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On Complete Spacelike Hypersurfaces with Constant Scalar Curvature in the De Sitter Space

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ABSTRACT

Let M^n be a complete spacelike hypersurface with constant normalized scalar curvature R in the de Sitter Space S_1^{n+1} . Let H the mean curvature and suppose that $\bar{R} = (R - 1) > 0$ and $\bar{R} \leq \sup H^2 \leq C_{\bar{R}}$, where $C_{\bar{R}}$ is a constant depending only on R and n . It is proved that either $\sup H^2 = \bar{R}$ and M^n is totally umbilical, or $\sup H^2 = C_{\bar{R}}$ and M^n is the hyperbolic cylinder $H^1(1 - \coth^2 r) \times S^{n-1}(1 - \tanh^2 r)$.

Key words: hyperbolic cylinder, spacelike hypersurfaces, de Sitter space.

1. INTRODUCTION

The study of spacelike hypersurfaces with constant scalar curvature in the de Sitter space S_1^{n+1} is related to an analogue of Goddard's conjecture for the second elementary symmetric polynomial in the principal curvatures; more precisely: "Let M^n be a complete spacelike hypersurface with constant scalar curvature immersed in de Sitter space S_1^{n+1} . Then M^n is totally umbilical".

S. Montiel (Montiel 1996) described the hyperbolic cylinders $H^1(1 - \coth^2 r) \times S^{n-1}(1 - \tanh^2 r)$, $n \geq 3$, in S_1^{n+1} with constant mean curvature $H^2 = \frac{4(n-1)}{n^2}$ and normalized constant scalar curvature $R = 1 + \frac{1}{n}(2 + (n-2)\tanh^2 r)$.

In (Zheng 1995), (Zheng 1996) and (Cheng & Ishikawa 1998) partial results were obtained. Recently, Haizhong Li (Li 1997) (and also S. Montiel in a more general spacetime (Montiel 1999)) obtained the following result: "Let M^n be a compact spacelike hypersurface immersed into the de Sitter space S_1^{n+1} with normalized constant scalar curvature R satisfying $R \geq 0$. Then M is totally umbilical".

In this note we announce the following

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THEOREM 1.1. *Let M^n be an n -dimensional complete ($n \geq 3$) spacelike hypersurface immersed into the de Sitter space S_1^{n+1} with constant scalar curvature R such that $\bar{R} = R - 1 > 0$ and suppose that $\bar{R} \leq \sup H^2 \leq C_{\bar{R}}$ where*

$$C_{\bar{R}} = \frac{1}{n} \left((n-1)^2 \frac{n\bar{R}-2}{n-2} + 2(n-1) + \frac{n-2}{n\bar{R}-2} \right).$$

Then

- (i) $\sup H^2 = \bar{R}$ and M is totally umbilic or
- (ii) $\sup H^2 = C_{\bar{R}}$ and M is isometric to $H^1(1 - \coth^2 r) \times S^{n-1}(1 - \tanh^2 r)$.

2. PRELIMINARIES

Let R_1^{n+2} be the real vector space R^{n+2} endowed with the Lorentzian metric $\langle \cdot, \cdot \rangle$ given by $\langle v, w \rangle = -v_0 w_0 + v_1 w_1 + \dots + v_{n+1} w_{n+1}$ that is, $R_1^{n+2} = L^{n+2}$ is the Lorentz-Minkowski $(n+2)$ -dimensional space. We define the de Sitter space as the following hyperquadric of R_1^{n+2} : $S_1^{n+1} = \{x \in R_1^{n+2}; |x|^2 = 1\}$. The induced metric $\langle \cdot, \cdot \rangle$ makes S_1^{n+1} into a Lorentz manifold with constant sectional curvature 1. Let M^n be a n -dimensional orientable manifold, complete and let $f: M^n \rightarrow S_1^{n+1} \subset L^{n+2}$ be a spacelike immersion of M^n into the de Sitter S_1^{n+1} . Choose a unit normal η along f and denote by $A: T_p M \rightarrow T_p M$ the linear map of the tangent space $T_p M$ at the point $p \in M$, associated to the second fundamental form of f along η ,

