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Parameter induction in continuous univariate distributions: Well-established G families

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ABSTRACT

The art of parameter(s) induction to the baseline distribution has received a great deal of attention in recent years. The induction of one or more additional shape parameter(s) to the baseline distribution makes the distribution more flexible especially for studying the tail properties. This parameter(s) induction also proved helpful in improving the goodness-of-fit of the proposed generalized family of distributions. There exist many generalized (or generated) G families of continuous univariate distributions since 1985. In this paper, the well-established and widely-accepted G families of distributions like the exponentiated family, Marshall-Olkin extended family, beta-generated family, McDonald-generalized family, Kumaraswamy-generalized family and exponentiated generalized family are discussed. We provide lists of contributed literature on these well-established G families of distributions. Some extended forms of the Marshall-Olkin extended family and Kumaraswamy-generalized family of distributions are proposed.

Key words: Beta-distribution, exponentiated family, Kumaraswamy distribution, Marshall-Olkin family, McDonald distribution, reliability properties.

INTRODUCTION

There has been an increased interest in developing generalized (or generated) G families of distributions by introducing one or more additional shape parameter(s) to the base-line distribution. There is no doubt that the popularity and the use of Euler-beta and -gamma functions in some G families of distributions have attracted the attention of statisticians, mathematicians, scientists, engineers, economists, demographers and other applied researchers. One reason might be the computational and analytical facilities available in programming softwares like R (packages), *ox5*, Python, Matlab, Maple and Mathematica, through which researchers can easily tackle problems involved in computing incomplete-beta and -gamma functions in G families. The second reason is the tail properties of G distributions that can easily be explored by inducing one or more additional shape parameter(s) to the baseline distribution. Thirdly, this parameter(s) induction has also proved to be helpful in improving the goodness-of-fit of the proposed G family of distributions. Fourthly, G families have the ability to fit skewed data better than existing

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distributions (Pescim et al. 2010). Lastly, the Kumaraswamy G family of distributions can generate effective models for censored data (Cordeiro and de Castro 2011).

There exists many generalized (or generated) G family of distributions like Azzalini's skewed family (Azzalini 1985), Marshall-Olkin extended (MOE) family (Marshall and Olkin 1997), exponentiated family (EF) of distributions (Gupta et al. 1998), beta-generated (beta G) family (Eugene et al. 2002, Jones 2004a), Ferreira and Steel's skewed family (Ferreira and Steel 2006), transmuted family (Shaw and Buckley 2007, Aryal and Tsokos 2009, 2011), Gupta and Gupta's skewed family (Gupta and Gupta 2008), gamma-generated (GG) families (Zografos and Balakrishnan 2009, Ristić and Balakrishnan 2012, Torabi and Montazari 2012, Nadarajah et al. 2015), transformed-transformer (T-X) family (Alza-Atreh 2011), Kumaraswamy generalized (Kw G) family (Cordeiro and de Castro 2011, Nadarajah et al. 2012a, Hussain 2013), generalized beta generated (GBG) or McDonald generalized (Mc G) family (Alexander et al. 2012), beta extended G family (Cordeiro et al. 2012f), Kummer beta generalized family (Pescim et al. 2012), exponentiated transformed-transformer family (ET-X) (Alzaghal et al. 2013), exponentiated generalized (Exp G) family (Cordeiro et al. 2013e), geometric exponential-Poisson family (Nadarajah et al. 2013a), truncated-exponential skew-symmetric family (Nadarajah et al. 2013c), logistic-generated (Lo G) family (Torabi and Montazari 2014), Marshall-Olkin extended family (Alshangiti et al. 2014), log-gamma generated (LG G) families, (Amini et al. 2014), Weibull G family (Bourguignon et al. 2014), Libby-Novick beta family (Cordeiro et al. 2014e), truncated negative-binomial family (Nadarajah et al. 2014a), modified beta G family (Nadarajah et al. 2014b) and exponentiated exponential-Poisson family (Ristić and Nadarajah 2014). These G families of distributions have received a great deal of attention in recent years. In this paper, we discuss the EF, MOE, beta G , Mc G , Exp G and Kw G families of distributions and provide additional literature (in chronological order) on these six families of distributions. We also propose some extended forms of the Kw G families of distributions by introducing one more additional shape parameter(s).

Because of the length of this paper, we have not given details like probabilistic interpretations, analytical properties, estimation methods, simulation algorithms and applications. These details can be obtained from the cited references.

The rest of the paper is organized as follows. In Section 2, the EF of distributions is defined and a list of contributed work is presented. In Section 3, we describe the MOE family and propose one generalized MOE family of distributions. The contributed literature on the MOE family is also presented. In Section 4, the beta G family of distributions is discussed. The contributions to the beta G family of distributions are also listed in this section. In Section 5, the McDonald distributions and Mc G families of distributions are described. The contributed work on Mc G families of distributions is also presented. Section 6 consists of Kumaraswamy distributions and Kw G families of distributions. Some new types of the Kumaraswamy distribution and Kw G families of distributions are proposed. The contributed work on the Kw G family of distributions is also listed in this section. Section 7 ends the paper with some final remarks.

EXPONENTIATED FAMILY (EF) OF DISTRIBUTIONS

The genesis of this family can be traced back to the first half of the nineteenth century when Gompertz (1825) and Verhulst (1838, 1845, 1847) used the cumulative distribution function (cdf) $G(t) = (1 - \rho e^{-\lambda t})^\alpha$ for $t > \lambda^{-1} \log \rho$, where ρ , α and λ are positive real numbers. Ahuja and Nash (1967) introduced the generalized Gompertz-Verhulst family of distributions to study growth curve mortality. Gompertz-Verhulst's cdf was the first member of the EF of distributions. The exponentiated exponential (EE) distribution is its particular case

for $\rho = 1$. The properties and estimation methods for parameters of the EF of distributions have been studied by many authors, see Mudholkar and Srivastava (1993), Mudholkar and Hutson (1996), Mudholkar et al. (1995), Gupta and Kundu (1999, 2001a, b, 2007), Pal et al. (2006), Nadarajah and Kotz (2006a), Nadarajah (2011) and Nadarajah et al. (2013b). The EF of distributions is also known as Lehmann alternatives (LAs) (Lehmann 1953) or proportional reversed hazard rate model (PHRM) (see Gupta et al. 1998, Gupta and Gupta 2007, Martínez-Florez et al. 2013), while other authors referred to the EF of distributions as max-stable family (Sarabia and Castillo 2005) and F^α -distributions (Gupta et al. 1998, Al-Hussaini, 2010a, b, 2012, Shakil and Ahsanullah 2012, Hamedani 2013 and Ghitany et al. 2013).

In literature there exist four different ways for obtaining the EF of distributions.

LEHMANN ALTERNATIVE 1 (LA1)

The method of Lehmann alternative 1 (LA1) (due to Lehmann (1953)) has received a great deal of attention in developing the EF of distributions.

If $G(z)$ is the cdf of the baseline distribution, then an EF of distributions is defined by taking the α th-power of $G(z)$ as

$$F(z) = G(z)^\alpha, \quad (2.1)$$

where $\alpha > 0$ is a positive real parameter. The variable z can take any of the form $z = x$ or $z = x - \mu$ or $z = \frac{x-\mu}{\sigma}$ or $z = k \left(\frac{x-\mu}{\sigma}\right)$ or $z = k \left(\frac{x-\mu}{\sigma}\right)^{\frac{1}{\delta}}$. The probability density function (pdf) corresponding to (2.1) is

$$f(z) = \alpha g(z) G(z)^{\alpha-1}, \quad (2.2)$$

where $g(z) = dG(z)/dz$ denotes the pdf of G . For any lifetime random variable t , the survival (reliability) function (sf), $\bar{F}(t)$, the hazard (failure) rate function (hrf), $h(t)$, the reversed hazard rate function (rhrf), $r(t)$, and the cumulative hazard rate function (chrf), $H(t)$, associated with (2.1) and (2.2) are

$$\begin{aligned} \bar{F}(t) &= 1 - G(t)^\alpha, \\ h(t) &= \alpha g(t) G(t)^{\alpha-1} [1 - G(t)^\alpha]^{-1}, \\ r(t) &= \alpha g(t) G(t)^{-1}, \end{aligned}$$

and

$$H(t) = -\log [1 - G(t)^\alpha].$$

LEHMANN ALTERNATIVE 2 (LA2)

The method of Lehmann alternative 2 (LA2) (due to Lehmann (1953)) has received less attention.

