \mathcal{T}_x \ \forall\, x \in M.$$

The sheaf $\mathcal{T}$ is called the *tangent sheaf* of the foliation. We will denote a foliation by $\mathcal{F}$ or by $\mathcal{F}(\mathcal{T})$ when a reference to its tangent sheaf is needed.

The *singular set* of $\mathcal{F} = \mathcal{F}(\mathcal{T})$ is the analytic set $S = \mathrm{Sing}(\mathcal{F})$ defined as the singular set of the sheaf $\Theta/\mathcal{T}$, which on its turn consists of the points where the stalks are not free modules over the structural sheaf $\mathcal{O}$. The dimension of $\mathcal{F}$ is defined as the rank of the locally free part of $\mathcal{T}$. The locally free sheaf $\mathcal{T}_{|M\setminus S}$ is the sheaf of sections of a rank $p$ vector bundle $T$, which is a subbundle of $TM_{|M\setminus S}$. The involutiveness of $\mathcal{T}$ implies that the distribution of $p$-dimensional subspaces of $TM$ induced by $T$ on $M \setminus S$ is integrable, that is, there exists a regular holomorphic foliation on $M \setminus S$ such that the tangent space to the leaf passing through each point $x \in M \setminus S$ is $T_x$, the fiber of $T$ over $x$. This is the so-called Theorem of Frobenius.

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We say that a foliation is *reduced* if $\mathcal{T}$ is *full*. This means that, whenever $U \subset M$ is an open subset and $\mathbf{v}$ is a holomorphic section of $\Theta_{|U}$ such that $\mathbf{v}_x \in \mathcal{T}_x \; \forall \, x \in U \cap (M \setminus S)$, then $\mathbf{v}_x \in \mathcal{T}_x$ holds also for $x$ in $U \cap S$. We remark that, given an involutive sheaf $\mathcal{T}$, which induces a foliation with singular set $S$, there is a unique sheaf $\widehat{\mathcal{T}}$ that is both full and involutive, and such that $\widehat{\mathcal{T}}_{|M\setminus S} = \mathcal{T}_{|M\setminus S}$. We can therefore restrict our attention to full involutive sheaves as a way to avoid artificial singularities (see Baum and Bott 1972, Suwa 1998).

We can describe foliations in a dual way by means of differential forms. Let $\Omega = \mathcal{O}(T^*M)$ be the cotangent sheaf of the $m$-dimensional complex manifold $M$. Let $\mathcal{C}$ be an analytic coherent subsheaf of $\Omega$ of rank $p$, where $1 \leq p \leq n-1$, which satisfies the integrability condition:

$$d\mathcal{C}_x \subset (\Omega \wedge \mathcal{C})_x \; \forall \, x \in M \setminus \mathrm{Sing}(\mathcal{C}).$$

The sheaf $\mathcal{C}$ defines a singular holomorphic foliation denoted by $\mathcal{F} = \mathcal{F}(\mathcal{C})$. The singular set of $\mathcal{F}$, denoted by $\mathrm{Sing}(\mathcal{F})$, is equal to $\mathrm{Sing}(\Omega/\mathcal{C})$. On $M \setminus S(\mathcal{F})$, the sheaf $\mathcal{C}$ is the sheaf of sections of a rank $n-p$ vector subbundle of $T^*M$. The local sections of this subbundle are holomorphic 1-forms whose kernels, at each point $x$, define a subspace of $T_xM$, which is the tangent space at $x$ of a regular foliation of codimension $p$ on $M \setminus \mathrm{Sing}(\mathcal{F})$.

We say that $\mathcal{F} = \mathcal{F}(\mathcal{C})$ is *reduced* if $\mathcal{C}$ is *full*, that is, whenever $U \subset M$ is an open subset and $\omega$ is a holomorphic section of $\Omega_{|U}$ with the property that $\omega_x \in \mathcal{C}_x \; \forall \, x \in U \setminus \mathrm{Sing}(\mathcal{F})$, then $\omega_x \in \mathcal{C}_x$ for all points $x$ in $U$. The sheaf $\mathcal{C}$ is called the *conormal sheaf* of $\mathcal{F}$.

Both definitions of foliation that we have just introduced are related as follows. Let $\mathcal{T}$ be the tangent sheaf of a foliation $\mathcal{F} = \mathcal{F}(\mathcal{T})$ of dimension $p$. Define

$$\mathcal{T}^a \;=\; \left\{\omega \in \Omega\,;\; i_{\mathbf{v}}\omega = 0 \;\forall\, \mathbf{v} \in \mathcal{T}\right\} \;\subset\; \Omega,$$

where $i_{\mathbf{v}}$ denotes the contraction by the germ of vector field $\mathbf{v}$. We have that $\mathcal{T}^a$ is the conormal sheaf of a codimension $m-p$ foliation $\mathcal{F}^a = \mathcal{F}(\mathcal{T}^a)$. We clearly have $\mathrm{Sing}(\mathcal{F}^a) \subset \mathrm{Sing}(\mathcal{F})$. Furthermore, $\mathcal{F}^a$ is a reduced foliation.

Similarly, given $\mathcal{C}$ the conormal sheaf of a codimension $m-p$ foliation $\mathcal{F} = \mathcal{F}(\mathcal{C})$ on $M$, we define

$$\mathcal{C}^a \;=\; \left\{\mathbf{v} \in \Theta\,;\; i_{\mathbf{v}}\omega = 0 \;\forall\, \omega \in \mathcal{C}\right\} \;\subset\; \Theta.$$

Then, $\mathcal{C}^a$ is the tangent sheaf of a foliation $\mathcal{F}^a = \mathcal{F}(\mathcal{C}^a)$. We have that $\mathrm{Sing}(\mathcal{F}^a) \subset \mathrm{Sing}(\mathcal{F})$ and that $\mathcal{F}^a$ is a reduced foliation.

