On the use, by Einstein, of the Principle of Dimensional Homogeneity, in three problems of the Physics of Solids*

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ABSTRACT

Einstein, in 1911, published an article on the application of the principle of dimensional homogeneity to three problems of the physics of solids: the characteristic frequency of the atomic nets of crystalline solids as a function of their moduli of compressibility or of their melting points, and the thermal conductivity of crystalline insulators. Recognizing that the physical dimensions of temperature are not the same as those of energy and heat, Einstein had recourse to the artifice of replace that physical parameter by its product by the Boltzmann constant, so obtaining correct results. But nowadays, with the new basic quantities "Thermodynamic Temperature θ (unit-Kelvin)", "Electric Current I (unit Ampère)" and "Amount of Substance MOL (unit-mole)", incorporated to the SI International System of Units, in 1960 and 1971, the same results are obtained in a more direct and coherent way. At the time of Einstein's article only three basic physical quantities were considered – length L, mass M, and time T. He ignored the Π theorem of dimensional analysis diffused by Buckingham three years later, and obtained the "Π numbers" by trial and error. In the present paper is presented a revisitation of the article of Einstein, conducted by the modern methodology of dimensional analysis and theory of physical similitude.

Key words: characteristic frequency of solids, thermal conductivity, dimensional homogeneity.

INTRODUCTION

Aus Dimensionalbetrachtungen kann man bekanntlich zunächst allgemeine funktionelle Zusamenhänge zwischen physikalischen Grössen finden, wenn man alle physikalischen Grössen kennt, welche in dem betreffenden Zusamenhang vorkommen" (Einstein 1911)".

"It is well known that from dimensional considerations it is possible to closely find general relationships between physical quantities when one knows all the physical quantities that participate in the referred relationship", wrote Einstein in his article "Elementare Betrachtungen über die

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thermische Molekularbewegung in festen Körper" – "Elementary considerations on the thermal agitation in solid bodies", published in 1911, six years after his fundamental articles on the special theory of relativity, the light quanta and the Brownian motion, and one year before the theory of general relativity. The "very close (zunächst) difference" between the approximate formulation of a physical relationship given by dimensional analysis and the exact one, is merely a dimensionless numerical factor that, if necessary, can be determined by experience or theoretical considerations. According to Einstein, this factor is generally of the order of magnitude "one" i.e., smaller than 10. He gives as an example, the determination of the period of small oscillations of the pendulum, T proportional to $(l/g)^{1/2}$, or $T = C(l/g)^{1/2}$, with $C \approx 6.3$. The theoretical value of C is 2π.

The Principle of Dimensional Homogeneity is a consequence of the condition that every mathematical expression which expresses a physical law or describes a physical process must be invariant in relation with any changes of the fundamental units of measure. Its utilization by scientists has been very fecund in basic sciences like mathematics and physics, – geometry, general mechanics, mechanics of deformable solids and of fluids, thermology, electromagnetism – and even biology. Maxwell, the creator of the symbolic dimensional formulae, was based in dimensional considerations in the electromagnetic theory of light and in the prediction of the electromagnetic waves, – later experimentally confirmed by Hertz (Maxwell 1868). The engineering applications are also very important – like strength of materials, structural engineering, hydraulics, soil mechanics, aerodynamics (Carneiro 1996). In the present article it is shown, by the valuable testimony of Einstein, that the applications to the physics of solids are also significant.

CHARACTERISTIC FREQUENCY OF THE ATOM OF A SOLID AS A FUNCTION OF THE COEFFICIENT OF COMPRESSIBILITY

The most simple possibility to determine, by dimensional considerations, the characteristic frequency ν of the atom of a solid crystalline body, according to Einstein, is that the mechanism of vibration is determined by the following quantities: the mass m_a of the atom (dimension M), the distance d between two neighbour atoms (dimension L), and the resistance opposed by two neighbour atoms to any change of the distance between them. That resistance is measured by the coefficient of compressibility χ valid for elastic deformations (dimension L T^2/M, the inverse of the modulus of elasticity).

