Anais da Academia Brasileira de Ciências (2016) 88(4): 2053-2065
(Annals of the Brazilian Academy of Sciences)
Printed version ISSN 0001-3765 / Online version ISSN 1678-2690
http://dx.doi.org/10.1590/0001-3765201620150085
www.scielo.br/aabc

On reduced L^2 cohomology of hypersurfaces in spheres with finite total curvature

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Manuscript received on February 6, 2015; accepted for publication on June 17, 2016

ABSTRACT

In this paper, we prove that the dimension of the second space of reduced L^2 cohomology of M is finite if M is a complete noncompact hypersurface in a sphere $\mathbb{S}^{n+1}$ and has finite total curvature $(n \geq 3)$.

Key words: total curvature, reduced L^2 cohomology, hypersurface in sphere, L^2 harmonic 2-form.

INTRODUCTION

For a complete manifold M^n, the p-th space of reduced L^2-cohomology is defined, for $0 \leq p \leq n$ in Carron (2007). It is interesting and important to discuss the finiteness of the dimension of these spaces. Carron (1999) proved that if M^n $(n \geq 3)$ is a complete noncompact submanifold of $\mathbb{R}^{n+p}$ with finite total curvature and finite mean curvature (i. e., the L^n-norm of the mean curvature vector is finite), then each p-th space of reduced L^2-cohomology on M has finite dimension, for $0 \leq p \leq n$. The reduced L^2 cohomology is related with the L^2 harmonic forms (Carron 2007). In fact, several mathematicians studied the space of L^2 harmonic p-forms for $p = 1, 2$. If M^n $(n \geq 3)$ is a complete minimal hypersurface in $\mathbb{R}^{n+1}$ with finite index, Li and Wang (2002) proved that the dimension of the space of the L^2 harmonic 1-forms M is finite and M has finitely many ends. More generally, Zhu (2013) showed that: suppose that N^{n+1} $(n \geq 3)$ is a complete simply connected manifold with non-positive sectional curvature and M^n is a complete minimal hypersurface in N with finite index. If the bi-Ricci curvature satisfies

$$b - \overline{Ric}(X, Y) + \frac{1}{n}|A|^2 \geq 0,$$

for all orthonormal tangent vectors X, Y in T_pN for $p \in M$, then the dimension of the space of the L^2 harmonic 1-forms M is finite. Furthermore, following the idea of Cheng and Zhou (2009), Zhu (2013) gave a result on finitely many ends of complete manifolds with a weighted Poincaré inequality by use of the

AMS Classification: 53C20, 53C40.
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space of L^2 harmonic functions. Cavalcante et al. (2014) discussed a complete noncompact submanifold M^n $(n \geq 3)$ isometrically immersed in a Hadamard manifold N^{n+p} with sectional curvature satisfying $-k^2 \leq K_N \leq 0$ for some constant k and showed that if the total curvature is finite and the first eigenvalue of the Laplacian operator of M is bounded from below by a suitable constant, then the dimension of the space of the L^2 harmonic 1-forms on M is finite. Fu and Xu (2010) studied a complete submanifold M^n in a sphere $\mathbb{S}^{n+p}$ with finite total curvature and bounded mean curvature and proved that the dimension of the space of the L^2 harmonic 1-forms on M is finite. Zhu and Fang (2014) proved Fu-Xu's result without the restriction on the mean curvature vector and therefore obtained that the first space of reduced L^2-cohomology on M has finite dimension. Zhu (2011) studied the existence of the symplectic structure and L^2 harmonic 2-forms on complete noncompact manifolds by use of a special version of Bochner formula.

Motivated by above results, we discuss a complete noncompact hypersurface M^n in a sphere $\mathbb{S}^{n+1}$ with finite total curvature in this paper. We obtain the following finiteness results on the space of all L^2 harmonic 2-forms and the second space of reduced L^2 cohomology:

Theorem 1. *Let M^n $(n \geq 3)$ be an n-dimensional complete noncompact oriented manifold isometrically immersed in an $(n+1)$-dimensional sphere $\mathbb{S}^{n+1}$. If the total curvature is finite, then the space of all L^2 harmonic 2-forms has finite dimension.*

Corollary 2. *Let M^n $(n \geq 3)$ be an n-dimensional complete noncompact oriented manifold isometrically immersed in $\mathbb{S}^{n+1}$. If the total curvature is finite, then the dimension of the second space of reduced L^2 cohomology of M is finite.*

Remark 3. *Under the same condition of Corollary 2, we conjecture that the p-th space of reduced L^2 cohomology of M has finite dimension for $3 \leq p \leq n-3$.*

