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On The Existence of Levi Foliations

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ABSTRACT

Let $L \subset \mathbb{C}^2$ be a real 3 dimensional analytic variety. For each regular point $p \in L$ there exists a unique complex line l_p on the space tangent to L at p . When the field of complex line

$$p \mapsto l_p$$

is completely integrable, we say that L is Levi variety. More generally; let $L \subset M$ be a real subvariety in an holomorphic complex variety M . If there exists a real 2 dimensional integrable distribution on L which is invariant by the holomorphic structure J induced by M , we say that L is a Levi variety. We shall prove:

Theorem. *Let \mathcal{L} be a Levi foliation and let \mathcal{F} be the induced holomorphic foliation. Then, \mathcal{F} admits a Liouvillian first integral.*

In other words, if \mathcal{L} is a 3 dimensional analytic foliation such that the induced complex distribution defines an holomorphic foliation \mathcal{F} ; that is, if \mathcal{L} is a Levi foliation; then \mathcal{F} admits a Liouvillian first integral—a function which can be constructed by the composition of rational functions, exponentiation, integration, and algebraic functions (Singer 1992). For example, if f is an holomorphic function and if θ is real a 1-form on \mathbb{R}^2 ; then the pull-back of θ by f defines a Levi foliation $\mathcal{L} : f^*\theta = 0$ which is tangent to the holomorphic foliation $\mathcal{F} : df = 0$.

This problem was proposed by D. Cerveau in a meeting (see Fernandez 1997).

Key words: Levi foliations, holomorphic foliations, singularities, Levi varieties.

ANNOUNCEMENT

Let \mathcal{L} be a Levi foliation and let \mathcal{F} be the holomorphic foliation tangent to \mathcal{L} . Note that if h is an holomorphic function such that \mathcal{F} is h -invariant ($h^*\mathcal{F} = \mathcal{F}$); then \mathcal{L} is also h -invariant ($h^*\mathcal{L} = \mathcal{L}$). We shall mainly use that property in order to prove

THEOREM. *Let \mathcal{L} be a Levi foliation and let \mathcal{F} be the induced holomorphic foliation. Then \mathcal{F} admits a Liouvillian first integral.*

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We proceed as follows:

We first show that if \mathcal{L} is a Levi foliation, there exists analytic real functions g_1, g_2 such that: if $G = g_1 + ig_2$, then the Levi foliation is defined by

$$\mathcal{L} : \overline{G}\omega + G\overline{\omega} = 0,$$

where ω is an holomorphic 1-form so that $\omega = 0$ defines the holomorphic foliation \mathcal{F} tangent to the Levi foliation \mathcal{L} . We then verify that; if \mathcal{F}^* is the holomorphic foliation obtained from \mathcal{F} after a finite number of blow-ups, there exists a Levi foliation \mathcal{L}^* tangent to \mathcal{F}^* . Therefore, by Seidenberg Theorem (Seidenberg 1968), we analyse the foliation \mathcal{F}^* for which all singularities are reduced.

Let D denote the divisor obtained on the process of reducing the singularity and let D_j denote the irreducible curves with normal crossings such that $D = \cup D_j$. We consider the induced Levi foliation on sections transversal to the holomorphic foliation through each component D_j of the divisor. We show that the holomorphic diffeomorphisms for which the Levi foliation is invariant must satisfy an equation on one variable of the type

$$h'(z) = t \frac{F(h)}{F}; \quad t \in \mathbb{R} \quad (*)$$

We can then find an holomorphic coordinate system y on the section such that

$$F(y) = \frac{y^{k+1}}{1 - \lambda y^k}.$$

We refer to such coordinate system as a normalizable coordinate system. We verify that it is unique up to homographies.

If either $\lambda \neq 0$ or $k = 0$, then $t = 1$ for all solutions h of the differential equation (*). Furthermore, if $k = 0$, then the group of solutions of the differential equation is a linear group. On both cases we have an abelian group for the group of solutions of (*). We can already conclude:

THEOREM A. *Let p be a singularity of the foliation*

$$\mathcal{F} : \omega = \lambda x dy + y dx + \{\text{higher order terms}\} = 0 \quad \lambda \in \mathbb{R}^* - \mathbb{Q}.$$

Suppose there exists a Levi foliation \mathcal{L} tangent to \mathcal{F} . Then the singularity is analytically equivalent to a linear singularity.