So, the problem presents 4 variables: the incognito or dependent variable ν and the independent variables m_a, d and χ. Applying the principle of dimensional homogeneity Einstein obtained, probably by the trial and error method, an expression – product of powers of the independent variables, with the same dimension as the dependent one, and wrote

$$\nu = C(dm_a^{-1}\chi^{-1})^{1/2}$$

This is the same as to equalize to the numerical factor C, a dimensionless number, the unique Π number of the problem. But the independent variables m_a and d are not directly given and Einstein

finally modified his formula taking the molar volume v_m, the density ρ and the atomic or molar mass M_a as known physical properties of the substance, and N_A – number of Avogrado, as physical universal constant:

$$m_a = M_a/N_a, \quad d = (v_m/N_A)^{1/3} = [(M_a/N_A)/\rho]^{1/3}$$

The new formula is then:

$$v = C N_A^{1/3} M_a^{-1/3} \rho^{-1/6} \chi^{-1/2} \tag{1}$$

Einstein would obtain this formula directly, if the new basic quantity MOL were already incorporated to the SI System and if he employed the modern methodology of dimensional analysis, based on the Π theorem of Vaschy-Buckingham (Carneiro 1993) and on the "dimensional matrix", whose lines correspond to the basic quantities and columns to the parameters of the problem, and whose elements are their "exponents of dimension"

	M_a	ρ	χ	N_A	v
L	0	−3	1	0	0
M	1	1	−1	0	0
T	0	0	2	0	−1
MOL	−1	0	0	−1	0

That matrix has a rank 4 and 5 columns. According to the Π theorem it will be only one Π number. The exponents of the variables in that Π number are easily obtainable by the multiplication of the negative of the inverse of the square matrix of the first 4 columns, by the vector of the last one (Carneiro 1996, Langhaar 1951, Brand 1957). The unique Π number so obtained is $\Pi = v N_A^{-1/3} M_a^{1/3} \rho^{1/6} \chi^{1/2}$ and the solution of the problem is $\Pi = C$, C being a numerical factor. According to Einstein, based on previous theoretical considerations, $C \cong 1/3$.

RELATIONSHIP BETWEEN THE MELTING POINT OF SOLID BODIES AND THE CHARACTERISTIC FREQUENCY OF THE ATOMS

Referring to the "formula of Lindemann", Einstein takes the melting point as a third characteristic physical magnitude of the substance, that is so determined as the mass of a atom, the distance between two neighbour atoms and the characteristic frequency (Einstein 1911). "The melting temperature θ_s", says Einstein, "is not usable in dimensional considerations, because it cannot directly be measurable in the C.G.S. System". "Then I take instead of θ_s the energy-magnitude $\tau = R\theta_s/N_A$, where τ is one third of the energy of a atom at the melting point, according to the kinetic theory of heat., R the constant of gases, and N_A the number of atoms in a atom-gram (Avogadro)" (R/N_A is the constant of Boltzmann). Taking the same steps as in the precedent section, with τ instead of χ, he obtains the formula

$$v = C(\tau m_a^{-1} d^{-2})^{1/2}$$

and replacing τ by $R\theta_s/N_A$, m_a by M_a/N_A and d by $[(M_a/N_A)/\rho]^{1/3}$

$$\nu = C N_A^{1/3} \theta_s^{1/2} \rho^{1/3} M_a^{-5/6} R^{1/2} \tag{2}$$

Einstein did not fall into the famous error of Riabouchinsky, who insisted in his polemic against Lord Rayleigh that temperature is a magnitude with the same dimensions as heat and energy. He used an artifice to escape from the impasse. With the new basic magnitude "Thermodynamic Temperature" introduced in the SI System in 1960 Einstein would not need to use that ingenious artifice, and, adopting the new methodology of dimensional analysis, would write the dimensional matrix (Carneiro 1996):