PRELIMINARIES

In this section, we recall some relevant definitions and results. Suppose that M^n is an n-dimensional complete Riemannian manifold. The Hodge operator $* : \wedge^p(M) \to \wedge^{n-p}(M)$ is defined by

$$*e^{i_1} \wedge \cdots \wedge e^{i_p} = \mathrm{sgn}\sigma(i_1, i_2, \cdots, i_n) e^{i_{p+1}} \wedge \cdots \wedge e^{i_n},$$

where $\sigma(i_1, i_2, \cdots, i_n)$ denotes a permutation of the set $(i_1, i_2, \cdots, i_n)$ and sgnσ is the sign of σ. The operator $d^* : \wedge^p(M) \to \wedge^{p-1}(M)$ is given by

$$d^*\omega = (-1)^{(nk+k+1)} * d * \omega.$$

The Laplacian operator is defined by

$$\triangle\omega = -dd^*\omega - d^*d\omega.$$

A p-form ω is called L^2 harmonic if $\triangle\omega = 0$ and

$$\int_M \omega \wedge *\omega < +\infty.$$

We denote by $H^p(L^2(M))$ the space of all L^2 harmonic p-forms on M. Let

$$Z_2^p(M) = \{\alpha \in L^2(\wedge^p(T^*M)) : d\alpha = 0\}$$

and

$$D^p(d) = \{\alpha \in L^2(\wedge^p(T^*M)) : d\alpha \in L^2(\wedge^{p+1}(T^*M))\}.$$

We define the p-th space of reduced L^2 cohomology by

$$H_2^p(M) = \frac{Z_2^p(M)}{\overline{D^{p-1}(d)}}.$$

Suppose that $x : M^n \to \mathbb{S}^{n+1}$ is an isometric immersion of an n-dimensional manifold M in an $(n+1)$-dimensional sphere. Let A denote the second fundamental form and H the mean curvature of the immersion x. Let

$$\Phi(X,Y) = A(X,Y) - H\langle X,Y\rangle,$$

for all vector fields X and Y, where $\langle , \rangle$ is the induced metric of M. We say the immersion x has finite total curvature if

$$\|\Phi\|_{L^n(M)} < +\infty.$$

We state several results which will be used to prove Theorem 1.

Proposition 4. *(Carron 2007) Let (M,g) is a complete Riemannian manifold, then the space of L^2 harmonic p-forms $H^p(L^2(M))$ is isomorphic to the p-th space of reduced L^2 cohomology $H_2^p(M)$.*

Lemma 5. *(Li 1993) If (M^n,g) is a Riemannian manifold and $\omega = a_I\omega_I \in \wedge^p(M)$, then*

$$\triangle|\omega|^2 = 2\langle\triangle\omega,\omega\rangle + 2|\nabla\omega|^2 + 2\langle E(\omega),\omega\rangle,$$

where $E(\omega) = R_{k_\beta i_\beta j_\alpha i_\alpha} a_{i_1\cdots k_\beta\cdots i_p} e^{i_p} \wedge \ldots \wedge e^{j_\alpha} \wedge \ldots \wedge e^{i_1}$.

Proposition 6. *(Hoffman and Spruck 1974, Zhu and Fang 2014) Let M^n be a complete noncompact oriented manifold isometrically immersed in a sphere $\mathbb{S}^{n+1}$. Then*

$$\left(\int_M |f|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \le C_0\left(\int_M |\nabla f|^2 + n^2\int_M (H^2+1)f^2\right)$$

for each $f \in C_0^1(M)$, where C_0 depends only on n and H is the mean curvature of M in $\mathbb{S}^{n+1}$.

AN INEQUALITY FOR L^2 HARMONIC 2-FORMS

In this section, we show an inequality for L^2 harmonic 2-forms on hypersurfaces in a sphere $\mathbb{S}^{n+1}$, which plays an important role in the proof of main results.

Proposition 7. *Let M^n $(n \ge 3)$ be an n-dimensional complete noncompact hypersurface isometrically immersed in an $(n+1)$-dimensional sphere $\mathbb{S}^{n+1}$. If $\omega \in H^2(L^2(M))$, then*

$$h\triangle h \ge |\nabla h|^2 + 2h^2 - |\Phi|^2h^2 + \frac{3}{2}H^2h^2,$$

for $n = 3$ and

$$h\triangle h \ge \frac{1}{n-2}|\nabla h|^2 + 2(n-2)h^2 - \frac{n-2}{2}|\Phi|^2h^2 + nH^2h^2,$$

for $n \ge 4$, where $h = |\omega|$.

Proof. Suppose that $\omega \in H^2(L^2(M))$. Then we have

$$\triangle|\omega|^2 = 2|\nabla|\omega||^2 + 2|\omega|\triangle|\omega|. \tag{1}$$

By Lemma 5, we get that:

$$\begin{aligned}\triangle|\omega|^2 &= 2\langle\triangle\omega,\omega\rangle + 2|\nabla\omega|^2 + 2\langle E(\omega),\omega\rangle \\ &= 2|\nabla\omega|^2 + 2\langle E(\omega),\omega\rangle.\end{aligned} \tag{2}$$

Combining (1) with (2), we obtain that

$$|\omega|\triangle|\omega| = |\nabla\omega|^2 - |\nabla|\omega||^2 + \langle E(\omega),\omega\rangle. \tag{3}$$

There exists the Kato inequality for L^2 harmonic 2-forms as follows (Cibotaru and Zhu 2012, Wang 2002):

$$\frac{n-1}{n-2}|\nabla|\omega||^2 \leq |\nabla\omega|^2. \tag{4}$$

By (3) and (4), we get that

$$|\omega|\triangle|\omega| \geq \frac{1}{n-2}|\nabla|\omega||^2 + \langle E(\omega),\omega\rangle. \tag{5}$$