$$\langle AX, Y \rangle = -\langle \nabla_X Y, \eta \rangle,$$

where X and Y are tangent vector fields on M and ∇ is the connection on S_1^{n+1} . Let $\{e_1, \dots, e_n\}$ be an orthonormal basis which diagonalizes A with eigenvalues k_i of $T_p M$, i.e., $Ae_i = k_i e_i$, $i = 1, \dots, n$. We will denote by $H = \frac{1}{n} \sum k_i$ the mean curvature of f and $|A|^2 = \sum k_i^2$. In our case it is convenient to define a linear map $\phi: T_p M \rightarrow T_p M$ by

$$\langle \phi X, Y \rangle = \langle AX, Y \rangle - H \langle X, Y \rangle.$$

It is easily checked that $\text{trace}(\phi) = 0$ and that

$$|\phi|^2 = \frac{1}{2n} \sum (k_i - k_j)^2,$$

so that $|\phi|^2 = 0$ if and only if M^n is totally umbilical. Let $\mu_i = k_i - H$ be the eigenvalues of ϕ ; then $\sum_i \mu_i = 0$, and

$$\begin{aligned} |\phi|^2 &= \sum_i \mu_i^2 = |A|^2 - nH^2 \\ \sum_i k_i^3 &= nH^3 + 3H \sum_i \mu_i^2 - \sum_i \mu_i^3. \end{aligned} \tag{2.1}$$

The standard examples of spacelike umbilical hypersurfaces with constant mean curvature in the de Sitter space are given by

$$M^n = \{p \in S_1^{n+1} \mid \langle p, a \rangle = \tau\},$$

where $a \in R_1^{n+2}$, $|a|^2 = \rho = 1, 0, -1$ and $\tau^2 > \rho$. The corresponding mean curvature H of such surfaces satisfies

$$H^2 = \frac{\tau^2}{(\tau^2 - \rho)}$$

(Montiel 1988) and M^n is isometric to a hyperbolic space, an Euclidean space or a sphere according to ρ equal to 1, 0, -1 , respectively. On the other hand, hyperbolic cylinders are the hypersurfaces given by

$$M^n = \{p \in S_1^{n+1}; -p_o^2 + p_1^2 + \dots + p_k^2 = -\sinh^2 r\},$$

with $r \in R$ and $1 \leq k \leq n$.

Such hyperbolic cylinders have constant mean curvature

$$nH = [k \coth r + (n - k) \tanh r].$$

Thus, we have

$$H^2 \geq \frac{4(n-1)}{n^2}$$

and the equality is attained for $k = 1$ and $\coth^2 r = (n - 1)$. Their normalized scalar curvature is $R = 1 + \frac{1}{n}(2 + (n - 2) \tanh^2 r)$.

We point out that these examples have only two different constant principal curvatures at each point and one of them has multiplicity one. Moreover, they are isometric to the Riemannian product

$$H^1(1 - \coth^2 r) \times S^{n-1}(1 - \tanh^2 r).$$

The Gauss equation relates the scalar curvature, the mean curvature and the square of the norm of the second fundamental formula as follows:

$$n(n-1)(R-1) = n^2 H^2 - |A|^2. \quad (2.2)$$

Let $T = \sum_{i,j} T_{ij} \omega_i \otimes \omega_j$ be a symmetric tensor defined on M^n , where

$$T_{ij} = nH\delta_{ij} - h_{ij}.$$

Following (Cheng & Yau 1977), we introduce the operator L_1 associated to T acting on C^2 functions f on M^n by

$$L_1 f = \sum_{i,j} T_{ij} f_{ij} = \sum_{i,j} (nH\delta_{ij} - h_{ij}) f_{ij}. \quad (2.3)$$

Around a given point $p \in M$ we choose an orthonormal frame field $\{e_1, \dots, e_n\}$ with dual frame field $\{w_1, \dots, w_n\}$ so that $h_{ij} = k_i \delta_{ij}$ at p . We have the following computation by using (2.3) and Gauss equation (2.2) :

$$\begin{aligned} L_1(nH) &= nH \Delta(nH) - \sum_i k_i (nH)_{ii} \\ &= \frac{1}{2} \Delta(nH)^2 - \sum_i (nH)_i^2 - \sum_i k_i (nH) \\ &= \frac{1}{2} n(n-1) \Delta R + \frac{1}{2} \Delta |A|^2 - n^2 |\nabla H|^2 - \sum_i k_i (nH)_{ii}. \end{aligned} \quad (2.4)$$