PROOF. For if there exists a Levi foliation, the holonomy associated to the singularity must satisfy an equation as (*). If so, the order of F at 0 cannot be but 1; that is, $k=0$. The holonomy is linearizable; as a result, so is the singularity (Mattei & Moussu 1980).

We still have to consider the case $\lambda = 0$. There are solutions for which $t \neq 1$, $(h'(0))^k = \frac{1}{t} \in \mathbb{R}$. These solutions are necessarily linearizable, but not those for which $t = 1$. The latter, though, also determine an abelian group. We shall then describe the abelian group of solutions of (*) for $t = 1, k > 0$.

We can take an holomorphic coordinate system (x, y) such that the group of solutions of the differential equation is in normalizable coordinate system on each transversal section $x = cte$.

For an holomorphic vector field X , let $\exp X$ denote its exponential application, that is, its flow for $t = 1$:

$$\exp(\xi(z) \frac{\partial}{\partial z})(z) = z + f_1(z) + \frac{1}{2}f_2(z) + \frac{1}{3!}f_3(z) + \dots$$

satisfying

$$\begin{cases} f_1 = \xi, \\ f_n = \xi f'_{n-1}. \end{cases}$$

If h is a diffeomorphism which satisfies

$$h'(z) = \frac{h^{k+1}}{1 - \lambda h^k} \frac{1 - \lambda y^k}{y^{k+1}}$$

then the k -th iterate of h ; h^k , is tangent to the identity. There exists μ such that h^k is the exponential of the vector field:

$$Y = 2\pi i \mu \frac{y^{k+1}}{1 - \lambda y^k} \frac{\partial}{\partial y};$$

that is

$$h^k(w) = \exp(2\pi i \mu \frac{y^{k+1}}{1 - \lambda y^k} \frac{\partial}{\partial y})(w).$$

Consequently

$$h(w) = \exp(2\pi i \frac{\mu}{k} \frac{y^{k+1}}{1 - \lambda y^k} \frac{\partial}{\partial y})(\epsilon w); \epsilon^k = 1.$$

If

$$X = x \frac{\partial}{\partial x} + y f(x, y) \frac{\partial}{\partial y}$$

is the vector field which defines the holomorphic foliation; then the holonomy application is defined by

$$\exp 2\pi i X.$$

We have found two linear independent vector fields— X, Y that define h . Therefore; they commute:

$$[X, Y] = 0.$$

We can describe X to be so as to satisfy the commutability condition. We then show the local result:

THEOREM B. *Let p be a singularity of the foliation*

$$\mathcal{F} : \omega = \lambda x dy + y dx + \{\text{higher order terms}\} = 0, \lambda \in \mathbb{C}.$$

Suppose there exists a Levi foliation \mathcal{L} tangent to \mathcal{F} . Then the singularity is normalizable in the sense of Martinet and Ramis (1982), Martinet and Ramis (1983). In particular, ω admits an analytic integrating factor.

PROOF. If $\lambda \in \mathbb{C} - \mathbb{R}$, the singularity is linearizable by Poincaré's Theorem. If $\lambda \in \mathbb{R} - \mathbb{Q}$, we have proved (Theorem A) that is also a linearizable singularity. Thus, we have to prove the result for $\lambda \in \mathbb{Q}$; since the singularity is a reduced one, $\lambda \in \mathbb{Q}_+$. Let

$$-2\pi i Y(x_0, y) = -2\pi i \mu(x_0)^k \frac{y^{k+1}}{1 - \lambda \mu(x_0)^k y^k} \frac{\partial}{\partial y}$$

be the vector field whose exponential application determines the holonomy application on x_0 . If there are two invariant curves through the singularity, then the vector field that defines the holomorphic distribution can be written as $x \frac{\partial}{\partial x} + y f(x, y) \frac{\partial}{\partial y}$. By solving the commutability condition $[X, Y] = 0$:

$$\begin{aligned} 0 &= \left[x \frac{\partial}{\partial x} + y f(x, y) \frac{\partial}{\partial y}, \mu(x)^k \frac{y^{k+1}}{1 - \lambda \mu(x)^k y^k} \frac{\partial}{\partial y} \right] \\ &= \left(y \frac{1}{(1 - \lambda \mu(x)^k y^k)^2} d(\mu(x)^k y^k) \cdot (x, y f) - y \frac{\partial f}{\partial y} \mu(x)^k \frac{y^{k+1}}{1 - \lambda \mu(x)^k y^k} \right) \frac{\partial}{\partial y}. \end{aligned}$$