If $\mathcal{T}$ is the tangent sheaf of a foliation $\mathcal{F} = \mathcal{F}(\mathcal{T})$, then $\mathcal{T}^r = (\mathcal{T}^a)^a$ is the tangent sheaf of a reduced foliation $\mathcal{F}^r = \mathcal{F}(\mathcal{T}^r)$. As a consequence of the definitions, we have that $\mathcal{T}$ is a subsheaf of $\mathcal{T}^r$. Thus,

$$\mathrm{Sing}(\mathcal{F}^r) = \mathrm{Sing}(\Theta/\mathcal{T}^r) \subset \mathrm{Sing}(\Theta/\mathcal{T}) = \mathrm{Sing}(\mathcal{F}).$$

Furthermore, on $M \setminus \mathrm{Sing}(\mathcal{F})$, the regular foliation induced by $\mathcal{T}^r$ coincides with the one induced by $\mathcal{T}$. We also notice that reduced foliations are stable by this reduction process: if $\mathcal{T}$ is full, then $\mathcal{T}^r = \mathcal{T}$. In a similar way, a reduction process can be defined for a foliation defined by a conormal sheaf.

Let $\mathcal{F} = \mathcal{F}(\mathcal{T})$ be a foliation with tangent sheaf $\mathcal{T}$. If $\mathcal{F}$ is reduced, then $\mathrm{codim}\,\mathrm{Sing}(\mathcal{F}) \geq 2$. The converse holds when $\mathcal{T}$ is locally free. The equivalent is true for a foliation $\mathcal{F} = \mathcal{F}(\mathcal{C})$ defined by its conormal sheaf $\mathcal{C}$: If $\mathcal{F}$ is reduced, then $\operatorname{codim}\operatorname{Sing}(\mathcal{F}) \geq 2$, and both facts are equivalent when $\mathcal{C}$ is locally free. A proof for these facts can be found in [Su1, Lemma 5.1].

DEFINITION. The foliations $\mathcal{F}_{i_1}, \dots, \mathcal{F}_{i_k}$ on the $m$-dimensional holomorphic manifold $M$ form a *flag of foliations* if

(i) $\mathcal{F}_{i_l}$ is reduced $\forall\, l = 1, \dots, k$.

(ii) $1 \leq i_1 < \dots < i_k < m$ and $\dim \mathcal{F}_{i_l} = i_l \;\forall\, l = 1, \dots, k$.

(iii) $\mathcal{T}_{i_l}$ is a subsheaf of $\mathcal{T}_{i_{l+1}}$ $\forall\, l = 1, \dots, k-1$, where $\mathcal{T}_i$ is the tangent sheaf of $\mathcal{F}_i$.

In the definition, we say that $\mathcal{F}_{i_r}$ *leaves* $\mathcal{F}_{i_s}$ *invariant* or that $\mathcal{F}_{i_s}$ is *invariant by* $\mathcal{F}_{i_r}$ whenever $i_r < i_s$. This terminology is due to the fact that, for $x \in M \setminus (\operatorname{Sing}(\mathcal{F}_{i_r}) \cup \operatorname{Sing}(\mathcal{F}_{i_s}))$, the inclusion relation $T_x\mathcal{F}_{i_r} \subset T_x\mathcal{F}_{i_s}$ holds, giving that the leaves of $\mathcal{F}_{i_r}$ are contained in leaves of $\mathcal{F}_{i_s}$. We will use the notation $\mathcal{F}_{i_r} \prec \mathcal{F}_{i_s}$.

Let $\mathcal{F}_i$ and $\mathcal{F}_j$ be foliations of dimensions $i < j$ on a complex manifold $M$ such that $\mathcal{F}_i \prec \mathcal{F}_j$. The tangent sheaves of these foliations satisfy $\mathcal{T}_i \subset \mathcal{T}_j$ where " $\subset$ " means subsheaf. We produce conormal sheaves by taking annihilators: $\mathcal{C}_i = (\mathcal{T}_i)^a$ and $\mathcal{C}_j = (\mathcal{T}_j)^a$ . This gives $\mathcal{C}_j \subset \mathcal{C}_i$. By taking annihilators again, since our sheaves are full, we have $\mathcal{T}_i = \mathcal{C}_i^a \subset \mathcal{C}_j^a = \mathcal{T}_j$. That is, $\mathcal{F}_i \prec \mathcal{F}_j$ if and only if $\mathcal{T}_i \subset \mathcal{C}_j^a$. In terms of local sections, this is equivalent to the following: whenever $\mathbf{v}$ is a local vector field tangent to $\mathcal{F}_i$ and $\omega$ is a local integrable 1-form tangent to $\mathcal{F}_j$, then $i_{\mathbf{v}}\omega = 0$. As a consequence, since the singular set of a foliation is a proper analytic set, we have

PROPOSITION 1. *Let $\mathcal{F}_i$ and $\mathcal{F}_j$ be reduced foliations of dimensions $i < j$ on a complex manifold $M$. Then, $\mathcal{F}_i \prec \mathcal{F}_j$ if and only if $T_x\mathcal{F}_i \subset T_x\mathcal{F}_j$ holds for every $x \in M \setminus (Sing(\mathcal{F}_i) \cup Sing(\mathcal{F}_j))$.*

We now recall some facts about the structure of the singular set of a foliation (see Yoshizaki 1998 and Suwa 1998 as well). Let, as above, $\mathcal{F}$ be a reduced foliation of dimension $p$, with tangent sheaf $\mathcal{T}$, on an $m$-dimensional complex manifold $M$. For each $x \in M$ let

$$T(x) = \{\mathbf{v}(x);\ \mathbf{v} \in \mathcal{T}_x\}$$

be the subspace of $T_xM$ formed by the directions induced by $\mathcal{T}_x$. For each integer $k$ with $0 \leq k \leq p$, we define

$$S^{(k)} = \{x \in M;\ \dim T(x) \leq k\}.$$

Then, $S^{(k)}$ is an analytic variety in $M$ and we have a filtration

$$S^{(0)} \subset \dots \subset S^{(p-1)} \subset S^{(p)},$$

where $S^{(p)} = M$ and $S^{(p-1)} = \operatorname{Sing}(\mathcal{F})$ is the singular set of $\mathcal{F}$. It is proved in (Yoshizaki 1998) that, for each $k = 0, \dots, p$, there is a Whitney stratification $\{M_\alpha\}_{\alpha \in A_k}$ of $S^{(k)}$ such that, for any $\alpha \in A_k$ and $x \in M_\alpha$, the inclusion $T(x) \subset T_xM_\alpha$ holds. Moreover, $\mathcal{F}$ induces a non-singular foliation of dimension $k$ on $M_\alpha \setminus S^{(k-1)}$ whose tangent space at $x \in M_\alpha$ is $T(x)$.