On the other hand, using Simons Formula (see Zheng 1995) we get

$$\frac{1}{2} |A|^2 = |\nabla A|^2 + n \sum_i k_i H_{ii} + \frac{1}{2} \sum_{i,j} R_{ijij} (k_i - k_j)^2. \quad (2.5)$$

From (2.4) and (2.5), we have:

$$L_1(nH) = \frac{1}{2} n(n-1) \Delta R + |\nabla A|^2 - n^2 |\nabla H|^2 + \frac{1}{2} \sum_{i,j} R_{ijij} (k_i - k_j)^2. \quad (2.6)$$

3. SKETCH OF THE PROOF OF THE THEOREM

Since R is constant, by (2.6) we obtain

$$\begin{aligned} L_1(nH) &= |\nabla A|^2 - n^2 |\nabla H|^2 + \frac{1}{2} \sum_{i,j} (1 + k_i k_j) (k_i - k_j)^2 \\ &= |\nabla A|^2 - n^2 |\nabla H|^2 + \frac{1}{2} n \sum_i k_i^2 + \frac{1}{2} n \sum_j k_j^2 \\ &\quad - \sum_{i,j} k_j k_i + \frac{1}{2} \sum_{i,j} k_i^3 k_j + \frac{1}{2} \sum_{i,j} k_i k_j^3 - \sum_{i,j} k_i^2 k_j^2. \end{aligned}$$

Making $i = j$, we then have

$$L_1(nH) = |\nabla A|^2 - n^2 |\nabla H|^2 + n |A|^2 - n^2 H^2 + |A|^4 - nH \sum_i k_i^3. \quad (3.1)$$

Using (2.1) in (3.1) we obtain

$$L_1(nH) = |\nabla A|^2 - n^2 |\nabla H|^2 + n |\phi|^2 + |\phi|^4 - nH \sum_i \mu_i^3 - 3nH^2 |\phi|^2 - n^2 H^4. \quad (3.2)$$

Then,

$$\begin{aligned} L_1(nH) &= |\nabla A|^2 - n^2 |\nabla H|^2 + n |\phi|^2 + |\phi|^4 + n^2 H^4 + 2nH^2 |\phi|^2 \\ &\quad - 3nH^2 |\phi|^2 - n^2 H^4 - nH \sum_i \mu_i^3. \end{aligned} \quad (3.3)$$

This yields

$$L_1(nH) = |\nabla A|^2 - n^2 |\nabla H|^2 + |\phi|^2 (n - nH^2 + |\phi|^2) - nH \sum_i \mu_i^3. \quad (3.4)$$

We need to estimate $tr(\phi^3)$ in (3.4). First we recall an algebraic lemma (Okumura 1974) which asserts

$$-\frac{(n-2)}{\sqrt{n(n-1)}}(|\phi|^2)^{3/2} \leq \sum_i \mu_i^3 = tr(\phi^3) \leq \frac{(n-2)}{\sqrt{n(n-1)}}(|\phi|^2)^{3/2} \quad (3.5)$$

and the equality holds on the right hand side if and only if

$$\mu_1 = \dots = \mu_{n-1} = -\sqrt{\frac{1}{n(n-1)}}|\phi| \quad \text{and} \quad \mu_n = \sqrt{\frac{n-1}{n}}|\phi|.$$

Using (3.5) in (3.4), we obtain

$$L_1(nH) \geq |\nabla A|^2 - n^2 |\nabla H|^2 + |\phi|^2 (n - nH^2 + |\phi|^2) - \frac{n(n-2)}{\sqrt{n(n-1)}}|H||\phi|. \quad (3.6)$$