Let $f(x, y) = f(x, 0) + g(x, y)$, then f must be as to satisfy

$$\begin{cases} f(x, 0) = \frac{\mu'(x)x}{\mu(x)}, \\ \frac{\partial}{\partial y} \log g = k \frac{1}{y(1 - \lambda \mu^k y^k)} = k \frac{\partial}{\partial y} \log \left(\frac{y}{(1 - \lambda \mu^k y^k)^{\frac{1}{k}}} \right); \end{cases}$$

which leads us to

$$f(x, y) = \frac{\mu'(x)x}{\mu(x)} + \delta(x) \frac{y^k}{1 - \lambda \mu(x)^k y^k}.$$

The foliation on the punctured neighborhood is defined by the following 1-form

$$\omega = x dy + y \left(\frac{\mu'(x)x}{\mu(x)} - \delta(x) \frac{y^k}{1 - \lambda \mu^k y^k} \right) dx$$

or still by

$$\begin{aligned} \frac{\mu}{x} \omega &= \mu dy + y \left(1 - \frac{\delta}{\mu' x} \frac{y^k \mu^k}{1 - \lambda \mu^k y^k} \right) d\mu \\ &= \frac{\mu^{k+1} y^{k+1}}{1 - \lambda x^k y^k} \left(\frac{1 - \lambda \mu^k y^k}{\mu^k y^k} \frac{d(\mu^k y^k)}{(\mu^k y^k)^2} + \frac{\delta}{x} dx \right). \end{aligned}$$

Necessarily δ has an holomorphic extension through 0 and μ^k has either an holomorphic or a meromorphic extension through 0. If it were meromorphic, the singularity would not be a reduced one, contradicting our hypotheses. The extension is then an holomorphic one. We have then a normal form for either cases:

If $\mu^k \in \mathcal{O}^*$, we have a saddle-node; if $\mu^k \in \mathcal{O} - \mathcal{O}^*$ and let p be the order of the zero of f at 0, we have a resonant singularity.

If there is only one invariant curve through the singularity; the singularity is a saddle-node and the invariant curve is $y = 0$. Therefore the vector field that defines the holomorphic distribution can be written as $X = (x + h(y))\frac{\partial}{\partial x} + yf(x, y)\frac{\partial}{\partial y}$, $f(0) = 0$. The holonomy is defined by the exponential application of the vector field $\frac{x}{x+h(y)}X = x\frac{\partial}{\partial x} + \frac{yf(x, y)}{x+h(y)}\frac{\partial}{\partial y}$. The commutability condition $[\frac{x}{x+h(y)}X, Y] = 0$ implies that

$$\frac{x}{x+h(y)}[X, Y] = \left(d\frac{x}{x+h(y)}.Y\right)X.$$

By solving the equation just above, we obtain that $\frac{1}{f}$ must be an holomorphic function which contradicts $f(0) = 0$. \square

Following, we prove results that will allow us to relate the first integrals obtained on the neighborhood of each component D_j .

THEOREM C. *Let p be a singularity of the foliation $\mathcal{F} : \omega = 0$ and*

$$\omega = f dF \text{ is an holomorphic 1-form}$$

where F is a Liouvillian function and f is an holomorphic integrating factor of ω . There exists a Levi foliation defined by

$$\mathcal{L} : \bar{f}(fdF) + f(\bar{f}\bar{dF}).$$

Furthermore, if p is not a linearizable resonant singularity, then any other Levi foliation must be of the type:

$$\mathcal{L}_\lambda : \lambda \bar{f}(fdF) + \bar{\lambda} f(\bar{f}\bar{dF}).$$

Note that $\Re(\lambda F)$ is a first integral of the Levi foliation \mathcal{L}_λ . We can then show:

COROLLARY. *Let p be a singularity of the holomorphic foliation $\mathcal{F} : \omega = 0$. Let F_j be Liouvillian functions and let f_j be holomorphic functions such that*

$$\omega = f_j dF_j.$$

Suppose there exists a Levi foliation \mathcal{L} tangent to \mathcal{F} and suppose that $\Re(F_1), \Re(F_2)$ are first integrals of \mathcal{L} . Then:

$$\frac{dF_j}{F_j} = \frac{dF_i}{F_i}.$$

PROOF. Follows from $dF_i = \frac{f_j}{f_i} dF_j$ and $d(F_i + \bar{F}_i) \wedge d(F_j + \bar{F}_j) = 0$.