If $G(z)$ is the cdf and $\bar{G}(z) = 1 - G(z)$ is the sf of the baseline distribution, then an EF of distributions is defined by taking one minus the α th-power of $\bar{G}(z)$ as

$$F(z) = 1 - [\bar{G}(z)]^\alpha,$$

where α is a positive real parameter. The LA2 cdf may also be written as

$$F(z) = 1 - [1 - G(z)]^\alpha. \quad (2.3)$$

The pdf corresponding to (2.3) is

$$f(z) = \alpha g(z) [1 - G(z)]^{\alpha-1}. \quad (2.4)$$

For any lifetime random variable t , the sf, hrf, rhf and chrf associated with (2.3) and (2.4) are

$$\begin{aligned}\bar{F}(t) &= [1 - G(t)]^\alpha, \\ h(t) &= \alpha g(t) [1 - G(t)]^{-1}, \\ r(t) &= \alpha g(t) [1 - G(t)]^{\alpha-1} \{1 - [1 - G(t)]^\alpha\}^{-1},\end{aligned}$$

and

$$H(t) = -\alpha \log [1 - G(t)].$$

Nadarajah and Kotz (2003, 2006a), Nadarajah (2006) and Rao et al. (2013) used the LA2 approach for introducing exponentiated Fréchet, exponentiated Gumbel and exponentiated log-logistic distributions. For more applications of the LA2 approach, the reader is referred to Abd-Elfattah and Omima (2009), Abd-Elfattah et al. (2010), Rao et al. (2012, 2013), and Al-Nasser and Al-Omari (2013).

USING TRANSFORMATION $z = \log(x)$, $x > 0$

Nadarajah (2005a) developed exponentiated distributions by applying the transformation $z = \log(x)$ to (2.3). The cdf, pdf and the hrf of the exponentiated distribution are

$$\begin{aligned}F(x) &= 1 - [1 - G(e^x)]^\alpha, \\ f(x) &= \alpha e^x g(e^x) [1 - G(e^x)]^{\alpha-1}\end{aligned}$$

and

$$h(x) = \alpha e^x g(e^x) [1 - G(e^x)]^{-1}.$$

USING TRANSFORMATION $z = -\log(x)$, $x > 0$

Nadarajah (2005b) developed exponentiated distributions by applying the transformation $z = -\log(x)$ to (2.3). The cdf, pdf and the hrf of the exponentiated distribution are

$$\begin{aligned}F(x) &= [1 - G(e^{-x})]^\alpha, \\ f(x) &= \alpha e^{-x} g(e^{-x}) [1 - G(e^{-x})]^{\alpha-1}\end{aligned}$$

and

$$h(x) = \alpha e^{-x} g(e^{-x}) [1 - G(e^{-x})]^{\alpha-1} \{1 - [1 - G(e^{-x})]^\alpha\}^{-1}.$$

A list of papers on the EF of distributions is presented in Table I.

TABLE I
Contributed work on the EF of distributions.

S.No.	Pioneer year	Distribution	Author(s)
1	1967	Exponentiated exponential distribution	Ahuja and Nash (1967) Gupta et al. (1998) Gupta and Kundu (1999, 2001a, b, 2007) Nadarajah (2011) Venkatesan and Sundaram (2011)
2	1993	Exponential Weibull distribution	Mudholkar and Srivastava (1993) Mudholkar et al. (1995) Mudholkar and Hutson (1996) Gupta et al. (1998) Jiang and Murthy (1999)

TABLE I (continuation)

S.No.	Pioneer year	Distribution	Author(s)
			Nassar and Eissa (2003)
			Choudhury (2005)
			Nadarajah and Gupta (2005)
			Singh et al. (2005)
			Pal et al. (2006)
			Ahmed et al. (2008)
			Saleem and Abo-Kasem (2011)
			Mazucheli et al. (2012)
			Qian (2012)
			Barrios and Dios (2012)
			Nadarajah et al. (2013b)
3	1998	Exponentiated gamma distribution	Gupta et al. (1998)
			Nadarajah and Kotz (2006a)
			Nadarajah and Gupta (2007)
			Shawky and Bakoban (2008, 2009, 2012)
4	1998	Exponentiated Pareto distribution	Gupta et al. (1998)
			Nadarajah (2005a)
			Shawky and Abu-Zinadah (2009)
			Afify (2010)
5	2001	Exponentiated Rayleigh distribution	Surles and Padgett (1998, 2001, 2005)
			Raqab (1998)
			Kundu and Raqab (2005)
			Raqab and Kundu (2006)
			Raqab and Madi (2009, 2011)
			Abd-Elfattah (2011)
6	2003	Exponentiated Fréchet distribution	Nadarajah and Kotz (2003)
			Nadarajah and Kotz (2006a)
			Abd-Elfattah and Omima (2009)
			Abd-Elfattah et al. (2010)
			Jamjoom and Al-Saiary (2012)
			Al-Nasser and Al-Omari (2013)
			Marwa et al. (2013)
7	2004	Exponentiated generalized Pareto distribution	Adeyemi and Adebajji (2004)
8	2005	Exponentiated beta distribution	Nadarajah (2005b)
9	2006	Exponentiated generalized extreme value distribution	Adeyemi and Adebajji (2006)
10	2006	Exponentiated log-logistic distribution	Rosaiah et al. (2006)
			Aslam and Jun (2010)
			Rao et al. (2012, 2013)
11	2006	Exponentiated Gumbel distribution	Nadarajah (2006)
			Nadarajah and Kotz (2006a)
			Shirke and Kakade (2007)
			Kakade et al. (2008)
			Persson and Rydén (2010)
12	2006	Exponentiated log-normal distribution	Shirke and Kakade (2006)
			Raja and Mir (2011)
13	2008	Exponential modified Weibull distribution	Carrasco et al. (2008)
			Elbatal (2011)
14	2009	Exponentiated extreme value distribution	Cho et al. (2009)
15	2011	Exponentiated Lindley distribution	Nadarajah et al. (2011)
16	2011	Extended generalized exponential distribution	Kundu and Gupta (2011)

TABLE I (continuation)

S.No.	Pioneer year	Distribution	Author(s)
17	2011	Exponentiated Burr XII distribution	Al-Hussaini and Hussein (2011a, b) Maswadah (2013)
18	2011	Exponentiated generalized gamma distribution	Cordeiro et al. (2011a)
19	2011	Exponentiated generalized inverse Gaussian distribution	Lemonte and Cordeiro (2011)
20	2012	Exponentiated inverted Weibull distribution	Flaih et al. (2012) Kim et al. (2012) Hassan (2013) Aljuaid (2013)
21	2012	Exponentiated Kumaraswamy distribution	Kumar (2012)
22	2012	Exponentiated Lomax distribution	Abdul-Moniem and Abdel-Hameed (2012)
23	2012	Exponentiated Gompertz distribution	El-Gohary (2012)
24	2013	Exponentiated modified Weibull extension distribution	Sarhan and Apaloo (2013)
25	2013	Exponentiated generalized linear exponential distribution	Sarhan et al. (2013)
26	2013	Exponentiated Dagum distribution	Khan (2013)
27	2013	Exponentiated sinh Cauchy distribution	Cooray (2013)
28	2015	Exponentiated geometric distribution	Chakraborty and Gupta (2015)

MARSHALL-OLKIN EXTENDED (MOE) FAMILY OF DISTRIBUTIONS

Marshall and Olkin (1997) proposed a flexible semi-parametric family of distributions and defined a new sf $\bar{F}^{MO}(x)$ by introducing an additional parameter $\alpha > 0$. Marshall and Olkin (1997) called α a tilt parameter and interpreted α in terms of the behavior of the hrfs of \bar{F}^{MO} and G . Their ratio is increasing in t for $\alpha \geq 1$ and decreasing in t for $0 < \alpha < 1$. Nanda and Das (2012) reinterpreted α as a tilt parameter since the hrf of the new family is shifted below ($\alpha \geq 1$) or above ($0 < \alpha \leq 1$) the hrf of the underlying distribution. Specifically, for all $t \geq 0$, $h^{MO}(t) \leq h(t)$ when $\alpha \geq 1$, and $h^{MO}(t) \geq h(t)$ when $0 < \alpha \leq 1$, where $h^{MO}(t)$ and $h(t)$ are the hrfs of the MOE and baseline distributions.