If $V$ is an analytic subvariety of $M$ with singular set $\mathrm{Sing}(V)$, we say that $V$ is *invariant* by $\mathcal{F}$ if $T(x) \subset T_x V$ holds for each $x \in V \setminus \mathrm{Sing}(V)$. The above discussion says, in particular, that the analytic set $\mathrm{Sing}(\mathcal{F})$ is invariant by $\mathcal{F}$. We obtain:

THEOREM 1. *Let $M$ be a complex manifold of dimension $n$, and let $\mathcal{F}$ and $\mathcal{G}$ be foliations of dimensions $i$ and $j$, where $1 \leq i < j < n$, such that $\mathcal{F} \prec \mathcal{G}$. Then, Sing$(\mathcal{G})$ is invariant by $\mathcal{F}$.*

This has the following simple consequence:

COROLLARY 1. *Let $M$ be a complex manifold of dimension $n$, and let $\mathcal{F}$ be a foliation of dimension one. If $\mathcal{G}$ is a foliation of dimension $i > 1$ such that $\mathcal{F} \prec \mathcal{G}$, then the isolated points of Sing$(\mathcal{G})$ are contained in Sing$(\mathcal{F})$.*

## FLAGS OF FOLIATIONS ON $\mathbb{P}^n$

In this section we consider, on the projective space $\mathbb{P}^n = \mathbb{P}^n_{\mathbb{C}}$ of dimension $n \geq 3$, a foliation $\mathcal{F}$ of dimension one and a foliation $\mathcal{G}$ of codimension one. Let us suppose that $\mathcal{F}$ leaves $\mathcal{G}$ invariant, that is, $\mathcal{F} \prec \mathcal{G}$ in our notation. If $\mathcal{T}$ is the tangent sheaf of $\mathcal{F}$, then $\mathcal{T} = \mathcal{O}(1-d)$, where $d \geq 0$. This number $d$ is the *degree* of $\mathcal{F}$, which is the degree of the variety of tangencies between $\mathcal{F}$ and a generic hyperplane $H \subset \mathbb{P}^n$. Now, if $\mathcal{C}$ is the cotangent sheaf of $\mathcal{G}$, then $\mathcal{C} = \mathcal{O}(-2-\tilde{d})$, where $\tilde{d} \geq 0$ is the *degree* of $\mathcal{G}$ and counts the number of tangencies, considering multiplicities , between $\mathcal{G}$ and a generic line $L \subset \mathbb{P}^n$.

The study of genericity properties of the set of foliations in $\mathbb{P}^n$ without invariant algebraic varieties is known as the *Jouanolou problem*. It was considered by many authors, such as J. P. Jouanolou, A. Lins Neto, M. Soares, X. Gomez-Mont, L. G. Mendes and M. Sebastiani, among others. We consider here the following result by S. C. Coutinho and J. V. Pereira (see Coutinho and Pereira 2006), Theorem 1.1 and the remark after its proof): if $\mathcal{F}ol_n(1,d)$ denotes the space of foliations on $\mathbb{P}^n$ of dimension one and degree $d$, then, for $d \geq 2$, there is a very generic set $\Im(1,d) \subset \mathcal{F}ol_n(1,d)$ such that if $\mathcal{F} \in \Im(1,d)$, then $\mathcal{F}$ does not admit proper invariant algebraic subvarieties of non-zero dimension. Here *very generic* means that its complementary set is contained in a countable union of hypersurfaces. In the case of invariant algebraic curves, $\Im(1,d)$ can be taken to be open and dense in $\mathcal{F}ol_n(1,d)$, as a consequence of a result by A. Lins Neto and M. Soares (see Lins Neto and Soares 1996, Soares 1993).

Let now $\mathcal{F}$ be a foliation of dimension one and degree $d \geq 2$ on $\mathbb{P}^n$, $n \geq 3$. Suppose that there is a foliation $\mathcal{G}$ of codimension one on $\mathbb{P}^n$ such that $\mathcal{F} \prec \mathcal{G}$. We recall that the singular set of a codimension one foliation on $\mathbb{P}^n$ necessarily has at least one component of codimension two (see Jouanolou 1979). So, by Theorem 1, if $\mathrm{Sing}(\mathcal{F})$ has codimension greater than two, then the components of dimension $n-2$ in $\mathrm{Sing}(\mathcal{G})$ are invariant by $\mathcal{F}$. This implies that $\mathcal{F}$ lies outside the subset $\Im(1,d) \subset \mathcal{F}ol_n(1,d)$ above. We recall that the foliations in $\mathcal{F}ol_n(1,d)$ with isolated singularities form a generic set. Thus, for $n \geq 3$, the set of foliations $\mathcal{F} \in \mathcal{F}ol_n(1,d)$ such that codim $\mathrm{Sing}(\mathcal{F}) > 2$ contains a generic set. This allows us to conclude the following:

THEOREM 2. *The set of foliations of dimension one and degree $d \geq 2$ on $\mathbb{P}^n$, $n \geq 3$, which do not leave invariant a foliation of codimension one, is very generic. When $n = 3$, this set contains a subset that is open and dense in $\mathcal{F}ol_n(1,d)$.*

We say that a foliation $\mathcal{F}$ of dimension one on $\mathbb{P}^n$ admits a *rational first integral* if there is a rational function $\Phi$ in $\mathbb{P}^n$ such that the leaves of $\mathcal{F}$ are contained in the level surfaces of $\Phi$. In homogeneous coordinates in $\mathbb{P}^n$, by writing $\Phi = P/Q$, where $P$ and $Q$ are homogeneous polynomials of the same degree, this means that the 1-form $QdP - PdQ$ induces a codimension one foliation on $\mathbb{P}^n$ that is invariant by $\mathcal{F}$. This gives:

COROLLARY 2. *The set of foliations of dimension one and degree $d \geq 2$ on $\mathbb{P}^n$, $n \geq 3$, which do not admit rational first integral, is very generic. When $n = 3$, this set contains a subset that is open and dense in $\mathcal{F}ol_n(1, d)$.*

**PENCIL OF FOLIATIONS ON $\mathbb{P}^n$**

Let us now consider $\mathcal{F}ol_n(n-1, d)$, the space of foliations of codimension one and degree $d$ on $\mathbb{P}^n$. Such foliations are given, in homogeneous coordinates $X = (X_0 : X_1 : \cdots : X_n) \in \mathbb{P}^n$, by holomorphic 1-forms of the type $\omega = \sum_{i=0}^{n} A_i(X)dX_i$, where each $A_i$ is a homogeneous polynomial of degree $d+1$, satisfying the following:

(i) $\omega \wedge d\omega = 0$ (integrability);

(ii) $i_{\mathbf{r}}\omega = \sum_{i=0}^{n} X_i A_i(X) = 0$, where $\mathbf{r} = X_0\partial/\partial X_0 + \cdots X_n\partial/\partial X_n$ is the radial vector field (Euler condition);

(iii) codim $\mathrm{Sing}(\omega) \geq 2$,

where $\mathrm{Sing}(\omega) = \{A_0 = A_1 = \cdots = A_n = 0\}$ is the singular set of $\omega$. We consider $\mathbb{P}^N$ the projectivization of the space of polynomial forms in $\mathbb{C}^{n+1}$ with homogeneous coefficients of degree $d+1$. Here

$$N = (n+1)\binom{n+d+2}{n+1} - 1.$$

Then, in Zariski's topology, $\mathcal{F}ol_n(n-1, d)$ is an open set of an algebraic subvariety $\overline{\mathcal{F}ol_n(n-1, d)}$ of $\mathbb{P}^N$. We remark that the elements in the border

$$\partial\mathcal{F}ol_n(n-1, d) = \overline{\mathcal{F}ol_n(n-1, d)} \setminus \mathcal{F}ol_n(n-1, d)$$

are integrable 1-forms satisfying Euler condition, but having a singular set of codimension one.

Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two distinct foliations on $\mathbb{P}^n$ induced, in homogeneous coordinates, by integrable 1-forms $\omega_1$ and $\omega_2$. The 2-form $\omega_1 \wedge \omega_2$ might be zero on a set of codimension one, which corresponds to the set of tangencies between $\mathcal{G}_1$ and $\mathcal{G}_2$. If $f = 0$ denotes the homogeneous polynomial equation for this set, we write $\omega_1 \wedge \omega_2 = f\theta$, for some 2-form $\theta$ whose coefficients are homogeneous polynomials and whose singular set has codimension two or greater. Since

$$i_{\mathbf{r}}(\omega_1 \wedge \omega_2) = i_{\mathbf{r}}\omega_1 \wedge \omega_2 - \omega_1 \wedge i_{\mathbf{r}}\omega_2 = 0$$

we have $i_{\mathbf{r}}\theta = 0$, so the field of $(n-1)$-planes on $\mathbb{C}^{n+1}$ defined by $\theta$ goes down to an integrable field of $n-2$-planes on $\mathbb{P}^n$ whose singular set has codimension two or greater. This defines a foliation $\mathcal{F}$ of codimension two on $\mathbb{P}^n$, which leaves both $\mathcal{G}_1$ and $\mathcal{G}_2$ invariant. Following the terminology on (Ghys 1991), $\mathcal{F}$ is called the *axis* of $\mathcal{G}_1$ and $\mathcal{G}_2$.

A line of the space $\mathbb{P}^N$, which is entirely contained in $\overline{\mathcal{F}ol_n(n-1,d)}$ and whose generic element is in $\mathcal{F}ol_n(n-1,d)$, is called a *pencil* of foliations. Remark that two foliations in $\mathcal{F}ol_n(n-1,d)$ represented by 1-forms $\omega_1$ and $\omega_2$ define a pencil of foliations if and only if $\omega = \omega_1 + t\omega_2$ is integrable for all $t \in \mathbb{C}$. This means

$$\begin{aligned} 0 = \omega \wedge d\omega &= (\omega_1 + t\omega_2) \wedge (d\omega_1 + td\omega_2) \\ &= t(\omega_1 \wedge d\omega_2 + \omega_2 \wedge d\omega_1), \end{aligned}$$

which is equivalent to

$$\omega_1 \wedge d\omega_2 + \omega_2 \wedge d\omega_1 = 0. \tag{1}$$

One value of $t \in \mathbb{C} \setminus \{0\}$ for which $\omega_1 + t\omega_2$ is integrable is sufficient for assuring condition (1). So, if three foliations are on a line, then they define a pencil of foliations. Of course, given a pencil of foliations in $\overline{\mathcal{F}ol_n(n-1,d)}$, a foliation $\mathcal{F}$ of codimension two is intrinsically associated to it as being the axis of any two foliations in the pencil. It leaves invariant all the foliations in the pencil.

For foliations of codimension one on $\mathbb{P}^3$ there is a conjecture due to M. Brunella, which asserts that, if $\mathcal{G}$ is such a foliation, then one of the alternatives holds:

(a) $\mathcal{G}$ leaves an algebraic surface invariant;

(b) $\mathcal{G}$ is invariant by a holomorphic foliation $\mathcal{F}$ by algebraic curves.