Now, we give the estimate of the term $\langle E(\omega),\omega\rangle$. Let $\omega_1 = b_{i_1i_2}e^{i_2}\wedge e^{i_1} \in \wedge^2(M)$ and $\omega_2 = c_{i_1i_2}e^{i_2}\wedge e^{i_1} \in \wedge^2(M)$, where $b_{i_1i_2} = -b_{i_2i_1}$ and $c_{i_1i_2} = -c_{i_2i_1}$. By Lemma 5, we obtain that

$$\begin{aligned}E(\omega_1) &= R_{k_1i_1j_1i_1}b_{k_1i_2}e^{i_2}\wedge e^{j_1} + R_{k_2i_2j_2i_2}b_{i_1k_2}e^{j_2}\wedge e^{i_1} \\ &\quad + R_{k_2i_2j_1i_1}b_{i_1k_2}e^{i_2}\wedge e^{j_1} + R_{k_1i_1j_2i_2}b_{k_1i_2}e^{j_2}\wedge e^{i_1} \\ &= Ric_{k_1j_1}b_{k_1i_2}e^{i_2}\wedge e^{j_1} + Ric_{k_2j_2}b_{i_1k_2}e^{j_2}\wedge e^{i_1} \\ &\quad + R_{k_2i_2j_1i_1}b_{i_1k_2}e^{i_2}\wedge e^{j_1} + R_{k_1i_1j_2i_2}b_{k_1i_2}e^{j_2}\wedge e^{i_1}.\end{aligned}$$

So, we get that

$$\begin{aligned}\langle E(\omega_1),\omega_2\rangle =& Ric_{k_1j_1}b_{k_1i_2}c_{j_1i_2} + Ric_{k_2j_2}b_{i_1k_2}c_{i_1j_2} \\ & + R_{k_2i_2j_1i_1}b_{i_1k_2}c_{j_1i_2} + R_{k_1i_1j_2i_2}b_{k_1i_2}c_{i_1j_2},\end{aligned}$$

which implies that

$$\begin{aligned}\langle E(\omega),\omega\rangle =& Ric_{k_1j_1}a_{k_1i_2}a_{j_1i_2} + Ric_{k_2j_2}a_{i_1k_2}a_{i_1j_2} \\ & + R_{k_2i_2j_1i_1}a_{i_1k_2}a_{j_1i_2} + R_{k_1i_1j_2i_2}a_{k_1i_2}a_{i_1j_2}.\end{aligned} \tag{6}$$

By Gauss equation, we have that

$$R_{ijkl} = (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + h_{ik}h_{jl} - h_{il}h_{jk}.$$

A direct computation shows that

$$Ric_{k_1j_1} = (n-1)\delta_{k_1j_1} + nHh_{k_1j_1} - h_{k_1i}h_{ij_1}; \tag{7}$$

$$Ric_{k_2j_2} = (n-1)\delta_{k_2j_2} + nHh_{k_2j_2} - h_{k_2i}h_{ij_2}; \tag{8}$$

$$R_{k_2i_2j_1i_1} = (\delta_{k_2j_1}\delta_{i_2i_1} - \delta_{k_2i_1}\delta_{i_2j_1}) + h_{k_2j_1}h_{i_2i_1} - h_{k_2i_1}h_{i_2j_1} \tag{9}$$

and

$$R_{k_1i_1j_2i_2} = (\delta_{k_1j_2}\delta_{i_1i_2} - \delta_{k_1i_2}\delta_{i_1j_2}) + h_{k_1j_2}h_{i_1i_2} - h_{k_1i_2}h_{i_1j_2}. \tag{10}$$

Since the curvature operator E is linear and zero order, and hence tensorial, it is sufficient to compute $\langle E(\omega), \omega\rangle$ at a point p. We can choose an orthonormal frame $\{e_i\}$ such that $h_{ij} = \lambda_i\delta_{ij}$ at p. Obviously,

$$nH = \lambda_1 + \cdots + \lambda_n.$$

By (6)-(10), we have

$$\begin{aligned}
\langle E(\omega), \omega\rangle =&(n-1)\sum(a_{j_1i_2})^2 + \sum nH\lambda_{k_1}(a_{k_1i_2})^2 - \sum\lambda_{k_1}^2(a_{k_1i_2})^2\\
&+(n-1)\sum(a_{i_1j_2})^2 + \sum nH\lambda_{k_2}(a_{i_1k_2})^2 - \sum\lambda_{k_2}^2(a_{i_1k_2})^2\\
&+\sum a_{i_1j_1}a_{j_1i_1} - \sum\lambda_{k_2\lambda_{i_2}}(a_{k_2i_2})^2\\
&+\sum a_{j_2i_2}a_{i_2j_2} - \sum\lambda_{j_2\lambda_{i_2}}(a_{j_2i_2})^2\\
=&2\sum_{i\neq j}\big((n-2) + (\lambda_1+\cdots+\lambda_n)\lambda_i - \lambda_i^2 - \lambda_i\lambda_j\big)(a_{ij})^2.
\end{aligned}$$