For any baseline pdf $g(t)$, cdf $G(t) = P(T \leq t)$ and sf $\bar{G}(t) = P(T > t)$ of the baseline distribution, the sf $\bar{F}^{MO}(t)$ of the MOE family of distributions is defined by

$$\bar{F}^{MO}(t) = \frac{a\bar{G}(t)}{1 - \bar{a}\bar{G}(t)} = \frac{a\bar{G}(t)}{G(t) + a\bar{G}(t)} \quad \text{or} \quad \frac{a[1 - G(t)]}{a + \bar{a}\bar{G}(t)}, \quad (3.1)$$

where $-\infty < t < \infty$, $\alpha > 0$ and $\bar{a} = 1 - \alpha$. The cdf and pdf associated with (3.1) are

$$F^{MO}(t) = \frac{\bar{G}(t)}{1 - \bar{a}\bar{G}(t)} = \frac{\bar{G}(t)}{G(t) + \bar{a}\bar{G}(t)} \quad \text{or} \quad \frac{1 - G(t)}{a + \bar{a}\bar{G}(t)},$$

and

$$f^{MO}(t) = \frac{ag(t)}{[1 - \bar{a}\bar{G}(t)]^2} \quad \text{or} \quad \frac{ag(t)}{[a + \bar{a}\bar{G}(t)]^2},$$

where $-\infty < t < \infty$, $\alpha > 0$ and $\bar{a} = 1 - \alpha$. If $\alpha = 1$, then we have $\bar{F}^{MO}(t) = \bar{G}(t)$. Other reliability measures like the hrf, rhrf and chrh associated with (3.1) are

$$h^{MO}(t) = \frac{f^{MO}(t)}{\bar{F}^{MO}(t)} = \frac{g(t)}{\bar{G}(t)} \frac{1}{[1 - \bar{a}\bar{G}(t)]} = \frac{h(t)}{1 - \bar{a}\bar{G}(t)} \quad \text{or} \quad \frac{h(t)}{a + \bar{a}\bar{G}(t)},$$

$$r^{MO}(t) = \frac{f^{MO}(t)}{F^{MO}(t)} = a \frac{g(t)}{\bar{G}(t)} \frac{1}{[1 - \bar{a}\bar{G}(t)]} = \frac{ah(t)}{1 - \bar{a}\bar{G}(t)} \quad \text{or} \quad \frac{ah(t)}{a + \bar{a}\bar{G}(t)},$$

and

$$H^{MO}(t) = -\log \left[\frac{a\bar{G}(t)}{1 - \bar{a}\bar{G}(t)} \right] \quad \text{or} \quad -\log \left\{ \frac{a[1 - G(t)]}{a + \bar{a}\bar{G}(t)} \right\},$$

where $h(t)$ is the hrf of the baseline distribution.

Note that if we define

$$\bar{F}^{MO}(t) = \frac{\bar{G}(t)}{1 - \bar{a}\bar{G}(t)}$$

then

$$F^{MO}(t) = \frac{a\bar{G}(t)}{1 - \bar{a}\bar{G}(t)} \quad \text{and} \quad f^{MO}(t) = \frac{g(t)}{[1 - \bar{a}\bar{G}(t)]^2},$$

For more general results on the MOE family of distributions, the reader is referred to Barakat et al. (2009), Jose (2011), Krishna (2011), Barreto-Souza et al. (2013) and Cordeiro et al. (2014c).

EXISTING GENERALIZED MOE FAMILY OF DISTRIBUTIONS

In this section, we describe existing generalized Marshall-Olkin families of distributions.

Jayakumar and Mathew (2008) proposed a generalization of the Marshall and Olkin (1997) family of distributions (by using the LA1 approach) as

$$\bar{F}^{GMO}(t) = \left[\frac{a\bar{G}(t)}{1 - \bar{a}\bar{G}(t)} \right]^\theta, \quad (3.2)$$

where $-\infty < t < \infty$, $\alpha > 0$, and $\theta > 0$ is an additional shape parameter. When $\theta = 1$, $\bar{F}^{GMO}(t) = \bar{F}^{MO}(t)$. The cdf and the pdf associated with (3.2) are

$$F^{GMO}(t) = 1 - \left[\frac{a\bar{G}(t)}{1 - \bar{a}\bar{G}(t)} \right]^\theta,$$

and

$$f^{GMO}(t) = \theta \left[\frac{a\bar{G}(t)}{1 - \bar{a}\bar{G}(t)} \right]^{\theta-1} \left\{ \frac{ag(t)}{[1 - \bar{a}\bar{G}(t)]^2} \right\}.$$

Other reliability measures like the hrf, rhf and chrf associated with (3.2) are

$$h^{GMO}(t) = \theta \frac{g(t)}{\bar{G}(t)} \frac{1}{1 - \bar{a}\bar{G}(t)} = \theta h(t) \frac{1}{1 - \bar{a}\bar{G}(t)}, \quad \text{or} \quad \frac{\theta h(t)}{a + \bar{a}\bar{G}(t)},$$

$$r^{GMO}(t) = \frac{\theta a^\theta g(t) G(t)^{\theta-1}}{[1 - \bar{a}\bar{G}(t)]^\theta - a^\theta G(t)^\theta},$$

and

$$H^{GMO}(t) = -\log \left\{ 1 - \left[\frac{a\bar{G}(t)}{[1 - \bar{a}\bar{G}(t)]} \right]^\theta \right\},$$

where $h(t)$ is the hrf of the baseline distribution.

A NEW GENERALIZED MOE FAMILY OF DISTRIBUTIONS

Here, we propose another generalization of the Marshall and Olkin (1997) family of distributions. Using the LA2 approach to the sf of the MOE family of distributions, we obtain

$$\bar{F}^{G2MO}(t) = 1 - \left[1 - \frac{a\bar{G}(t)}{1 - \bar{a}\bar{G}(t)} \right]^\theta, \quad (3.3)$$

where $-\infty < t < \infty$, $\alpha > 0$, and $\theta > 0$ is the additional shape parameter. When $\theta = 1$, $\bar{F}^{G2MO}(t) = \bar{F}^{MO}(t)$. The cdf and the pdf associated with (3.3) are

$$F^{G2MO}(t) = \left[1 - \frac{a\bar{G}(t)}{1 - \bar{a}\bar{G}(t)} \right]^\theta,$$

and

$$f^{G2MO}(t) = \theta \left[1 - \frac{a\bar{G}(t)}{1 - \bar{a}\bar{G}(t)} \right]^{\theta-1} \left\{ \frac{ag(x)}{[1 - \bar{a}\bar{G}(t)]^2} \right\}.$$

After simplification, the above pdf can be rewritten as

$$f^{G2MO}(t) = \frac{\theta ag(t)G(t)^{\theta-1}}{1 - \bar{a}\bar{G}(t)} \quad \text{or} \quad \frac{\theta ag(t)G(t)^{\theta-1}}{a + \bar{a}\bar{G}(t)}.$$

Other reliability measures like the hrf, rhrf and chrf associated with (3.3) are

$$h^{G2MO}(t) = \frac{\theta ag(t)G(t)^{\theta-1}}{[a + \bar{a}\bar{G}(t)]^{\theta+1}} \left\{ 1 - \left[1 - \frac{a\bar{G}(t)}{[1 - \bar{a}\bar{G}(t)]} \right]^\theta \right\}^{-1},$$

$$r^{G2MO}(t) = \theta a \frac{g(t)}{G(t)} \frac{1}{a + \bar{a}\bar{G}(t)} = \frac{\theta a r(t)}{a + \bar{a}\bar{G}(t)},$$

and

$$H^{G2MO}(t) = -\log \left\{ 1 - \left[1 - \frac{a\bar{G}(t)}{[1 - \bar{a}\bar{G}(t)]} \right]^\theta \right\},$$

where $r(t)$ is the rhrf of the baseline distribution.