In (b) we mean that the closure of each leaf of $\mathcal{F}$ is an algebraic curve. In (Cerveau 2002), the following result is proved:

THEOREM 3. *Let $\mathcal{G}$ be a foliation of codimension one on $\mathbb{P}^3$, which is an element of a pencil of foliations. Then, $\mathcal{G}$ satisfies (a) or (b) above.*

It is worth remarking that, in Cerveau's proof, the foliation $\mathcal{F}$ that appears in alternative (b) is the axis of the pencil and is given by two independent rational first integrals. We next prove the following simple lemma:

LEMMA 1. *Let $\mathcal{F}$ be a foliation of codimension two on $\mathbb{P}^n$, which leaves invariant three foliations of codimension one induced, in homogeneous coordinates, by integrable polynomial 1-forms $\omega_1$, $\omega_2$ and $\omega_3$. Then, there are non-zero homogeneous polynomials $\alpha_1, \alpha_2$ and $\alpha_3$, relatively prime two by two, such that*

$$\alpha_3\omega_3 = \alpha_1\omega_1 + \alpha_2\omega_2. \tag{2}$$

PROOF. We write $\omega_1 \wedge \omega_3 = f_1\theta$, where $\theta$ is a polynomial 2-form that induces $\mathcal{F}$, having singular set of codimension at least two, and $f_1$ is a non-zero homogeneous polynomial. Similarly, we have $\omega_2 \wedge \omega_3 = -f_2\theta$, for some non-zero homogeneous polynomial $f_2$. We thus have

$$\left(\frac{1}{f_1}\omega_1 + \frac{1}{f_2}\omega_2\right) \wedge \omega_3 = 0.$$

This implies that there is a rational function $\Phi$ such that

$$\omega_3 \; = \; \Phi\left(\frac{1}{f_1}\,\omega_1 + \frac{1}{f_2}\,\omega_2\right).$$

By canceling denominators, we get homogeneous polynomials $\alpha_1, \alpha_2$ and $\alpha_3$, which satisfy (2). Finally, a common factor for two of these polynomials would be a factor of the third and, so, could be canceled. We can thus suppose that $\alpha_1, \alpha_2$ and $\alpha_3$ relatively prime two by two. $\square$

Before proceeding we make a simple remark: if $\omega$ is an integrable 1-form with homogeneous coefficients of the same degree $d+1$ inducing a foliation in $\mathcal{F}ol_n(n-1, d)$, and $\alpha$ is a homogeneous polynomial of degree $k$, then $\tilde{\omega} = \alpha\omega$ is also integrable. Of course, if $\alpha$ is non-constant, then $\tilde{\omega}$ has a codimension one component in its singular set. It will be regarded as representing an element of $\overline{\mathcal{F}ol_n(n-1, d+k)}$. Actually, it is an element in the border $\partial\mathcal{F}ol_n(n-1, d+k)$, if $k > 0$.

LEMMA 2. *Let $\omega_1$ and $\omega_2$ be 1-forms in $\mathbb{C}^{n+1}$ with homogeneous polynomial coefficients of the same degree, defining different distributions of $n$-planes in the sense that $\omega_1 \wedge \omega_2$ is not identically zero. Suppose also that the singular sets of $\omega_1$ and $\omega_2$ do not have a common component of codimension one. Then, the generic element of the pencil of 1-forms*

$$\{t_1\omega_1 + t_2\omega_2;\ (t_1 : t_2) \in \mathbb{P}\}$$

*has singular set of codimension two or greater.*

PROOF. Let us write

$$\omega_1 = \sum_{i=0}^{n} A_i dX_i \quad \text{and} \quad \omega_2 = \sum_{i=0}^{n} B_i dX_i,$$

where $A_i$ and $B_i$ are homogeneous polynomial of the same degree. Suppose that the result is false. Then, for all values of $t \in \mathbb{C}$ but a finite number, the 1-form $\omega_t = \omega_1 + t\omega_2$ has a component of codimension one in its singular set. For such a $t$, take $g_t = 0$ as an equation of this component, where $g_t$ is non-constant reduced homogeneous polynomial. Fix $i, j$, with $0 \le i, j \le n$. We have that both $A_i + tB_i$ and $A_j + tB_j$ vanish over $\{g_t = 0\}$. If $g_t$ is a factor of neither $B_i$ nor $B_j$, then we have that $A_i/B_i = t = A_j/B_j$ over $\{g_t = 0\}$, which means that $A_iB_j - A_jB_i$ vanishes over $\{g_t = 0\}$. The same will be true if $g_t$ is a factor of $B_i$ (or $B_j$), since, in this case, it will also be a factor of $A_i$ (or $A_j$). In any case, we have that $g_t$ is a factor of $A_iB_j - A_jB_i$. Finally, the hypothesis on the singular sets of $\omega_1$ and $\omega_2$ implies that, by varying $t$, there are infinitely many different polynomials $g_t$. This gives that $A_iB_j - A_jB_i = 0$, that is $A_i/B_i = A_j/B_j = \Phi$, where $\Phi$ is a rational function of degree zero. Doing this to all values of $i$ and $j$, we get $\omega_1 = \Phi\,\omega_2$, which is a contradiction with the fact that $\omega_1 \wedge \omega_2 \neq 0$. $\square$

It is worth mentioning the following result, which is a corollary of the above lemma:

COROLLARY 3. *Let $\omega_1$ and $\omega_2$ be integrable 1-forms in $\mathbb{C}^{n+1}$ with polynomial coefficients of the same degree $d+1$, such that $\omega_1 \wedge \omega_2 \neq 0$. Suppose that the pencil of 1-forms*

$$\{t_1\omega_1 + t_2\omega_2;\ (t_1 : t_2) \in \mathbb{P}\}$$

*lies entirely in $\partial\mathcal{F}ol_n(n-1,d)$. Then, the singular sets of the elements of this pencil have a common component of codimension one.*

We have the following result:

PROPOSITION 2. *Let $\mathcal{F}$ be a foliation of codimension two on $\mathbb{P}^n$ that leaves invariant three foliations of codimension one. Then, $\mathcal{F}$ leaves invariant a whole pencil of foliations.*