Since R is constant, by (Alencar et al. 1993)

$$|A|^2 - n^2 |\nabla H|^2 \geq 0. \quad (3.7)$$

By Gauss equation (2.1) we know that

$$|\phi|^2 = |A|^2 - nH^2 = \frac{n-1}{n}(|A|^2 - n\bar{R}). \quad (3.8)$$

Using (3.8) and (3.7) in (3.6) we obtain

$$L_1(nH) \geq \frac{n-1}{n}(|A|^2 - n\bar{R})P_H(|\phi|), \quad (3.9)$$

where P_H is a polynomial given by

$$P_H(|\phi|) = (n - nH^2 + |\phi|^2) - \frac{(n-2)}{\sqrt{n(n-1)}}|H||\phi|. \quad (3.10)$$

By (3.8) we may write the above polynomial as

$$\begin{aligned} P_{\bar{R}}(|A|) &= n - 2(n-1)\bar{R} + \frac{n-2}{n}|A|^2 \\ &\quad - \frac{(n-2)}{n}\sqrt{(n(n-1)\bar{R} + |A|^2)(|A|^2 - n\bar{R})}. \end{aligned} \quad (3.11)$$

Therefore (3.9) becomes

$$L_1(nH) \geq \frac{n-1}{n}(|A|^2 - n\bar{R})P_{\bar{R}}(|A|). \quad (3.12)$$

Using the hypothesis that $\bar{R} \leq \sup H^2 \leq C_{\bar{R}}$, one proves that

$$P_{\bar{R}}(\sqrt{\sup |A|^2}) \geq 0. \quad (3.13)$$

On the other hand,

$$\begin{aligned} L_1(nH) &= \sum_{i,j} (nH\delta_{ij} - nh_{ij})(nH)_{ij} = \sum_i (nH - nh_{ii})(nH)_{ii} \\ &= n \sum_i H(nH)_{ii} - n \sum_i k_i(nH)_{ii} \leq n(|H|_{\max} - C)\Delta(nH), \end{aligned} \quad (3.14)$$

where $|H|_{\max}$ is the maximum of the mean curvature H in M and $C = \min k_i$ is the minimum of the principal curvatures in M .

Now, we need the maximum principle at infinity for complete manifolds by Omori and Yau (Omori 1967):

"Let M^n be an n -dimensional complete Riemannian manifold whose Ricci curvature is bounded from below. Let f be a C^2 -function bounded from below on M^n . Then for each $\varepsilon > 0$ there exists a point $p_\varepsilon \in M$ such that

$$|\nabla f|(p_\varepsilon) < \varepsilon, \quad \Delta f(p_\varepsilon) > -\varepsilon \quad \text{and} \quad \inf f \leq f(p_\varepsilon) < \inf f + \varepsilon." \quad (3.15)$$

The hypothesis $\bar{R} \leq \sup H^2 \leq C_{\bar{R}}$ together with Gauss equation implies that $Ric_M \geq (n-1) - \frac{nH^2}{4}$, so the Ricci curvature is bounded below. Thus we may apply Omori and Yau's result to the function

$$f = \frac{1}{\sqrt{1 + (nH)^2}}.$$

We have

$$|\nabla f|^2 = \frac{1}{4} \frac{|\nabla(nH)|^2}{(1 + (nH)^2)^2} \quad (3.16)$$

and

$$\Delta f = -\frac{1}{2} \frac{\Delta(nH)}{(1 + (nH))^{\frac{3}{2}}} + \frac{3|\nabla(nH)|^2}{4(1 + nH)^{\frac{5}{2}}}. \quad (3.17)$$

Let $\{p_k\}$, $k \in N$, be a sequence of points in M given by (3.15) such that

$$\lim_{k \rightarrow \infty} f(p_k) = \inf f, \quad \Delta f(p_k) > -\frac{1}{k} \quad \text{and} \quad |\nabla f|^2(p_k) < \frac{1}{k^2}. \quad (3.18)$$

Using (3.18) in the two equations (3.16) and (3.17) and the fact that

$$\lim_{k \rightarrow \infty} (nH)(p_k) = \sup_{p \in M} (nH)(p),$$

we obtain

$$-\frac{1}{k} < -\frac{1}{2} \frac{\Delta(nH)}{(1+nH)^{\frac{3}{2}}}(p_k) + \frac{3}{k^2}(1+nH(p_k))^{\frac{3}{2}}. \quad (3.19)$$