The construction in (3.3) is similar to that due to Jayakumar and Mathew (2008). But there is an important distinction. Suppose that a system consists of θ independent components. Suppose too that each component has a lifetime with the sf given by $a\bar{G}(t) / [1 - \bar{a}\bar{G}(t)]$. Then (3.2) is the sf of the minimum of the lifetimes and (3.3) is the sf of the maximum of the lifetimes. So, (3.2) can be used to model the minimum of the lifetimes and (3.3) can be used to model the maximum of the lifetimes.

SEMI-TYPE PROCESSES BASED ON CHARACTERISTIC FUNCTION

In this section, we briefly discuss semi-Pareto, semi-Burr, semi-Laplace, semi-logistic and semi-Weibull distributions based on the characteristic function (cf) $\psi(t)$ of the baseline distribution. The concept of semi-type distributions arose from the minification process. Tavares (1980) defined a minification process as observations in a process generated by

$$X_n = k \min(X_{n-1}, \epsilon_n), \quad (3.4)$$

where $n \geq 1$, $k > 1$ is a constant and $\{\epsilon_n\}$ is an innovation process of independent and identically distributed random variables. Here, $\{X_n\}$ is called the first order autoregressive AR(1) minification process. There exists many modified minification processes.

Linnik (1963) introduced the α -Laplace distribution, a symmetric distribution defined on $(-\infty, \infty)$. For $\alpha=2$, the Linnik distribution reduces to the Laplace distribution. Pillai (1985) generalized the Linnik distribution and introduced the semi- α -Laplace distribution.

Yeh et al. (1988) modified (3.4) and introduced the first auto-regressive Pareto minification process having Pareto marginals. Arnold and Robertson (1989) introduced minification processes with logistic marginals. Pillai (1991) and Pillai et al. (1995) introduced semi-Pareto minification processes. Balakrishna (1998) investigated some properties and estimated the unknown parameters of Pillai's semi-Pareto minification process.

Pillai (1985) proposed the *semi- α Laplace* distribution. Its sf is

$$\bar{F}^{SLap}(t) = \frac{1}{1 + \psi(t)},$$

where $\psi(t)$ satisfies the functional equation

$$\psi(t) = \frac{1}{p} \psi\left(tp^{1/\alpha}\right), \quad (3.5)$$

where $\alpha > 0$ and $0 < p < 1$. The solution of (3.5) is $\psi(t) = |t|^\alpha \eta(t)$, where $\eta(t)$ is periodic in $\log |t|$. In the particular case $\eta(t) = c$, the *semi- α -Laplace* distribution reduces to the Linnik distribution.

A random variable T is said to have the *semi-Pareto* distribution if its sf is

$$\bar{F}^{SP}(t) = \frac{1}{1 + \psi(t)},$$

where $t > 0$ and $\psi(t)$ satisfies the functional equation

$$\psi(t) = \frac{1}{p} \psi\left(p^{1/\gamma}(t)\right), \quad (3.6)$$

where $0 < p < 1$, $t > 0$ and $\gamma > 0$. The solution of (3.6) is $\psi(t) = t^\gamma \eta(t)$, where $\eta(t)$ is periodic in $\log t$ with period $\left(\frac{-2\pi\gamma}{\log p}\right)$. Further details are in Pillai (1991) and Pillai et al. (1995).

If $\psi(t) = t^\gamma$ (that is for $\eta(t) = 1$), we obtain the *semi-Pareto distribution of type III* having the sf

$$\bar{F}^{SP3}(t) = \frac{1}{1 + t^\gamma},$$

where $t > 0$ and $\gamma > 0$. For details, see Chrapek et al. (1996), Balakrishna (1998) and Cifarelli et al. (2010).

A random variable T is said to have the *semi-Burr* distribution if its sf is

$$\bar{F}^{SB}(t) = \left[\frac{1}{1 + \psi(t)} \right]^\beta,$$

where $t > 0$, $\beta > 0$ and $\psi(t)$ satisfies the same functional as (3.6).

Cifarelli et al. (2010) expressed the sf of the *semi-Burr* distribution as

$$\bar{F}^{SB}(t) = \frac{1}{[1 + \psi(t)]^{b+1}},$$

where $\psi(t)$ satisfies the same functional as (3.6) and $b > 0$.

According to Arnold (1992) and Jayakumar and Mathew (2005), a random variable T is said to have the *semi-logistic* distribution if its sf is

$$\bar{F}^{SL}(t) = \frac{1}{1 + \psi(t)},$$

where $\psi(t)$ is a nondecreasing and right-continuous function satisfying

$$\psi(t) = \frac{1}{p} \psi \left(t + \frac{1}{\sigma} \log p \right), \quad (3.7)$$

where $0 < p < 1$, $t > 0$, and $\sigma > 0$.

According to Jose (1994) and Thomas and Jose (2005), a random variable T is said to have the *semi-Weibull* distribution if its sf is

$$\bar{F}^{SW}(t) = \exp [-\psi(t)],$$

where $\psi(t)$ satisfies the functional equation

$$p\psi(t) = \psi \left(p^{1/\gamma}(t) \right), \quad (3.8)$$

where $\gamma > 0$ and $0 < p < 1$. Note that (3.8) yields the iterative solution

$$p^n \psi(t) = \psi \left(p^{n/\gamma}(t) \right).$$

Solving (3.8), we have $\psi(t) = t^\gamma h(t)$, where $h(t)$ is periodic in $\log t$ with period $\left(\frac{-2\pi\gamma}{\log p} \right)$.

More details are in Thomas and Jose (2005).

SEMI-TYPE MARSHALL-OLKIN DISTRIBUTIONS BASED ON CHARACTERISTIC FUNCTION

Using (3.1), various authors have proposed *Marshall-Olkin semi-type* distributions from the baseline cf $\psi(t)$.

Alice and Jose (2003) introduced the *Marshall-Olkin semi-Pareto* (MOSP) distribution with sf

$$\bar{F}^{MOSP3}(t) = \frac{1}{1 + \frac{1}{a}\psi(t)},$$

and established geometric extreme stability. Thomas and Jose (2005) and Alice and Jose (2005b) introduced the *Marshall-Olkin semi-Weibull* distribution with sf

$$\bar{F}^{MOSW}(t) = \frac{a}{e^{\psi(t)} - (1 - a)},$$

where $t > 0$ and $\alpha > 0$. Jayakumar and Mathew (2008) proposed the *Marshall-Olkin semi-Burr* (GMOSB) distribution as that defined by the sf

$$\bar{F}^{GMOSB}(t) = \left[\frac{a}{a + \psi(t)} \right]^\beta = \left[\frac{1}{1 + \frac{1}{a}\psi(t)} \right]^\beta = [\bar{F}^{MOSP3}(t)]^\beta,$$

where $\alpha > 0$, $\beta > 0$ and $\psi(t)$ satisfies the same functional as (3.7).