PROOF. Suppose that the codimension one foliations are induced in homogeneous coordinates by 1-forms $\omega_1$, $\omega_2$ and $\omega_3$. In view of the previous lemma, there are homogeneous polynomials $\alpha_1$, $\alpha_2$ and $\alpha_3$, such that

$$\alpha_3\omega_3 \;=\; \alpha_1\omega_1 + \alpha_2\omega_2. \tag{3}$$

If $\omega_1$ and $\omega_2$ lies in a pencil of foliations, the result is done. Otherwise, we necessarily have that either $\alpha_1$ or $\alpha_2$ is non-constant. Expression (3) gives that the integrable 1-form $\alpha_3\omega_3$ lies in the pencil generated by the integrable 1-forms $\alpha_1\omega_1$ and $\alpha_2\omega_2$. Thus, this whole pencil is composed by integrable 1-forms. Finally, even though the generators of this pencil may lie in $\partial\mathcal{F}ol_n(n-1,d)$, where $d+1$ is the degree of $\alpha_i\omega_i$, its generic element lies in $\mathcal{F}ol_n(n-1,d)$. This is a consequence of Lemma 2 above. Therefore, $\alpha_1\omega_1$ and $\alpha_2\omega_2$ generate a pencil of foliations whose axis is $\mathcal{F}$. $\square$

The above proposition together with Theorem 3 give:

COROLLARY 4. *Let $\mathcal{F}$ be a foliation of dimension one on $\mathbb{P}^3$. Suppose that no hypersurface in $\mathbb{P}^3$ is invariant by $\mathcal{F}$. Then, the number of foliations of codimension one invariant by $\mathcal{F}$ is at most two.*

PROPOSITION 3. *Let $\mathcal{F}$ be a foliation of codimension two on $\mathbb{P}^n$ that leaves invariant a pencil of foliations in $\mathcal{F}ol_n(n-1,d)$. Suppose that, outside this pencil, there is another foliation $\mathcal{G}$ of codimension one and degree at least $d$ that leaves $\mathcal{F}$ invariant. Then, $\mathcal{F}$ admits a rational first integral.*

PROOF. Suppose that the pencil of foliations is generated by the 1-forms $\omega_1$ and $\omega_2$, and that $\mathcal{G}$ is induced by the 1-form $\omega_3$. Lemma 1 assures the existence of homogeneous polynomials $\alpha_1$, $\alpha_2$ and $\alpha_3$, two by two without common factors, such that

$$\alpha_3\omega_3 \;=\; \alpha_1\omega_1 + \alpha_2\omega_2. \tag{4}$$

Since $\mathcal{G}$ does not lie in the pencil of foliations generated by $\omega_1$ and $\omega_2$, we have that $\alpha_1$ and $\alpha_2$ are non-constant. The integrability condition applied to $\alpha_3\omega_3$ reads

$$0 \;=\; \alpha_3\omega_3 \wedge d(\alpha_3\omega_3) \;=\; (\alpha_1\omega_1 + \alpha_2\omega_2) \wedge (d\alpha_1 \wedge \omega_1 + d\alpha_2 \wedge \omega_2 + \alpha_1 d\omega_1 + \alpha_2 d\omega_2),$$

which gives

$$(\alpha_2 d\alpha_1 - \alpha_1 d\alpha_2) \wedge \omega_1 \wedge \omega_2 \;=\; 0, \tag{5}$$

where we used that $\omega_1 \wedge d\omega_2 + \omega_2 \wedge d\omega_1 = 0$. The rational function $\alpha_1/\alpha_2$, which is non-constant since $\alpha_1$ and $\alpha_2$ are non-constant and without common factor, is thus a rational first integral for $\mathcal{F}$. $\square$

**POLAR CLASSES**

We now consider an $r$-dimensional foliation $\mathcal{F}$ defined on a projective manifold $M \subset \mathbb{P}^n$ of dimension $m$. Let $\mathcal{T}$ be the tangent sheaf of $\mathcal{F}$. For each $x \in M \setminus \mathrm{Sing}(\mathcal{F})$, there is a unique $r$-dimensional plane $T_x^{\mathbb{P}}\mathcal{F} \subset \mathbb{P}^n$ passing through $x$ with direction $T_x\mathcal{F} \subset T_xM$.

Let us fix

$$\mathcal{D} : L_n \subset L_{n-1} \subset \cdots \subset L_j \subset \cdots L_1 \subset L_0 = \mathbb{P}^n, \qquad (*)$$

a flag of codimension $j$ linear subspaces $L_j \subset \mathbb{P}^n$.

For $k = 1, \dots, r+1$, the $k$-th *polar locus* of $\mathcal{F}$ with respect to $\mathcal{D}$ is defined as

$$P_k^{\mathcal{F}} \ = \ Cl\{x \in M \setminus \mathrm{Sing}(\mathcal{F});\ \dim(T_x^{\mathbb{P}}\mathcal{F} \cap L_{r-k+2}) \geq k-1\},$$

where the closure $Cl$ is taken in $M$. We remark that a point $x \in M \setminus \mathrm{Sing}(\mathcal{F})$ belongs to $P_k^{\mathcal{F}}$ if and only if the subspaces of $\mathbb{C}^{n+1}$ corresponding to $T_x^{\mathbb{P}}\mathcal{F}$ and to $L_{r-k+2}$ do not span $\mathbb{C}^{n+1}$. It follows straight from the definition that

$$P_i^{\mathcal{F}} \subset P_j^{\mathcal{F}} \ \text{ if } \ i > j.$$

Let $A_k(M)$ denote the Chow group of $M$, where $k$ stands for the complex dimension. In (Mol 2006, Proposition 3.3), it is proved that, for a generic choice of a flag $\mathcal{D}$ and for $k = 1, \dots, r+1$, the set $P_k^{\mathcal{F}}$ is empty or is an analytic variety of pure codimension $k$ whose class $\left[P_k^{\mathcal{F}}\right] \in A_{m-k}(M)$ is independent of the flag, where $A_{m-k}(M)$ stands for the Chow group of $M$ of complex dimension $m-k$. We then have a well-defined class that is called *polar class* of $\mathcal{F}$. The *polar degrees* of $\mathcal{F}$ are the degrees of these polar classes. We denote them by $\rho_k^{\mathcal{F}} = \deg\left[P_k^{\mathcal{F}}\right]$, $k = 1, \dots, r+1$.