Hence

$$\frac{\Delta(nH)}{(1+nH)^{\frac{3}{2}}}(p_k) < \frac{2}{k} \left(\frac{1}{\sqrt{1+(nH)(p_k)}} + \frac{3}{k} \right). \quad (3.20)$$

On the other hand, by (3.12) and (3.14), we have

$$\frac{n-1}{n}(|A|^2 - n\bar{R})P_{\bar{R}}(|A|) \leq L_1(nH) \leq n(|H|_{\max} - C)\Delta(nH). \quad (3.21)$$

At points p_k of the sequence given in (3.18), this becomes

$$\begin{aligned} \frac{n-1}{n}(|A|^2(p_k) - n\bar{R})P_{\bar{R}}(|A|(p_k)) &\leq L_1(nH(p_k)) \\ &\leq n(|H|_{\max} - C)\Delta(nH)(p_k). \end{aligned} \quad (3.22)$$

Making $k \rightarrow \infty$ and using (3.20) we have that the right hand side of (3.22) goes to zero, so by (3.15) either $\frac{n-1}{n}(\sup |A|^2 - n\bar{R}) = 0$ or $P_{\bar{R}}(\sqrt{\sup |A|^2}) = 0$. But by (3.8) $|\phi|^2 = \frac{n-1}{n}(|A|^2 - n\bar{R})$ and so $\sup |\phi|^2 = \frac{n-1}{n}(\sup |A|^2 - n\bar{R}) = 0$, then $|\phi|^2 = 0$, proving that M^n is totally umbilical.

If $P_{\bar{R}}(\sqrt{\sup |A|^2}) = 0$, it can be proved that $\sup H^2 = C_{\bar{R}}$ and so the equality holds on the right hand side of (3.9) and we obtain

$$L_1(n \sup H) = \frac{n-1}{n}(\sup |A|^2 - n\bar{R})P_{\bar{R}}(\sqrt{\sup |A|^2}).$$

One proves that the equality also holds in Okumura's lemma (3.5). After reenumeration, we finally have

$$k_1 = k_2 = \dots = k_{n-1}, \quad k_1 \neq k_n, \text{ where } k_1 = \tanh r \text{ and } k_n = \coth r.$$

Therefore, M^n is isometric to $H^1(1 - \coth^2 r) \times S^{n-1}(1 - \tanh^2 r)$, finishing the proof.

REFERENCES

- ALENCAR H, DO CARMO MP & COLARES AG. 1993. Stable hypersurfaces with constant scalar curvature, *Math Z* **213**: 117-131.
- CHENG QM & ISHIKAWA S. 1998. Spacelike hypersurface with constant scalar curvature, *Manuscripta Math* **95**(4): 499-505.
- CHENG SY & YAU ST. 1977. Hypersurfaces with constant scalar curvature. *Math Ann* **225**: 195-204.
- LI H. 1997. Global rigidity theorems of hypersurfaces, *Arkiv Mat* **35**: 327-351.
- MONTIEL S. 1988. An integral inequality for compact spacelike hypersurfaces in the de Sitter space and applications to the case of constant mean curvature, *Indiana Univ Math J* **37**: 909-917.

- MONTIEL S. 1996. A characterization of hyperbolic cylinders in the de Sitter space, *Tohoku Math J* **48**: 23-31.
- MONTIEL S. 1999. Uniqueness of spacelike hypersurfaces of constant mean curvature in foliated spacetimes, *Math Ann* **314**: 529-553.
- OMORI H. 1967. Isometric immersions of Riemannian manifolds, *J Math Soc Japan* **19**: 205-214.
- OKUMURA M. 1974. Hypersurfaces and a pinching problem on the second fundamental tensor, *Amer J Math* **96**: 207-213.
- ZHENG Y. 1995. On Spacelike hypersurfaces in the de Sitter space, *Ann of Global Anal Geom* **13**: 317-332.
- ZHENG Y. 1996. Spacelike hypersurfaces with constant scalar curvature in the de Sitter space, *Diff Geom Appl* **6**: 51-54.