If $0 < \alpha < 1$ and $\varphi(t)$ is a valid cf then

$$\psi(\varphi(t)) = \frac{a \varphi(t)}{1 - (1 - a) \varphi(t)}$$

is also a valid cf. Using this fact, Krishna and Jose (2011) defined the *Marshall-Olkin generalized asymmetric Laplace* distribution as that having the cf

$$\psi(t) = \frac{a}{\left(1 - \frac{it}{\lambda_1}\right)^{\beta_1} \left(1 + \frac{it}{\lambda_2}\right)^{\beta_2} + a - 1},$$

where $i = \sqrt{-1}$, $0 < \alpha < 1$, $\lambda_1 > 0$, $\lambda_2 > 0$, $\beta_1 > 0$ and $\beta_2 > 0$. George and George (2013) defined the *Marshall-Olkin Esscher transformed Laplace* distribution as that having the cf

$$\psi(t) = \left[1 + \frac{1}{a} \left(\frac{t^2}{1 - \theta^2} - \frac{2it\theta}{1 - \theta^2} \right) \right]^{-1} = \left\{ 1 + \frac{1}{\lambda^2} [t^2 - 2it\theta] \right\}^{-1},$$

where $0 < \alpha \leq 1$, $|\theta| < 1$, $\lambda = \sqrt{a(1 - \theta^2)}$, $k = \frac{\lambda}{\theta + \sqrt{\lambda + \theta^2}}$, $\lambda > 0$ and $k > 0$. Jose and Uma (2009) defined the Marshall-Olkin Linnik and Mittag-Leffler distributions as those having the cfs

$$\psi(t) = \frac{\beta}{(1 + |t|^\alpha)^v + \beta - 1}$$

and

$$\psi(t) = \frac{\beta}{\beta - s^\alpha}$$

respectively, where $v > 0$, $0 < \alpha \leq 2$, and $\beta > 0$.

A list of papers on the MOE family is presented in Table II.

TABLE II
Contributed work on the MOE family of distributions.

S.No.	Pioneer year	Distribution	Author(s)
1	1997	MOE exponential distribution	Marshall and Olkin (1997) Alice and Jose (2004b) Parikh et al. (2008) Salah et al. (2009) Bdair (2011) Gopal and Damodaran (2011) Krishna (2011) Rao et al. (2011) Salah (2012) Pushkarna et al. (2013)
2	1997	MOE Weibull distribution	Marshall and Olkin (1997) Jose and Alice (2001) Hirose (2002) Ghitany et al. (2005) Zhang and Xie (2007) Caroni (2010) Srivastava and Kumar (2011) Athar et al. (2012) Cordeiro and Lemonte (2013)
3	2003	MOE Pareto-III distribution	Alice and Jose (2003)
4	2003	MOE Pareto-III distribution	Alice and Jose (2003)
5	2004	MOE Pareto-I distribution	Alice and Jose (2004a) Ghitany (2005)
6	2005	MOE logistic distribution	Alice and Jose (2005a) Kumar (2013)
7	2005	MO semi-logistic distribution	Alice and Jose (2005a)

TABLE II (continuation)

S.No.	Pioneer year	Distribution	Author(s)
8	2005	MO semi-Weibull distribution	Alice and Jose (2005b) Thomas and Jose (2005)
9	2005	MOE Fréchet distribution	Jose and Alice (2005) Krishna (2011) Krishna et al. (2013)
10	2007	MOE Lomax distribution	Ghitany et al. (2007) Gupta et al. (2010)
11	2007	MOE linear failure rate distribution	Ghitany and Kotz (2007)
12	2007	MOE gamma distribution	Ristić et al. (2007) Jose (2009)
13	2008	MOE q-Weibull distribution	Naik et al. (2008) Jose et al. (2010)
14	2008	MOE Burr distribution	Jayakumar and Mathew (2008) El-Bassiouny and Abdo (2010)
15	2008	MOE semi-Burr distribution†	Jayakumar and Mathew (2008)
16	2008	MOE semi-Pareto III distribution†	Jayakumar and Mathew (2008)
17	2009	MOE Linnik distribution†	Jose and Uma (2009)
18	2009	MOE Mittag-Leffler distribution†	Jose and Uma (2009)
19	2009	MOE beta distribution	Jose et al. (2009)
20	2011	MOE uniform distribution	Krishna (2011) Jose and Krishna (2011)
21	2011	MOE Gumbel distribution	Jose (2011)
22	2011	MOE generalized asymmetric Laplace distribution†	Krishna (2011) Krishna and Jose (2011)
24	2013	MOE Zipf distribution	Pérez-Casany and Casellas (2013)
25	2013	MOE power log-normal distribution	Gui (2013a)
26	2013	MOE log-logistic distribution	Gui (2013b)
27	2013	MOE quasi-Lindley distribution	Gui (2013c)
28	2013	MOE Esscher transformed Laplace distribution†	George and George (2013)

† MO extension based on cf.

BETA DISTRIBUTIONS AND EXISTING BETA G FAMILIES OF DISTRIBUTIONS

Consider the cdf of a beta random variable of type 1 with two shape parameters a and b given by

$$F^{B1}(x) = P(X \leq x) = I_x(a, b) = \frac{B_x(a, b)}{B(a, b)} = \frac{1}{B(a, b)} \int_0^x x^{a-1} (1-x)^{b-1} dx, \quad (4.1)$$

where $a > 0$, $b > 0$, $x \in (0, 1)$, $B_t(a, b) = \int_0^t t^{a-1} (1-t)^{b-1} dt$ is the incomplete beta function, $It(a, b)$ is the

incomplete beta function ratio and $B(a, b) = \int_0^1 t^{a-1} (1-t)^{b-1} dt = \frac{\Gamma(a) \Gamma(b)}{\Gamma(a+b)}$ is the beta function.

The pdf corresponding to (4.1) is

$$f^{B1}(x) = \frac{1}{B(a, b)} x^{a-1} (1-x)^{b-1},$$

where $a > 0$, $b > 0$, $x \in (0, 1)$.

Similarly, the cdf of a beta random variable of type 2 with parameters a and b is

$$F^{B2}(y) = P(Y \leq y) = I_{2y}(a, b) = \frac{B_{2y}(a, b)}{B2(a, b)} = \frac{1}{B2(a, b)} \int_0^y \frac{y^{a-1}}{(1+y)^{a+b}} dy, \quad (4.2)$$

where $a > 0, b > 0, y > 0$, $B_{2t}(a, b) = \int_0^t t^{a-1} (1+t)^{-(a+b)} dt$ is the incomplete beta function, $I_{2t}(a, b)$ is the

incomplete beta function ratio and $B2(a, b) = \int_0^\infty t^{a-1} (1+t)^{-(a+b)} dt = \frac{\Gamma(a) \Gamma(b)}{\Gamma(a+b)}$ is the beta function.

The pdf corresponding to (4.2) is

$$f^{B2}(y) = \frac{1}{B2(a, b)} \frac{y^{a-1}}{(1+y)^{(a+b)}},$$

where $a > 0, b > 0$, and $y > 0$. The beta type 2 distribution is also known as inverted beta distribution as it can be obtained from (4.1) by the transformation $Y = \frac{X}{1-X}$.

Cardeño et al. (2005) introduced the beta type 3 distribution by transforming $Z = \frac{Y}{2-Y}$ in (4.1). The cdf of a beta random variable of type 3 with parameters a and b is

$$F^{B3}(z) = P(Z \leq z) = I_{3z}(a, b) = \frac{B_{3z}(a, b)}{B3(a, b)} = \frac{1}{B3(a, b)} \int_0^z \frac{z^{a-1} (1-z)^{b-1}}{(1+z)^{(a+b)}} dz, \quad (4.3)$$

where $a > 0, b > 0, z \in (0, 1)$, $B_{3t}(a, b) = \int_0^t t^{a-1} (1-t)^{b-1} (1+t)^{-(a+b)} dt$ is the incomplete beta function,

$I_{3t}(a, b)$ is the incomplete beta function ratio and $B3(a, b) = \int_0^1 t^{a-1} (1-t)^{b-1} (1+t)^{-(a+b)} dt = \frac{\Gamma(a) \Gamma(b)}{\Gamma(a+b)}$ is the beta function. The pdf corresponding to (4.3) is

$$f^{B3}(z) = \frac{2^a}{B3(a, b)} \frac{z^{a-1} (1-z)^{b-1}}{(1+z)^{(a+b)}},$$

where $a > 0, b > 0$, and $z \in (0, 1)$.