Note that

$$|A|^2 = |\Phi|^2 + nH^2.$$

For $n = 3$, we have that

$$\begin{aligned}
\langle E(\omega), \omega\rangle &= 2\sum_{i\neq j}\big(1 + (\lambda_1+\lambda_2+\lambda_3)\lambda_i - \lambda_i^2 - \lambda_i\lambda_j\big)(a_{ij})^2\\
&= \sum_{i\neq j}\big(2 + (\lambda_1+\lambda_2+\lambda_3)(\lambda_i+\lambda_j) - (\lambda_i^2+\lambda_j^2) - 2\lambda_i\lambda_j\big)(a_{ij})^2\\
&= \sum_{i\neq j}\big(2 + \frac{1}{2}(3H)^2 - \frac{1}{2}\sum_{k=1,k\neq i,j}^{3}\lambda_k^2 - \frac{1}{2}(\lambda_i+\lambda_j)^2\big)(a_{ij})^2\\
&\geq \sum_{i\neq j}\big(2 + \frac{1}{2}(3H)^2 - \frac{1}{2}\sum_{k=1,k\neq i,j}^{3}\lambda_k^2 - (\lambda_i^2+\lambda_j^2)\big)(a_{ij})^2\\
&\geq \sum_{i\neq j}\big(2 + \frac{9}{2}H^2 - |A|^2\big)(a_{ij})^2\\
&= (2 + \frac{3}{2}H^2 - |\Phi|^2)|\omega|^2.
\end{aligned}$$

For $n \geq 4$, we obtain that

$$\begin{aligned}
\langle E(\omega), \omega\rangle &= 2\sum_{i\neq j}\big((n-2) + (\lambda_1 + \cdots + \lambda_n)\lambda_i - \lambda_i^2 - \lambda_i\lambda_j\big)(a_{ij})^2 \\
&= \sum_{i\neq j}\big(2(n-2) + (\lambda_1 + \cdots + \lambda_n)(\lambda_i + \lambda_j) - (\lambda_i^2 + \lambda_j^2) - 2\lambda_i\lambda_j\big)(a_{ij})^2 \\
&= \sum_{i\neq j}\big(2(n-2) + (\lambda_1 + \cdots + \widehat{\lambda_i} + \cdots + \widehat{\lambda_j} + \cdots + \lambda_n)(\lambda_i + \lambda_j)\big)(a_{ij})^2 \\
&= \sum_{i\neq j}\big(2(n-2) + \frac{1}{2}(nH)^2 - \frac{1}{2}(\sum_{k=1,k\neq i,j}^{n}\lambda_k)^2 - \frac{1}{2}(\lambda_i + \lambda_j)^2\big)(a_{ij})^2 \\
&\geq \sum_{i\neq j}\big(2(n-2) + \frac{1}{2}(nH)^2 - \frac{n-2}{2}(\sum_{k=1,k\neq i,j}^{n}\lambda_k^2) - (\lambda_i^2 + \lambda_j^2)\big)(a_{ij})^2 \\
&\geq \sum_{i\neq j}\big(2(n-2) + \frac{1}{2}(nH)^2 - \frac{n-2}{2}|A|^2\big)(a_{ij})^2 \\
&= \big(2(n-2) + \frac{1}{2}(nH)^2 - \frac{n-2}{2}|A|^2\big)|\omega|^2 \\
&= \big(2(n-2) + nH^2 - \frac{n-2}{2}|\Phi|^2\big)|\omega|^2.
\end{aligned}$$

By (5), we have that:

$$h\triangle h \geq |\nabla h|^2 + 2h^2 - |\Phi|^2h^2 + \frac{3}{2}H^2h^2,$$

for $n = 3$ and

$$h\triangle h \geq \frac{1}{n-2}|\nabla h|^2 + 2(n-2)h^2 - \frac{n-2}{2}|\Phi|^2h^2 + nH^2h^2,$$

for $n \geq 4$. □

Remark 8. *If ω is 1-form , then the term $E(\omega, \omega)$ is equal to $Ric(\omega, \omega)$. The corresponding estimate for this term was given by Leung (1992).*

PROOF OF MAIN RESULTS

In this section, we prove Theorem 1 and Corollary 2.

If η is a compactly supported piecewise smooth function on M, then

$$\begin{aligned}
div(\eta^2h\nabla h) &= \eta^2h\triangle h + \langle\nabla(\eta^2h), \nabla h\rangle \\
&= \eta^2h\triangle h + \eta^2|\nabla h|^2 + 2\eta h\langle\nabla\eta, \nabla h\rangle.
\end{aligned}$$

Integrating by parts on M, we obtain that

$$\int_M \eta^2h\triangle h + \int_M \eta^2|\nabla h|^2 + 2\int_M \eta h\langle\nabla\eta, \nabla h\rangle = 0. \tag{11}$$

Case I: $n = 3$. By Proposition 7 and (11), we obtain that

$$-2\int_M \eta h\langle\nabla\eta,\nabla h\rangle - 2\int_M \eta^2|\nabla h|^2 - 2\int_M \eta^2 h^2 + \int_M |\Phi|^2\eta^2h^2 - \frac{3}{2}\int_M H^2h^2\eta^2 \geq 0. \tag{12}$$