We are then able to show:

THEOREM D. *Let \mathcal{F} be an holomorphic foliation and \mathcal{L} be a Levi foliation tangent to \mathcal{F} . Suppose all singularities lie on an irreducible curve S ; which is \mathcal{F} -invariant. Then \mathcal{F} admits a Liouvillian first integral I defined on a neighborhood of S . Furthermore, $d(I + \bar{I})$ defines a Levi foliation tangent to \mathcal{F} .*

PROOF. To show the existence of a Liouvillian first integral of \mathcal{F} it is enough to show the existence of a Liouvillian first integral of the reduced foliation \mathcal{F}^* . Let $D = \cup D_j$ be the divisor obtained on the process of reducing the singularities. Let us fix a transversal section of \mathcal{F}^* through D_j . Since there exists a Levi foliation tangent to \mathcal{F}^* , there exists a normal coordinate system on the section so that the holonomy applications determined by the singularities on D_j satisfy (*).

For each D_j , we then find an holomorphic vector field Z_j that defines the foliation \mathcal{F}^* in a neighborhood of the divisor. Let Y be the holomorphic vector on each transversal section which defines the holonomies. To find Z_j , all we have to do is solve the equation

$$[Z_j, Y] = 0.$$

The vector field Z_j allows us to describe a Liouvillian first integral of the holomorphic foliation on a neighborhood of each irreducible component D_j of the divisor $D = \cup D_j$ obtained on the resolution of the singularity. Let F_j be a Liouvillian first integral of the holomorphic foliation \mathcal{F}^* on a neighborhood of the D_j such that $\Re(F_j)$ is a first integral of \mathcal{L}^* . By Theorem b, for each

$$p \in D_i \cap D_j$$

we have

$$\frac{dF_1}{F_1} = \frac{dF_2}{F_2}.$$

Therefore

$$\omega^* = \left\{ \frac{dF_i}{F_i} \right\}.$$

is a well defined closed 1-form. Thus

$$I = \exp \int \omega^*$$

is a Liouvillian first integral of the holomorphic foliation \mathcal{F}^* and there is a Levi foliation $d(I + \bar{I}) = 0$; *The Theorem* is thereby proved.

RESUMO

Seja $L \subset \mathbb{C}^2$ uma variedade real de dimensão 3. Para todo ponto regular $p \in L$ existe uma única reta complexa l_p no espaço tangente à L em p . Quando o campo de linhas complexas

$$p \mapsto l_p$$

é completamente integrável, dizemos que L é uma variedade de Levi. Mais geralmente, seja $L \subset M$ uma subvariedade real em uma variedade analítica complexa. Se existe uma distribuição real integrável de

dimensão 2 em L que é invariante pela estrutura holomorfa J induzida pela variedade complexa M , dizemos que L é uma variedade de Levi. Vamos provar:

Teorema. *Seja \mathcal{L} uma folheação de Levi e seja \mathcal{F} a folheação holomorfa induzida. Então \mathcal{F} tem integral primeira Liouvilliana.*

Em outras palavras, se \mathcal{L} é uma folheação real de dimensão 3 tal que a folheação holomorfa induzida define uma folheação holomorfa \mathcal{F} ; isto é, se \mathcal{L} é uma folheação de Levi; então \mathcal{F} admite uma integral primeira Liouvilliana – uma função que pode ser construída por composição de funções racionais, exponenciações, integrações e funções racionais (Singer 1992). Por exemplo, se f é uma função holomorfa e se θ é uma 1-forma real em \mathbb{R}^2 ; então o pull-back de θ por f define uma folheação de Levi: $\mathcal{L} : f^*\theta = 0$ a qual é tangente a folheação holomorfa $\mathcal{F} : df = 0$.

Este problema foi proposto por D. Cerveau em uma reunião (Fernandez 1997).

Palavras-chave: folheações de Levi, folheações holomorfas, singularidades, variedades de Levi.

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