Eugene et al. (2002) and Jones (2004a) replaced the upper limit x of the integral in (4.1) with $G(x)$. The resulting cdf of beta G family of distributions is

$$F^{BG}(x) = I_{G(x)}(a, b) = \frac{B_{G(x)}(a, b)}{B(a, b)} = \frac{1}{B(a, b)} \int_0^{G(x)} \omega^{a-1} (1-\omega)^{b-1} d\omega. \quad (4.4)$$

The pdf corresponding to (4.4) is

$$f^{BG}(x) = \frac{1}{B(a, b)} g(x) G(x)^{a-1} [1-G(x)]^{b-1}, \quad (4.5)$$

where $g(x) = dG(x)/dx$ denotes the pdf. The beta G family of distributions is also known as the beta logit family. For any lifetime random variable t , the sf, hrf, rhrf and chrf associated with (4.4) and (4.5) are

$$\begin{aligned} \bar{F}(t) &= 1 - I_{G(t)}(a, b) = \frac{B(a, b) - B_{G(t)}(a, b)}{B(a, b)}, \\ h(t) &= \frac{g(t)G(t)^{a-1} [1-G(t)]^{b-1}}{B(a, b) [I_{G(t)}(a, b)]} = \frac{g(t)G(t)^{a-1} [1-G(t)]^{b-1}}{B_{G(t)}(a, b)}, \end{aligned}$$

$$r(t) = \frac{g(t)G(t)^{a-1} [1 - G(t)]^{b-1}}{B(a, b) [1 - I_{G(t)}(a, b)]} = \frac{g(t)G(t)^{a-1} [1 - G(t)]^{b-1}}{[B(a, b) - B_{G(t)}(a, b)]},$$

and

$$H(t) = -\log \left[\frac{B(a, b) - B_{G(t)}(a, b)}{B(a, b)} \right].$$

A list of papers on the beta G family of distributions is given in Table III.

TABLE III
Contributed work on the beta G family of distributions.

S.No.	Pioneer year	Distribution	Author(s)
1	2002	Beta normal distribution	Eugene et al. (2002) Famoye et al. (2002) Eugene (2004) Famoye et al. (2004) Gupta and Nadarajah (2004) Jones (2004b) Rêgo et al. (2012)
2	2003	Beta exponential distribution	Maynard (2003) Nadarajah and Kotz (2006b)
3	2004	Beta gamma distribution	Kong (2004), Kong et al. (2007)
4	2004	Beta Gumbel distribution	Nadarajah and Kotz (2004)
5	2004	Beta Fréchet distribution	Nadarajah and Gupta (2004)
6	2005	Beta Weibull distribution	Barreto-Souza et al. (2011) Famoye et al. (2005) Lee et al. (2007) Zografos (2008) Cordeiro et al. (2011b, c) Sun (2011), Mdziniso (2012) Mahmoud and Mandouh (2012a, b, c)
7	2006	Beta Bessel distribution	Gupta and Nadarajah (2006)
8	2008	Beta Pareto distribution	Akinsete et al. (2008)
9	2008	Beta Rayleigh distribution	Akinsete and Lowe (2009)
10	2008	Beta Laplace distribution	Kozubowski and Nadarajah (2008) Cordeiro and Lemonte (2011a)
11	2009	Beta generalized logistic-IV distribution	Morais (2009) Morais et al. (2013)
12	2010	Beta modified Weibull distribution	Silva et al. (2010) Nadarajah et al. (2012b)
13	2010	Beta generalized half-normal distribution	Pescim et al. (2010)
14	2010	Beta generalized exponential distribution	Barreto-Souza et al. (2010)
15	2010	Beta Maxwell distribution	Amusan (2010)
16	2010	Beta hyperbolic secant distribution	Fischer and Vaughan (2010)
17	2010	Beta inverse Weibull distribution	Kersey (2010) Hanook et al. (2013)
18	2011	Beta Cauchy distribution	Alshawarbeh (2011) Alshawarbeh et al. (2012)
19	2011	Beta half-Cauchy distribution	Cordeiro and Lemonte (2011b)
20	2011	Beta Burr XII distribution	Paranaíba et al. (2011)
21	2011	Beta generalized Pareto distribution	Mahmoudi (2011)

TABLE III (continuation)

S.No.	Pioneer year	Distribution	Author(s)
			Nassar and Nada (2011)
22	2011	Beta Birnbaum-Sanders distribution	Cordeiro and Lemonte (2011c)
23	2012	Beta skew-normal distribution	Mameli (2012)
24	2012	Beta exponential-geometric distribution	Bidram (2012)
25	2012	Beta Moyal distribution	Cordeiro et al. (2012d)
26	2012	Beta generalized Weibull distribution	Singla et al. (2012)
27	2012	Beta exponentiated Pareto distribution	Zea et al. (2012)
28	2012	Beta power distribution	Cordeiro and Brito (2012)
29	2012	Beta linear failure rate distribution	Jafari and Mahmoudi (2012)
30	2012	Beta extended Weibull distribution	Cordeiro et al. (2012f)
31	2012	Beta truncated Pareto distribution	Lourenzutti et al. (2012)
32	2013	Beta Weibull-geometric distribution	Cordeiro et al. (2013f)
			Bidram et al. (2013)
33	2013	Beta generalized gamma distribution	Cordeiro et al. (2013a)
34	2013	Beta log-normal distribution	Castellars et al. (2013)
35	2013	Beta generalized Rayleigh distribution	Cordeiro et al. (2013b)
36	2013	Beta generalized logistic distribution	Morais et al. (2013)
37	2013	Beta exponentiated Weibull distribution	Cordeiro et al. (2013c)
38	2013	Beta Nakagami distribution	Shittu and Adepoju (2013)
39	2013	Beta Burr III distribution	Gomes et al. (2013)
40	2013	Beta Dagum distribution	Domma and Condino (2013)
41	2013	Beta Stoppa distribution	Mansoor (2013)
42	2013	Beta inverse Rayleigh distribution	Leˆao et al. (2013)
43	2014	Beta generalized inverse Weibull distribution	Baharith et al. (2014)
44	2014	Beta extended half-normal distribution	Cordeiro et al. (2014f)
45	2014	Beta log-logistic distribution	Lemonte (2014)

MCDONALD DISTRIBUTIONS AND MCDONALD *G* FAMILIES OF DISTRIBUTIONS

MCDONALD TYPE DISTRIBUTIONS

McDonald (1984) replaced the upper limit x of the integral in (4.1) with x^c , where c is an additional (third) shape parameter. The resulting cdf of the McDonald type (Mc) distribution is

$$F(x) = I_{x^c}(a, b) = \frac{B_{x^c}(a, b)}{B(a, b)} = \frac{1}{B(a, b)} \int_0^{x^c} x^{a-1} (1-x)^{b-1} dx, \quad (5.1)$$

where $a > 0$, $b > 0$ and $c > 0$ are the three shape parameters. The Mc distribution includes as special cases the beta type 1 distribution ($c = 1$) and the Kumaraswamy distribution ($a = 1$). The pdf corresponding to (5.1) is

$$f(x) = \frac{c}{B(a, b)} x^{ac-1} (1-x^c)^{b-1},$$

where $0 < x < 1$.

EXISTING MCDONALD *G* FAMILY OF DISTRIBUTIONS

For any baseline cdf $G(x)$, Alexander et al. (2012) replaced the upper limit x^c of the integral in (5.1) with $G(x)^c$. Lemonte and Cordeiro (2013) stated that this simple transformation facilitates the computation of several properties of the *G* family of distributions.