	θ_s	ρ	M_a	N_A	R	ν
L	0	−3	0	0	2	0
M	0	1	1	0	1	0
T	0	0	0	0	−2	−1
θ	1	0	0	0	−1	0
MOL	0	0	−1	−1	−1	0

From that matrix, a unique Π number is obtained

$$\Pi = \nu \theta_s^{-1/2} \rho^{-1/3} M_a^{5/6} N_A^{-1/3} R^{-1/2}$$

and the solution of the problem is $\Pi = C$. According to Einstein, based on the findings of Lindemann, $C \cong 3$. It is possible to obtain the same formula (2), but with a different value for C, with the hypothesis that at the melting point the amplitude of the vibrations of the atoms is equal to the mean distance between them (Leite Lopes 1992).

THERMAL CONDUCTIVITY OF CRYSTALINE INSULATORS

The third problem treated in the article of Einstein is the theoretical result obtained by Eucken that the thermal conductivity of crystalline insulators is approximately proportional to the inverse of the absolute temperature θ, and not independent of that temperature, Einstein says that it is possible to obtain this result by one "very interesting" dimensional consideration. But, by the same reasons of the precedent chapter, he takes, in the definition of the coefficient k of thermal conductivity, instead of θ, the product $\tau = R\theta/N_A$, whose dimension is -energy. The corresponding coefficient is designed by $k_{nat.}$ by Einstein. But a new difficulty arises. Instead of 3 independent variables, like in the two precedent problems, there are now 4: d, m, ν, and τ, besides the dependent variable $k_{nat.}$. With the deficient base L M T Einstein obtained then, by the trial and error method, two Π numbers, and wrote

$$k_{nat.} = C \cdot (d^{-1}\nu) \cdot \varphi(md^2\nu^2/\tau)$$

Einstein, for obtaining the result from Eucken, formulated the hypothesis that the function φ would be proportional to its argument, and wrote

$$k_{nat} = Cmdv^3\tau^{-1}$$

from that

$$k \quad \text{proportional to} \quad \theta^{-1}.$$

By the new SI System the dimensional matrix is no more deficient, and the temperature can be directly included into the parameters of the problem:

$$
\begin{array}{c|cccccc}
 & \rho & M_a & \nu & N_A & \theta & k \\
\hline
L & -3 & 0 & 0 & 0 & 0 & 1 \\
M & 1 & 1 & 0 & 0 & 0 & 1 \\
T & 0 & 0 & -1 & 0 & 0 & -3 \\
\theta & 0 & 0 & 0 & 0 & 1 & -1 \\
MOL & 0 & -1 & 0 & -1 & 0 & 0 \\
\end{array}
$$

From that matrix a unique Π number is obtained

$$\Pi = kM_a^{-4/3}\rho^{1/3}\nu^{-3}N_A^{1/3}\theta$$

and the solution is $\Pi = C$. Then k proportional to θ^{-1}. The results showed in this chapter can not be applied to the case of crystalline solids, like metals, that are good conductors of electricity, but only to crystalline insulators, because the thermal conductivity is strongly related with the electrical conductivity.

EXAMPLES OF APPLICATION OF THE FORMULAE

Substance	M_a ($\times 10^3$) kg	χ ($\times 10^{12}$) Pa^{-1}	θ_s $^\circ K$	ρ kg/m^3	characteristic ($\times 10^{-12}$) Formula(1) ($C = 1/3$)	frequency Formula(2) ($C = 3$)
iron	55.8	6.0	1808	7860	6.75	6.83
aluminium	27.0	14.0	931	2700	6.72	6.28
copper	63.6	8.0	1353	8960	5.47	5.51
nickel	28.0		2453	8900		12.90
silver	107.9		1233	10500		3.59
gold	197.2		1347	19310		2.79
platinum	195.1		2037	21400		3.37
silicon	28.1		1693	2420		7.90

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