Note that

$$-2\int_M \eta h\langle\nabla\eta,\nabla h\rangle \leq a_1\int_M \eta^2|\nabla h|^2 + \frac{1}{a_1}\int_M h^2|\nabla\eta|^2, \tag{13}$$

for any positive real number a_1. Now we give an estimate of the term $\int_M |\Phi|^2\eta^2h^2$ as follows: set $\phi_1(\eta) = \left(\int_{Supp\eta}|\Phi|^3\right)^{\frac{1}{3}}$. Then there exists

$$\begin{aligned}
&\int_M |\Phi|^2\eta^2h^2 \leq \left(\int_{Supp\eta}\left(|\Phi|^2\right)^{\frac{3}{2}}\right)^{\frac{2}{3}}\cdot\left(\int_M\left(\eta^2h^2\right)^3\right)^{\frac{1}{3}}\\
&= \phi_1(\eta)^2\cdot\left(\int_M(\eta h)^6\right)^{\frac{1}{3}}\\
&\leq C_0\phi_1(\eta)^2\cdot\left(\int_M|\nabla(\eta h)|^2 + 9\int_M(H^2+1)(\eta h)^2\right)\\
&\leq C_0\phi_1(\eta)^2\cdot\left((1+\frac{1}{b_1})\int_M h^2|\nabla\eta|^2 + (1+b_1)\int_M\eta^2|\nabla h|^2 + 9\int_M(H^2+1)(\eta h)^2\right),
\end{aligned} \tag{14}$$

for any positive real number b_1, where the second inequality holds because of Proposition 6. By (12)-(14), we obtain that

$$\mathcal{A}_1\int_M\eta^2|\nabla h|^2 + \mathcal{B}_1\int_M H^2\eta^2h^2 + \mathcal{C}_1\int_M\eta^2h^2 \leq \mathcal{D}_1\int_M h^2|\nabla\eta|^2, \tag{15}$$

where

$$\begin{aligned}
\mathcal{A}_1 &:= (2 - C_0\phi_1(\eta)^2) - (a_1 + b_1C_0\phi_1(\eta)^2),\\
\mathcal{B}_1 &:= \frac{3}{2} - 9C_0\phi_1(\eta)^2,\\
\mathcal{C}_1 &:= 2 - 9C_0\phi_1(\eta)^2
\end{aligned}$$

and

$$\mathcal{D}_1 := \frac{1}{a_1} + C_0\phi_1(\eta)^2(1+\frac{1}{b_1}).$$

Since the total curvature $\|\Phi\|_{L^3(M)}$ is finite, we can choose a fixed r_0 such that

$$\|\Phi\|_{L^3(M-B_{r_0})} < \delta_1 = \sqrt{\frac{1}{12C_0}}.$$

Set

$$\begin{aligned}
\tilde{\mathcal{A}}_1 &:= (2 - C_0\delta_1^2) - (a_1 + b_1C_0\delta_1^2),\\
\tilde{\mathcal{B}}_1 &:= \frac{3}{2} - 9C_0\delta_1^2,\\
\tilde{\mathcal{C}}_1 &:= 2 - 9C_0\delta_1^2
\end{aligned}$$

and

$$\tilde{\mathcal{D}}_1 := \frac{1}{a_1} + C_0\delta_1^2(1+\frac{1}{b_1}).$$

Thus,

$$\tilde{\mathcal{A}}_1\int_M \eta^2|\nabla h|^2 + \tilde{\mathcal{B}}_1\int_M H^2\eta^2h^2 + \tilde{\mathcal{C}}_1\int_M \eta^2h^2 \le \tilde{\mathcal{D}}_1\int_M h^2|\nabla\eta|^2, \tag{16}$$

for any $\eta \in C_0^\infty(M - B_{r_0})$. By Proposition 6, we have

$$\begin{aligned}
&\frac{1}{C_0}\left(\int_M (\eta h)^6\right)^{\frac{1}{3}} \le \int_M |\nabla(\eta h)|^2 + 9\int_M (H^2+1)(\eta h)^2 \\
&\le (1+\frac{1}{c_1})\int_M h^2|\nabla\eta|^2 + (1+c_1)\int_M \eta^2|\nabla h|^2 + 9\int_M (H^2+1)(\eta h)^2,
\end{aligned} \tag{17}$$

for any positive real number c_1. By (16) and (17), we have

$$\begin{aligned}
&\frac{1}{C_0}\left(\int_M (\eta h)^6\right)^{\frac{1}{3}} \\
&\le (1+\frac{1}{c_1})\int_M h^2|\nabla\eta|^2 + (1+c_1)\int_M \eta^2|\nabla h|^2 + 9\int_M (H^2+1)(\eta h)^2 \\
&\le (1+\frac{1}{c_1} + (1+c_1)\frac{\tilde{\mathcal{D}}_1}{\tilde{\mathcal{A}}_1})\int_M h^2|\nabla\eta|^2 + (9-(1+c_1)\frac{\tilde{\mathcal{B}}_1}{\tilde{\mathcal{A}}_1})\int_M H^2\eta^2h^2 \\
&+ (9-(1+c_1)\frac{\tilde{\mathcal{C}}_1}{\tilde{\mathcal{A}}_1})\int_M \eta^2h^2.
\end{aligned} \tag{18}$$

Choose a sufficient large c_1 such that

$$9-(1+c_1)\frac{\tilde{\mathcal{B}}_1}{\tilde{\mathcal{A}}_1} < 0$$

and

$$9-(1+c_1)\frac{\tilde{\mathcal{C}}_1}{\tilde{\mathcal{A}}_1} < 0.$$

Then (18) implies that

$$(\int_M (\eta h)^6)^{\frac{1}{3}} \le \tilde{A}\int_M h^2|\nabla\eta|^2, \tag{19}$$

for any $\eta \in C_0^\infty(M - B_{r_0})$. where $\tilde{A}$ is a positive constant.