The resulting cdf $F(x)$ of the Mc-generalized family of distributions (Mc G) is

$$F(x) = I_{G(x)^c}(a, b) = \frac{B_{G(x)^c}(a, b)}{B(a, b)} = \frac{1}{B(a, b)} \int_0^{G(x)^c} \omega^{a-1} (1-\omega)^{b-1} d\omega, \quad (5.2)$$

where $I_{G(x)^c}(a, b)$ denotes the incomplete beta function ratio. The pdf corresponding to (5.2) is

$$f(x) = \frac{c}{B(a, b)} g(x) G(x)^{ac-1} [1 - G(x)^c]^{b-1}, \quad (5.3)$$

where $a > 0$, $b > 0$ and $c > 0$ are the three shape parameters. For a lifetime random variable t , the sf, hrf, rhrf and chrf associated with (5.2) and (5.3) are

$$\begin{aligned} \bar{F}(t) &= 1 - I_{G(t)^c}(a, b) = \frac{B(a, b) - B_{G(t)^c}(a, b)}{B(a, b)}, \\ h(t) &= \frac{c g(t) G(t)^{ac-1} [1 - G(t)^c]^{b-1}}{B(a, b) [I_{G(t)^c}(a, b)]} = \frac{c g(t) G(t)^{ac-1} [1 - G(t)^c]^{b-1}}{B_{G(t)^c}(a, b)}, \\ r(t) &= \frac{c g(t) G(t)^{ac-1} [1 - G(t)^c]^{b-1}}{B(a, b) [1 - I_{G(t)^c}(a, b)]} = \frac{c g(t) G(t)^{ac-1} [1 - G(t)^c]^{b-1}}{[B(a, b) - B_{G(t)^c}(a, b)]}, \end{aligned}$$

and

$$H(t) = -\log \left[\frac{B(a, b) - B_{G(t)^c}(a, b)}{B(a, b)} \right].$$

NOTES ON EXISTING MC G FAMILIES OF DISTRIBUTIONS

The three shape parameters a , b and c introduce skewness, kurtosis, and vary tail weights. The parameters control skewness and kurtosis through altering the tail entropy (Alexander et al. 2012). They also control skewness and kurtosis through adding entropy to the center of the baseline distribution (Alexander et al. 2012). Cordeiro et al. (2014b) mentioned that a and b are skewness parameters that control relative tail weights but not the peak, but c provides the control over the peak.

Alexander et al. (2012), Marciano et al. (2012), Cordeiro and Lemonte (2012, 2014), Cordeiro et al. (2012a, b, 2013d, 2014b), Lemonte and Cordeiro (2013) and Gomes et al. (2013a) used Mc G distributions for developing McDonald normal, McDonald (extended) exponential, McDonald gamma, McDonald inverted beta, McDonald arcsine, McDonald Weibull, McDonald Birnbaum-Sanders (fatigue life), McDonald Lomax, McDonald Burr XII and McDonald Burr III distributions. These authors believe that the Mc G family of distributions can fit skew data better than existing distributions. The Mc G family of distributions is most applicable when $G(x)$ and $g(x)$ take simple analytical forms.

The Mc G family of distributions reduces to the beta G family of distribution for $c = 1$ and to the Kw G family of distribution for $a = c$. Further, the Mc G family of distributions for $G(x) = x$ contains as particular cases the beta type 1 distribution ($c = 1$) and the Kumaraswamy distribution ($a = c$).

Zografos (2011) studied a family of distributions based on McDonald and Xu (1995)'s generalized beta distribution. This family was called the family of generalized beta generated (GBG) distributions.

A list of papers on the Mc G family of distributions is given in Table IV.

TABLE IV
Contributed work on the Mc G family of distributions.

S.No.	Pioneer year	Distribution	Author(s)
1	2010	McDonald Kumaraswamy distribution	Carrasco et al. (2010)
2	2012	McDonald exponential distribution	Cordeiro et al. (2012f)
3	2012	McDonald gamma distribution	Marciano et al. (2012)
4	2012	McDonald inverted beta distribution	Cordeiro and Lemonte (2012)
5	2012	McDonald normal distribution	Cordeiro et al. (2012a)
6	2012	McDonald extended exponential distribution	Cordeiro et al. (2012b)
7	2013	McDonald Burr XII distribution	Gomes et al. (2015)
8	2013	McDonald Burr III distribution	Gomes et al. (2015)
9	2013	McDonald Lomax distribution	Lemonte and Cordeiro (2013)
10	2013	McDonald Birnbaum-Sanders distribution	Cordeiro et al. (2013d)
11	2013	McDonald Fisk (log-logistic) distribution	Zubair (2013)
12	2013	McDonald Dagum distribution	Rajasooriya (2013)
			Oluyede and Rajasooriya (2013)
13	2013	McDonald modified Weibull distribution	Merovci and Elbatal (2013)
14	2014	McDonald arcsine distribution	Cordeiro and Lemonte (2014)
15	2014	McDonald Weibull distribution	Cordeiro et al. (2014b)
16	2015	McDonald Burr distribution	Cordeiro et al. (In press)

KUMARASWAMY DISTRIBUTIONS AND KUMARASWAMY G FAMILIES OF DISTRIBUTIONS

Kumaraswamy (1980) argued that the beta distribution does not fairly fit hydrological random variables like rainfall, daily stream flow, etc. Jones (2009) commented that “beta distribution is fairly tractable, but in some ways not fabulously so. In particular its distribution function is an incomplete beta function ratio and its quantile function the inverse thereof”. The Kumaraswamy (Kw) distribution is relatively much appreciated in comparison to the beta distribution, and has a simple form which can be unimodal, increasing, decreasing or constant, depending on the parameter values.

In this section, we give functional forms of Kw distributions. We also propose Kumaraswamy generalized families of distributions.

EXISTING KUMARASWAMY DISTRIBUTIONS

The Kw distribution has the cdf and the pdf specified by

$$F(x) = 1 - (1 - x^a)^b, \quad (6.1)$$

and

$$f(x) = a b x^{a-1} (1 - x^a)^{b-1}, \quad (6.2)$$

respectively, where $0 < x < 1$ and $a > 0, b > 0$ are both shape parameters.

EXISTING KUMARASWAMY G FAMILY OF DISTRIBUTIONS

For a baseline cdf $G(x)$ with pdf $g(x)$, Cordeiro and de Castro (2011) defined the Kw G distribution specified by the cdf and the pdf

$$F(x) = 1 - [1 - G(x)^a]^b, \quad (6.3)$$

and

$$f(x) = a b g(x) G(x)^{a-1} [1 - G(x)]^{b-1}, \quad (6.4)$$

where $x > 0$, $g(x) = dG(x) = dx$ and $a > 0$, $b > 0$ are shape parameters in addition to those in the baseline distribution. They partly govern skewness and vary tail weights. For a lifetime random variable t , the sf, hrf, rhrf and chrf associated with (6.3) and (6.4) are

$$\begin{aligned} \bar{F}(t) &= [1 - G(t)]^b, \\ h(t) &= a b g(t) G(t)^{a-1} [1 - G(t)]^{b-1}, \\ r(t) &= a b g(t) G(t)^{a-1} [1 - G(t)]^{b-1} \{1 - [1 - G(t)]^b\}^{-1}, \end{aligned}$$

and

$$H(t) = -b \log [1 - G(t)].$$

NOTES ON KUMARASWAMY G FAMILIES OF DISTRIBUTIONS

Equations (6.3) and (6.4) do not involve any special function like the beta function, incomplete beta function, incomplete beta ratio, gamma function, incomplete gamma function or the incomplete gamma ratio. Therefore, the generalization in (6.3) and (6.4) is computationally more efficient compared to beta G and Mc G families of distributions.

The Kw G families of distributions are more flexible than the baseline distribution in the sense that the families allow for greater flexibility of tail properties. Their second benefit is their ability to fit skew data that cannot be properly fitted by existing distributions.

NOTES ON KUMARASWAMY G FAMILIES OF DISTRIBUTIONS

Equations (6.3) and (6.4) do not involve any special function like the beta function, incomplete beta function, incomplete beta ratio, gamma function, incomplete gamma function or the incomplete gamma ratio. Therefore, the generalization in (6.3) and (6.4) is computationally more efficient compared to beta G and Mc G families of distributions.