EXAMPLE 1. Let $\mathcal{F}$ be a foliation of dimension one on $\mathbb{P}^n$. We have

$$x \in P_1^{\mathcal{F}} \cap \left(\mathbb{P}^n \setminus \mathrm{Sing}(\mathcal{F})\right) \Leftrightarrow \dim\left(T_x^{\mathbb{P}}\mathcal{F} \cap L_2\right) \geq 0.$$

This means that the hyperplane generated by $L_2$ and $x$ is tangent to $\mathcal{F}$ at $x$. The tangency locus between $\mathcal{F}$ and a non-invariant hyperplane $H \subset \mathbb{P}^n$ is a hypersurface in $H$ of degree $\deg(\mathcal{F})$. We then conclude that $P_1^{\mathcal{F}}$ is a hypersurface in $\mathbb{P}^n$ of degree $\deg(\mathcal{F})+1$, since $L_2 \subset P_1^{\mathcal{F}}$.

EXAMPLE 2. Let now $\mathcal{G}$ be a foliation of codimension one on $\mathbb{P}^n$ with $\mathrm{Sing}(\mathcal{G})$ of codimension at least two. If $X = (X_0 : X_1 : \cdots : X_n)$ is a system of homogeneous coordinates in $\mathbb{P}^n$, then $\mathcal{G}$ is induced by a polynomial 1-form $\omega = \sum_{i=0}^n A_i(X)dX_i$ with homogeneous coefficients of degree $\deg(\mathcal{G})+1$, which is integrable and satisfies the Euler condition. We have

$$x \in P_1^{\mathcal{G}} \cap \left(\mathbb{P}^n \setminus \mathrm{Sing}(\mathcal{G})\right) \Leftrightarrow \dim\left(T_x^{\mathbb{P}}\mathcal{G} \cap L_m\right) \geq 0,$$

that is, the hyperplane $T_x^{\mathbb{P}}\mathcal{G}$ contains the point $L_m$. Writing in homogeneous coordinates $L_m = (\alpha_0 : \alpha_1 : \ldots : \alpha_n)$, we have that $P_1^{\mathcal{G}}$ has equation

$$\alpha_0 A_0 + \alpha_1 A_1 + \cdots + \alpha_n A_n \ = \ 0$$

and we see that $P_1^{\mathcal{G}}$ is a hypersurface of degree $\deg(\mathcal{G})+1$.

EXAMPLE 3. Let us now examine $P_2^{\mathcal{G}}$, where $\mathcal{G}$ is a foliation of codimension one on $\mathbb{P}^n$. We have

$$x \in P_2^{\mathcal{G}} \cap \left(\mathbb{P}^n \setminus \mathrm{Sing}(\mathcal{G})\right) \Leftrightarrow \dim\left(T_x^{\mathbb{P}}\mathcal{G} \cap L_{m-1}\right) \geq 1.$$

Suppose that $\mathcal{G}$ is given in homogeneous coordinates in $\mathbb{P}^n$ by the polynomial 1-form $\omega$ of the previous example. The space $L_{m-1}$ is a line in $\mathbb{P}^n$, which we suppose to be generated by points of coordinates $(\alpha_0 : \alpha_1 : \cdots : \alpha_n)$ and $(\beta_1 : \beta_2 : \cdots : \beta_n)$. Thus, $P_2^{\mathcal{G}}$ is contained in the variety $V_2$ given by the pair of equations

$$\begin{cases} \alpha_0 A_0 + \alpha_1 A_1 + \cdots + \alpha_n A_n &= 0 \\ \beta_0 A_0 + \beta_1 A_1 + \cdots + \beta_n A_n &= 0 \end{cases}$$

We assume that $L_{m-1}$ is generic, so $V_2$ has pure codimension two and has degree $(\deg(\mathcal{G})+1)^2$. It contains two types of points. Outside $\mathrm{Sing}(\mathcal{G})$, the points of $V_2$ correspond to those of $P_2^{\mathcal{G}}$. On the other hand, since $\mathrm{Sing}(\mathcal{G}) \subset V_2$, the remaining points of $V_2$ are contained in the component of codimension two of $\mathrm{Sing}(\mathcal{G})$, which will be denoted by $S_2$. We then have $V_2 = P_2^{\mathcal{G}} \cup S_2$, and the two sets of this union do not have a common component of codimension two. We conclude that

$$\rho_2^{\mathcal{G}} + \deg(S_2) = (\deg(\mathcal{G})+1)^2.$$

Let $1 \leq i_1 < \cdots < i_k < m$ and $\mathcal{F}_{i_1} \prec \cdots \prec \mathcal{F}_{i_k}$ be a flag of foliations on the $m$-dimensional projective manifold $M \subset \mathbb{P}^n$. Fix a flag of linear subspaces of $\mathbb{P}^n$:

$$\mathcal{D} : L_n \subset L_{n-1} \subset \cdots \subset L_j \subset \cdots L_1 \subset L_0 = \mathbb{P}^n$$

For $i = i_1, \ldots, i_k$, let $P_k^{\mathcal{F}_i}$ be the $k$-th polar locus of $\mathcal{F}_i$ with respect to $\mathcal{D}$. We have the following result:

PROPOSITION 4. *Let $i < j$ be two integers of the list $i_1 < \cdots < i_k$. For integers $r$ and $s$ such that $r = 1, \ldots, i$ and $r + s \leq j$, it holds*

$$P_{r+s}^{\mathcal{F}_j} \subset P_r^{\mathcal{F}_i} \cap P_s^{\mathcal{F}_j}.$$

PROOF. We start by remarking that the inclusion $P_{r+s}^{\mathcal{F}_j} \subset P_s^{\mathcal{F}_j}$ follows immediately from the definition of polar locus. Thus, all we have to prove is that $P_{r+s}^{\mathcal{F}_j} \subset P_r^{\mathcal{F}_i}$. Let $x \in M \setminus (\mathrm{Sing}(\mathcal{F}_i) \cup \mathrm{Sing}(\mathcal{F}_j))$. We have

$$x \in P_{r+s}^{\mathcal{F}_j} \Leftrightarrow \dim\left(T_x^{\mathbb{P}}\mathcal{F}_j \cap L_{j-(r+s)+2}\right) \geq (r+s)-1$$

and

$$x \in P_r^{\mathcal{F}_i} \Leftrightarrow \dim\left(T_x^{\mathbb{P}}\mathcal{F}_i \cap L_{i-r+2}\right) \geq r-1\,.$$