Case II: $n \geq 4$. By Proposition 7 and (11), we obtain that

$$\begin{aligned}
&-2\int_M \eta h\langle\nabla\eta,\nabla h\rangle - \frac{n-1}{n-2}\int_M \eta^2|\nabla h|^2 - 2(n-2)\int_M \eta^2h^2 \\
&+ \frac{n-2}{2}\int_M |\Phi|^2\eta^2h^2 - n\int_M H^2h^2\eta^2 \ge 0.
\end{aligned} \tag{20}$$

Note that

$$-2\int_M \eta h\langle\nabla\eta,\nabla h\rangle \le a_2\int_M \eta^2|\nabla h|^2 + \frac{1}{a_2}\int_M h^2|\nabla\eta|^2, \tag{21}$$

for any positive real number a_2. We set $\phi_2(\eta) = \left(\int_{Supp\eta} |\Phi|^n\right)^{\frac{1}{n}}$ and obtain that

$$\begin{aligned}
&\int_M |\Phi|^2\eta^2h^2 \leq \left(\int_{Supp\eta} (|\Phi|^2)^{\frac{n}{2}}\right)^{\frac{2}{n}} \cdot \left(\int_M (\eta^2h^2)^{\frac{n}{n-2}}\right)^{\frac{n-2}{n}} \\
&= \phi_2(\eta)^2 \cdot \left(\int_M (\eta h)^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \\
&\leq C_0\phi_2(\eta)^2 \cdot \left(\int_M |\nabla(\eta h)|^2 + n^2\int_M (H^2+1)(\eta h)^2\right) \\
&\leq C_0\phi_2(\eta)^2 \cdot \left(\int_M (1+\frac{1}{b_2})h^2|\nabla\eta|^2 + (1+b_2)\eta^2|\nabla h|^2 + n^2\int_M (H^2+1)(\eta h)^2\right), \qquad (22)
\end{aligned}$$

for any positive real number b_2, where the second inequality holds because of Proposition 6. By (20)-(22), there exists

$$\mathcal{A}_2\int_M \eta^2|\nabla h|^2 + \mathcal{B}_2\int_M H^2\eta^2h^2 + \mathcal{C}_2\int_M \eta^2h^2 \leq \mathcal{D}_2\int_M h^2|\nabla\eta|^2, \qquad (23)$$

where

$$\begin{aligned}
\mathcal{A}_2 &:= \left(\frac{n-1}{n-2} - \frac{n-2}{2}C_0\phi_2(\eta)^2\right) - \left(a_2 + \frac{n-2}{2}b_2C_0\phi_2(\eta)^2\right), \\
\mathcal{B}_2 &:= n - \frac{n^2(n-2)}{2}C_0\phi_2(\eta)^2, \\
\mathcal{C}_2 &:= 2(n-2) - \frac{n^2(n-2)}{2}C_0\phi_2(\eta)^2
\end{aligned}$$

and

$$\mathcal{D}_2 := \frac{1}{a_2} + \frac{n-2}{2}(1+\frac{1}{b_2})C_0\phi_2(\eta)^2.$$

Since the total curvature $\|\Phi\|_{L^n(M)}$ is finite, we can choose a fixed r_0 such that

$$\|\Phi\|_{L^n(M-B_{r_0})} < \delta_2 = \sqrt{\frac{1}{n(n-2)C_0}}.$$

$$\begin{aligned}
\tilde{\mathcal{A}}_2 &:= \left(\frac{n-1}{n-2} - \frac{n-2}{2}C_0\delta_2^2\right) - \left(a_2 + \frac{n-2}{2}b_2C_0\delta_2^2\right), \\
\tilde{\mathcal{B}}_2 &:= n - \frac{n^2(n-2)}{2}C_0\delta_2^2, \\
\tilde{\mathcal{C}}_2 &:= 2(n-2) - \frac{n^2(n-2)}{2}C_0\delta_2^2
\end{aligned}$$

and

$$\tilde{\mathcal{D}}_2 := \frac{1}{a_2} + \frac{n-2}{2}(1+\frac{1}{b_2})C_0\delta_2^2.$$

Obviously, $\tilde{\mathcal{A}}_2, \tilde{\mathcal{B}}_2, \tilde{\mathcal{C}}_2$ and $\tilde{\mathcal{D}}_2$ are positive. Thus,

$$\tilde{\mathcal{A}}_2 \int_M \eta^2|\nabla h|^2 + \tilde{\mathcal{B}}_2 \int_M H^2\eta^2h^2 + \tilde{\mathcal{C}}_2 \int_M \eta^2h^2 \le \tilde{\mathcal{D}}_2 \int_M h^2|\nabla\eta|^2, \tag{24}$$

for any $\eta \in C_0^\infty(M - B_{r_0})$. Combining with Proposition 6, we get that

$$\begin{aligned}\frac{1}{C_0}(\int_M |\eta h|^{\frac{2n}{n-2}})^{\frac{n-2}{n}} &\le \int_M |\nabla(\eta h)|^2 + n^2\int_M (H^2+1)(\eta h)^2 \\ &\le (1+c_2)\int_M \eta^2|\nabla h|^2 + (1+\frac{1}{c_2})\int_M h^2|\nabla\eta|^2 + n^2\int_M (H^2+1)\eta^2h^2,\end{aligned} \tag{25}$$

for any positive real number c_2. By (24) and (25), we have

$$\begin{aligned}&\frac{1}{C_0}(\int_M |\eta h|^{\frac{2n}{n-2}})^{\frac{n-2}{n}} \\ &\le (1+\frac{1}{c_2}+(1+c_2)\frac{\tilde{\mathcal{D}}_2}{\tilde{\mathcal{A}}_2})\int_M h^2|\nabla\eta|^2 + (n^2-(1+c_2)\frac{\tilde{\mathcal{B}}_2}{\tilde{\mathcal{A}}_2})\int_M H^2\eta^2h^2 \\ &+ (n^2-(1+c_2)\frac{\tilde{\mathcal{C}}_2}{\tilde{\mathcal{A}}_2})\int_M \eta^2h^2.\end{aligned} \tag{26}$$