The Kw G families of distributions are more flexible than the baseline distribution in the sense that the families allow for greater flexibility of tail properties. Their second benefit is their ability to fit skew data that cannot be properly fitted by existing distributions.

A list of papers on the Kw G family of distributions is given in Table V.

TABLE V
Contributed work on Kumaraswamy G family of distributions.

S.No.	Pioneer year	Distribution	Author(s)
1	2010	Kumaraswamy Weibull distribution	Cordeiro et al. (2010)
2	2011	Kumaraswamy generalized gamma distribution	de Pascoa et al. (2011)
3	2011	Kumaraswamy skew-normal distribution	Kazemi et al. (2011)
			Mameli (2012)
			Mameli and Musio (2013)
4	2011	Kumaraswamy Gumbel minimum distribution	El-Sherpieny and Ahmed (2011)
5	2012	Kumaraswamy log-logistic distribution	de Santana et al. (2012)
			Muthulakshmi and Selvi (2013)
6	2012	Kumaraswamy Gumbel distribution	Cordeiro et al. (2012c)
7	2012	Kumaraswamy Birnbaum-Sanders distribution	Saulo et al. (2012)

TABLE V (continuation)

S.No.	Pioneer year	Distribution	Author(s)
8	2012	Kumaraswamy generalized half-normal distribution	Cordeiro et al. (2012e)
9	2012	Kumaraswamy inverse Weibull distribution	Shahbaz et al. (2012)
10	2012	Kumaraswamy normal distribution	Correa et al. (2012)
11	2012	Kumaraswamy generalized inverse Weibull distribution	Yang (2012)
12	2013	Kumaraswamy Pareto distribution	Bourguignon et al. (2013)
13	2013	Kumaraswamy generalized Pareto distribution	Nadarajah and Eljabri (2013)
14	2013	Kumaraswamy Burr XII distribution	Paranaíba et al. (2013)
15	2013	Kumaraswamy generalized extreme value distribution	Eljabri (2013)
16	2013	Kumaraswamy linear exponential distribution	Elbatal (2013a)
17	2013	Kumaraswamy generalized linear failure rate distribution	Elbatal (2013b)
18	2013	Kumaraswamy exponentiated Pareto distribution	Elbatal (2013c)
19	2013	Kumaraswamy Lomax distribution	Shams (2013)
20	2013	Exponentiated Fisk (log-logistic) distribution	Zubair (2013)
21	2014	Exponentiated generalized Burr III distribution	Zubair (2013)
22	2014	Kumaraswamy modified Weibull distribution	Cordeiro et al. (2014d)
23	2014	Kumaraswamy generalized Rayleigh distribution	Gomes et al. (2014)

NEW KUMARASWAMY TYPE DISTRIBUTION

Setting $X = 1 - Y$ in (6.1) and (6.2), we obtain a distribution specified by the cdf and the pdf

$$F(x) = 1 - [1 - (1 - x)^a]^b \quad (6.5)$$

and

$$f(x) = a b (1 - x)^{a-1} [1 - (1 - x)^a]^{b-1},$$

where $0 < x < 1$ and $a > 0, b > 0$ are the shape parameters.

OTHER KW G FAMILIES OF DISTRIBUTIONS

Replacing x with $G(x)$ in (6.5), we obtain a Kw G distribution specified by the cdf

$$F(x) = 1 - \{1 - [1 - G(x)]^a\}^b, \quad (6.6)$$

where $a > 0$ and $b > 0$ are both shape parameters. The pdf corresponding to (6.7) is

$$f(x) = a b g(x) [1 - G(x)]^{a-1} \{1 - [1 - G(x)]^a\}^{b-1}. \quad (6.7)$$

Equations (6.6) and (6.7) are the cdf and the pdf of the Exp G family of distributions recently proposed by Cordeiro et al. (2013e). For a lifetime random variable t , the sf, hrf, rhrf and chrh associated with (6.6) and (6.7) are

$$\begin{aligned} \bar{F}(t) &= \{1 - [1 - G(t)]^a\}^b, \\ h(t) &= a b g(t) [1 - G(t)]^{a-1} \{1 - [1 - G(t)]^a\}^{-1}, \\ r(t) &= a b g(t) [1 - G(t)]^{a-1} \{1 - [1 - G(t)]^a\}^{b-1} [1 - \{1 - [1 - G(t)]^a\}^b]^{-1}, \end{aligned}$$

and

$$H(t) = -b \log \{1 - [1 - G(t)]^a\}.$$

CONCLUSIONS

We first refer to some important surveys on the developments of continuous univariate distributions: Kotz and Vicari (2005) surveyed the developments in the theory of skewed continuous distributions; Gupta and Kundu (2009) described six different methods for the induction of shape and/or skewness parameter(s) in univariate probability distributions; Chakraborty and Hazarika (2011) surveyed the theoretical developments of the univariate skew-normal distribution, its extensions and generalizations; Lee et al. (2013) surveyed recent methods for generating families of univariate continuous distributions. They discussed five general methods for generating G families of distributions: (1) method for generating skewed distributions, (2) method for adding parameters (e.g., exponentiation), (3) beta G , (4) transformed-transformer (T-X) family, and (5) composite method. Recently, Nadarajah (2015a, 2015b) introduced the R package *Newdistsn* which computes the pdf, cdf, quantiles and random numbers for nineteen general families of distributions.

In this paper, we have discussed the well-established and widely used G families of distributions: the EF of distributions, the MOE distributions, the beta G distributions, the Mc G distributions, the Kw G distributions and the Exp G distributions. We have provided exhaustive lists of papers on these families of distributions. We have cited 28 papers on the EF of distributions, 28 papers on the MOE distributions, 45 papers on the beta G distributions, 16 papers on the Mc G distributions, 21 papers on the Kw G distributions and 2 papers on the Exp G distributions. The literature review in Lee et al. (2013) appears less detailed.

We have introduced several new families of distributions relating to the MOE distributions and the Kw G distributions. Of course, this is not an attempt to increase the frequency of articles on new families of distributions but rather to effectively explore real life phenomena through data sets available from different fields. We have noted that contributors (practitioners) have used different model selection criteria: the maximized log-likelihood $\ell(\hat{\theta})$, the Akaike Information Criterion (AIC), the Bayesian Information Criterion (BIC), the Consistent Akaike Information Criterion (CAIC), the Hannan-Quinn Information Criterion (HQIC), the Cramer-von-Mises (W^*), the Anderson-Darling (A^*), the Wald (W) statistic, the Kolmogorov-Smirnov (K-S) test and graphical inspection of the proximity of histograms to the fitted pdfs.

Tractability and effectiveness for modeling censored data require, among other things, closed form expressions for the cdf. So, the Kw G distributions can be tractable and effective models for censored data. The EF and MOE distributions can also be tractable and effective models for censored data, provided G is in closed form. However, beta G and Mc G distributions may not be tractable or effective models for censored data since their cdfs involve the incomplete beta function.

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RESUMO

O método de adicionar parâmetros a uma distribuição especificada tem sido bastante adotado nos últimos anos. A adição de um ou mais parâmetros de forma torna a distribuição gerada mais flexível especialmente no estudo de suas propriedades. Esse método tem se mostrado eficaz, também, na melhoria das estatísticas de adequação do ajuste da nova distribuição. Desde 1985, muitas famílias de distribuições contínuas geradas por esse método têm sido investigadas. Neste artigo, as famílias geradoras mais conhecidas de distribuições como a família estendida de Marshall-Olkin, a família beta, as famílias generalizadas de McDonald e Kumaraswamy e as famílias exponencializadas são discutidas. Apresentam-se as referências mais importantes dessas famílias. Algumas formas mais amplas da família estendida de Marshall-Olkin e da família generalizada de Kumaraswamy são propostas.

Palavras-chave: Distribuição beta, família exponencializada, distribuição de Kumaraswamy, família de Marshall-Olkin, distribuição de McDonald, propriedades da confiabilidade.

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