Since $\mathcal{F}_i$ and $\mathcal{F}_j$ are foliations in a flag, $T_x^{\mathbb{P}}\mathcal{F}_i$ is a subspace $T_x^{\mathbb{P}}\mathcal{F}_j$ of codimension $j-i$. Furthermore, $L_{i-r+2}$ has codimension

$$(i-r+2) - (j-(r+s)+2) = i-j+s$$

in $L_{j-(r+s)+2}$. Thus, if $\dim\left(T_x^{\mathbb{P}}\mathcal{F}_j \cap L_{j-(r+s)+2}\right) \geq (r+s)-1$, then

$$\dim\left(T_x^{\mathbb{P}}\mathcal{F}_i \cap L_{j-(r+s)+2}\right) \geq (r+s)-1)-(j-i)\,.$$

Thus,

$$\dim\left(T_x^{\mathbb{P}}\mathcal{F}_i \cap L_{i-r+2}\right) \geq \left((r+s)-1)-(j-i)\right)-(i-j+s) \;=\; r-1\,,$$

which finishes the proof. □

Let us now consider a flag of foliations $\mathcal{F}_{i_1} \prec \cdots \prec \mathcal{F}_{i_k}$ on $\mathbb{P}^n$, where $1 \leq i_1 < \cdots < i_k < n$.

THEOREM 4. *Let $i < j$ be two integers of the list $i_1 < \cdots < i_k$. For integers $r$ and $s$ such that $1 \leq r \leq i$ and $r + s \leq j$, with $j - i \neq s - r$, it holds*

$$\rho_{r+s}^{\mathcal{F}_j} \leq \rho_r^{\mathcal{F}_i} \rho_s^{\mathcal{F}_j}.$$

PROOF. This is a consequence of Bezout's Theorem ([Fu]). All we have to do is to prove that $P_r^{\mathcal{F}_i}$ and $P_s^{\mathcal{F}_j}$ can be chosen to be transverse. These polar loci are induced by $L_{i-r+2}$ and $L_{j-s+2}$, which are distinct linear spaces, since $j - i \neq s - r$. Thus, transversality occurs for generic choices of $L_{i-r+2}$ and of $L_{j-s+2}$ as a consequence of Piene's Transversality Lemma (see Piene 1978, Mol 2006). □

Taking into account the calculations made in Examples 1 and 2, Theorem 4 gives:

COROLLARY 5. *Let $\mathcal{F}$ and $\mathcal{G}$ be foliations on $\mathbb{P}^n$, $n \geq 3$, where $\mathcal{F}$ has dimension one and $\mathcal{G}$ has codimension one. Suppose that $\mathcal{F} \prec \mathcal{G}$. Then, the following inequality holds:*

$$\rho_2^{\mathcal{G}} \leq (\deg(\mathcal{F}) + 1)(\deg(\mathcal{G}) + 1), \tag{6}$$

*where $\deg(\mathcal{F})$ and $\deg(\mathcal{G})$ are the degrees of $\mathcal{F}$ and $\mathcal{G}$, respectively.*

As seen in Example 3, $\rho_2^{\mathcal{G}} + \deg(S_2) = (\deg(\mathcal{G}) + 1)^2$, where $S_2$ corresponds to the component of codimension two of $\mathrm{Sing}(\mathcal{G})$. Putting this in (6) gives

COROLLARY 6. *Let $\mathcal{F}$ and $\mathcal{G}$ be as in Corollary 5. Then,*

$$\deg(S_2) \geq (\deg(\mathcal{G}) + 1)(\deg(\mathcal{G}) - \deg(\mathcal{F})),$$

*where $S_2$ stands for the component of codimension two in $\mathrm{Sing}(\mathcal{G})$.*

EXAMPLE 4. Take $\tilde{\mathcal{G}}$ a foliation of degree $d$ on $\mathbb{P}^2$ defined in homogeneous coordinates by an 1-form $\tilde{\omega}$. Let $\Phi : \mathbb{P}^3 \to \mathbb{P}^2$ be a rational projection, for instance the one defined in homogeneous coordinates by

$$\Phi(X_0 : X_1 : X_2 : X_3) = (X_0 : X_1 : X_2).$$

Then, $\omega = \Phi^*\tilde{\omega}$ defines a foliation $\mathcal{G}$ of codimension one and of degree $d$ on $\mathbb{P}^3$. The linear fibration given by the levels of $\Phi$ is a foliation of dimension one on $\mathbb{P}^n$ whose degree is zero. It leaves $\mathcal{G}$ invariant. Corollary 6 gives in this case $\deg(S_2) \geq (d+1)d = d^2 + d$. However, $\mathrm{Sing}(\mathcal{G}) = \Phi^{-1}(\mathrm{Sing}(\tilde{\mathcal{G}}))$ is a finite family of lines. Thus, $\mathrm{Sing}(\mathcal{G}) = S_2$. In the generic situation, $\tilde{\mathcal{G}}$ has $d^2 + d + 1$ singularities (see Baum and Bott 1972), and we find $\deg(S_2) = d^2 + d + 1$, which is larger than the bound obtained.

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**RESUMO**

Uma bandeira de folheações holomorfas em uma variedade complexa $M$ é um objeto que consiste de um número finito de folheações holomorfas singulares em $M$ de dimensões crescentes tais que o feixe tangente de uma folheação fixa é subfeixe do feixe tangente de cada folheação de dimensão maior. Estudamos algumas propriedades básicas destes objetos e, em $\mathbb{P}^n_{\mathbb{C}}$, $n \geq 3$, estabelecemos condições necessárias para que uma folheação de dimensão menor deixe invariante folheações de codimensão um. Finalmente, ainda em $\mathbb{P}^n_{\mathbb{C}}$, encontramos quotas envolvendo graus das classes polares de folheações em uma bandeira.

**Palavras-chave:** folheações holomorfas, variedades polares, variedades invariantes.

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