We choose a sufficient large c_2 such that

$$n^2-(1+c_2)\frac{\tilde{\mathcal{B}}_2}{\tilde{\mathcal{A}}_2} < 0$$

and

$$n^2-(1+c_2)\frac{\tilde{\mathcal{C}}_2}{\tilde{\mathcal{A}}_2} < 0.$$

Then (26) implies that

$$(\int_M (\eta h)^{\frac{2n}{n-2}})^{\frac{n-2}{n}} \le \tilde{A}\int_M h^2|\nabla\eta|^2, \tag{27}$$

for any $\eta \in C_0^\infty(M - B_{r_0})$, where $\tilde{A}$ is a positive constant depending only on n.
By Case I and Case II, we have that

$$(\int_M (\eta h)^{\frac{2n}{n-2}})^{\frac{n-2}{n}} \le \tilde{A}\int_M h^2|\nabla\eta|^2, \tag{28}$$

for any $\eta \in C_0^\infty(M - B_{r_0})$, where $\tilde{A}$ is a positive constant depending only on n ($n \ge 3$).
Next, the proof follows standard techniques (after inequality (33) in Cavalcante et al. (2014) and uses a Moser iteration argument (lemma 11 in Li (1980)). We include a concise proof here for the sake of completeness. Choose $r > r_0 + 1$ and $\eta \in C_0^\infty(M - B_{r_0})$ such that

$$\begin{cases}\eta = 0 \;\; on \;\; B_{r_0} \cup (M - B_{2r}),\\ \eta = 1 \;\; on \;\; B_r - B_{r_0+1},\\ |\nabla\eta| < \tilde{c} \;\; on \;\; B_{r_0+1} - B_{r_0},\\ |\nabla\eta| \le \tilde{c}r^{-1} on \;\; B_{2r} - B_r,\end{cases}$$

for some positive constant $\tilde{c}$. Then (28) becomes that

$$(\int_{B_r-B_{r_0+1}} h^{\frac{2n}{n-2}})^{\frac{n-2}{n}} \leq \tilde{A}\int_{B_{r_0+1}-B_{r_0}} h^2 + \frac{\tilde{A}}{r^2}\int_{B_{2r}-B_r} h^2.$$

Letting $r \to \infty$ and noting that $h \in L^2(M)$, we obtain that

$$(\int_{M-B_{r_0+1}} h^{\frac{2n}{n-2}})^{\frac{n-2}{n}} \leq \tilde{A}\int_{B_{r_0+1}-B_{r_0}} h^2. \tag{29}$$

By Hölder inequality

$$\int_{B_{r_0+2}-B_{r_0+1}} h^2 \leq (\int_{B_{r_0+2}-B_{r_0+1}} h^{\frac{2n}{n-2}})^{\frac{n-2}{n}} \cdot (\int_{B_{r_0+2}-B_{r_0+1}} 1^{\frac{n}{2}})^{\frac{2}{n}},$$

we get that

$$\int_{B_{r_0+2}} h^2 \leq (1+\tilde{A}Vol(B_{r_0+2})^{\frac{2}{n}})\int_{B_{r_0+1}} h^2. \tag{30}$$

Set

$$\Psi = \begin{cases} |2-|\Phi|^2+\frac{3}{2}H^2|, & for\ \ n=3, \\ |2(n-2)-\frac{n-2}{2}|\Phi|^2+nH^2|, & for\ \ n\geq 4. \end{cases}$$

Fix $x \in M$ and take $\tau \in C_0^1(B_1(x))$. Proposition 7 implies that

$$h\triangle h \geq \alpha|\nabla h|^2 - \Psi h^2,$$

where

$$\alpha = \begin{cases} \frac{1}{2}, & for\ \ n=3, \\ \frac{1}{n-2}, & for\ \ n\geq 4. \end{cases}$$

Then, for $p>2$, there exists

$$\int_M \tau^2 h^{p-1}\triangle h \geq \alpha\int_M \tau^2 h^{p-2}|\nabla h|^2 - \int_M \tau^2\Psi h^p.$$

That is,

$$\begin{aligned} -2\int_{B_1(x)} \tau h^{p-1}\langle\nabla\tau,\nabla h\rangle \geq (\alpha+(p-1))&\int_{B_1(x)} \tau^2 h^{p-2}|\nabla h|^2 \\ &-\int_{B_1(x)} \tau^2\Psi h^p. \end{aligned} \tag{31}$$

Note that

$$\begin{aligned} -2\tau h^{p-1}\langle\nabla\tau,\nabla h\rangle &= -2\langle h^{\frac{p}{2}}\nabla\tau, \tau h^{\frac{p}{2}-1}\nabla h\rangle \\ &\leq \frac{1}{\alpha}h^p|\nabla\tau|^2 + \alpha\tau^2 h^{p-2}|\nabla h|^2. \end{aligned}$$

Combining with (31), we obtain that

$$(p-1)\int_{B_1(x)} \tau^2 h^{p-2}|\nabla h|^2 \leq \int_{B_1(x)} \Psi\tau^2 h^p + \frac{1}{\alpha}\int_{B_1(x)} |\nabla\tau|^2 h^p. \tag{32}$$

Combining Cauchy-Schwarz inequality with (32), we obtain that

$$\int_{B_1(x)} |\nabla(\tau h^{\frac{p}{2}})|^2 \leq \int_{B_1(x)} \mathcal{A}\Psi\tau^2 h^p + \mathcal{B}|\nabla\tau|^2 h^p, \tag{33}$$

where $\mathcal{A} = \frac{1}{p-1}(\frac{p^2}{4}+\frac{p}{2})$ and $\mathcal{B} = (1+\frac{p}{2}) + \frac{1}{\alpha(p-1)}(\frac{p^2}{4}+\frac{p}{2})$. Choose $f = \tau h^{\frac{p}{2}}$ in Proposition 6. Combining with (33), we obtain that

$$(\int_{B_1(x)} (\tau h^{\frac{p}{2}})^{\frac{2n}{n-2}})^{\frac{n-2}{2}} \leq p\mathcal{C}\int_{B_1(x)} (\tau^2 + |\nabla\tau|^2)h^p, \tag{34}$$

where $\mathcal{C}$ depends on n and $\sup_{B_1(x)}\Psi$. Set $p_k = \frac{2n^k}{(n-2)^k}$ and $\rho_k = \frac{1}{2} + \frac{1}{2^{k+1}}$ for $k = 0, 1, 2, \cdots$. Take a function $\tau_k \in C_0^\infty(B_{\rho_k(x)})$ satisfying:

$$\begin{cases} 0 \leq \tau_k \leq 1, \\ \tau_k = 1 \ \ on \ \ B_{\rho_{k+1}}(x), \\ |\nabla\tau_k| \leq 2^{k+3}. \end{cases}$$

Choosing $p = p_k$ and $\tau = \tau_k$ in (34), we obtain that

$$(\int_{B_{\rho_{k+1}}(x)} h^{p_{k+1}})^{\frac{1}{p_{k+1}}} \leq (\mathcal{C}p_k 4^{k+4})^{\frac{1}{p_k}}(\int_{B_{\rho_k}(x)} h^{p_k})^{\frac{1}{p_k}}. \tag{35}$$

By recurrence, we have

$$\|h\|_{L^{p_{k+1}}(B_{\frac{1}{2}}(x))} \leq \prod_{i=0}^{k} p_i^{\frac{1}{p_i}} 4^{\frac{i}{p_i}} (\mathcal{C}4^4)^{\frac{1}{p_i}} \|h\|_{L^2(B_1(x))} \leq \mathcal{D}\|h\|_{L^2(B_1(x))}, \tag{36}$$

where $\mathcal{D}$ is a positive constant depending only on n, $Vol(B_{r_0+2})$ and $\sup_{B_{r_0+2}}\Psi$. Letting $k \to \infty$, we get

$$\|h\|_{L^\infty(B_{\frac{1}{2}}(x))} \leq \mathcal{D}\|h\|_{L^2(B_1(x))}. \tag{37}$$

Now, choose $y \in \overline{B}_{r_0+1}$ such that $\sup_{B_{r_0+1}} h^2 = h(y)^2$. Note that $B_1(y) \subset B_{r_0+2}$. (37) implies that

$$\sup_{B_{r_0+1}} h^2 \leq \mathcal{D}\|h\|^2_{L^2(B_1(y))} \leq \mathcal{D}\|h\|^2_{L^2(B_{r_0+2})}. \tag{38}$$

By (30), we have

$$\sup_{B_{r_0+1}} h^2 \leq \mathcal{F}\|h\|^2_{L^2(B_{r_0+1})}, \tag{39}$$

where $\mathcal{F}$ depends only on n, $Vol(B_{r_0+2})$ and $\sup_{B_{r_0+2}}\Psi$. In order to show the finiteness of the dimension of $H^2(L^2(M))$, it suffices to prove that the dimension of any finite dimensional subspaces of $H^2(L^2(M))$ is bounded above by a fixed constant. Combining (39) with Lemma 11 in Li (1980), we show that $\dim H^2(L^2(M)) < +\infty$. By Proposition 4, we obtain that the dimension of the second space of reduced L^2 cohomology of M is finite.

Remark 9. *For the case of $n = 3$, Theorem 1 can also be obtained by a different method. In fact, Yau (1976) proved that if $\omega \in H^2(L^2(M))$, then ω is closed and coclosed. By use of the Hodge-$*$ operator, we obtain the dimensions of $H^2(L^2(M))$ and $H^1(L^2(M))$ are equal. By Theorem 1.1 in Zhu and Fang (2014), we obtain the desired result.*

ACKNOWLEDGMENTS

The author would like to thank professor Detang Zhou for useful suggestions. The work was partially supported by NSFC Grants 11471145, 11371309 and Qing Lan Project.

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