Anais da Academia Brasileira de Ciências (2002) 74(1): 33–35
(Annals of the Brazilian Academy of Sciences)
ISSN 0001-3765
www.scielo.br/aabc

Surfaces of Constant Mean Curvature in Euclidean 3-space Orthogonal to a Plane along its Boundary

PEDRO A. HINOJOSA

Centro de Ciências Exatas e da Natureza, Departamento de Matemática
Universidade Federal da Paraíba, João Pessoa, Brasil, and
Universidade Federal do Ceará, Centro de Ciências, Pós Graduação em Matemática
Campus do Pici, Bloco 914 – 60455-760 Fortaleza, Ce, Brasil

Manuscript received on October 25, 2001; accepted for publication on November 28, 2001;
presented by J. Lucas Barbosa

ABSTRACT

We consider compact surfaces with constant nonzero mean curvature whose boundary is a convex planar Jordan curve. We prove that if such a surface is orthogonal to the plane of the boundary, then it is a hemisphere.

Key words: surfaces with boundary, constant mean curvature, elliptic partial differential equation.

Let M be a compact surface inmersed in $\mathbb{R}^3$ with constant mean curvature H whose boundary $\partial M = \Gamma$ is a planar Jordan curve of length L. Let D be a planar region enclosed by Γ and let A be the area of D. Let us consider the cycle $M \cup D$ oriented in such a way that its orientation, along M, coincides with the one defined by the mean curvature vector. Let Y be a Killing vector field in $\mathbb{R}^3$ and n_D be a unitary vector field normal to D in the orientation of $M \cup D$. Let ν be the unitary co-normal vector field along $\partial M = \Gamma$ pointing inwards M. By the flux formula it is known that $|H| \leq \frac{L}{2A}$ where equality holds if and only if $\nu = n_D$. That is, if and only if ν is constant and orthogonal to D along Γ.

In this work we consider the case $|H| \leq \frac{L}{2A}$ and we show that, in the above conditions, if M is embedded and Γ is convex, then M is a hemisphere. Explicitly we prove that:

Theorem 1. *Let M be a compact embedded surface in $\mathbb{R}^3$ with constant mean curvature $H \neq 0$ whose boundary ∂M is a Jordan curve Γ in a plane $\mathbb{P} \subset \mathbb{R}^3$. Suposse that Γ is convex and M is perpendicular to the plane $\mathbb{P}$ along its boundary. Then M is a hemisphere of radius $\frac{1}{|H|}$.*

Correspondence to: Universidade Federal do Ceará
e-mail: hinojosa@mat.ufpb.br

This theorem generalizes a result obtained by Brito and Earp (Brito and Earp 1991). We succed in discarding their assumption that ∂M should be a circle of radius $\frac{1}{|H|}$.

A sketch of the proof of the theorem is as follows.

First, under the hypothesis of the theorem, M must be totally contained in one of the halfplanes determined by $\mathbb{P}$ (see (Brito et al. 1991), for example). Now let M^* be the reflection of M with respect to the plane $\mathbb{P}$. Since M is orthogonal to $\mathbb{P}$ along Γ, we have that $\widetilde{M} := M \cup M^*$ is a compact surface without boundary, embedded in $\mathbb{R}^3$. Note that *a priori* $\widetilde{M}$ is only of class C^1 along Γ. We will prove that $\widetilde{M}$ is at least of class C^3. In this way we are able to use a classical result due to Alexandrov (see (Hopf 1983), for example) in order to establish that $\widetilde{M}$ is a sphere and therefore M is a hemisphere.

The regularity of $\widetilde{M}$ along Γ is achieved by means of the theory of elliptic partial differential equations. Let p be any point in $\Gamma \subset \widetilde{M}$ and Ω be an open neighborhood of 0 in $T_p\widetilde{M}$ chosen in such a way that locally around p, $\widetilde{M}$ may be described as the graph of a function $u : \Omega \to \mathbb{R}$. For our purposes, it is suffices to consider Ω of class $C^{1,1}$.

It is clear that $u \in C^1(\Omega)$. So, ∇u is well-defined and continuous. Since Ω is bounded we have that $u \in W^{1,2}(\Omega)$.

Let us denote the linear space of k-times weakly differentiable functions by $W^k(\Omega)$. For $p \geq 1$ and k a non-negative integer, we let $W^{k,p}(\Omega) = \{u \in W^k(\Omega), D^\sigma u \in L^p(\Omega) \text{ for all } |\sigma| \leq k\}$.

The Hölder spaces $C^{k,\alpha}(\Omega)$ are defined as the subspaces of $C^k(\Omega)$ consisting of functions whose k-th order partial derivatives are locally Hölder continuous whith exponent α in Ω.

We define on Ω the following linear operators:

$$L_1 v := D_i(a^{ij} D_j v), \qquad v \in W^{1,2}(\Omega) \quad i, j = 1, 2$$

and

$$L_2 v := A^{ij} D_{ij} v, \qquad v \in W^{2,2}(\Omega) \quad i, j = 1, 2,$$

where the coefficients a^{ij} are given by

$$a^{11} = a^{22} = \frac{1}{1 + |\nabla u|^2}, \quad a^{12} = a^{21} = 0$$

and the coefficients A^{ij} are defined by $A^{11} = 1 + u_y^2$, $A^{12} = A^{21} = -u_x u_y$, $A^{22} = 1 + u_x^2$. Finally, the symbols D_i, D_{ij}, $i, j = 1, 2$ stand for partial differentiation.

We prove that u is a weak solution to the equation $L_1 u = 2H$. By the Corollary 8.36 (Gilbarg and Trudinger 1983) we have $u \in C^{1,\alpha}(\Omega)$. Moreover, by the Lebesgue's dominated convergence theorem and Lemma 7.24 (Gilbarg and Trudinger 1983) we can conclude that $u \in W^{2,p}(\Omega')$ for any subdomain $\Omega' \subset\subset \Omega$. Fixed $\Omega' \subset\subset \Omega$, we consider the equation

$$L_2 v = 2H \left(1 + |\nabla u|^2\right)^{\frac{3}{2}}. \tag{1}$$

We observe that $u \in W^{2,2}(\Omega')$. Thus, $L_2 u$ is well-defined. Moreover, we have that $L_2 u = 2H\left(1+|\nabla u|^2\right)^{\frac{3}{2}}$ in Ω'. It means that $u \in W^{2,2}(\Omega')$ is a solution to the equation (1) just above. Now using the Theorem 9.19 (Gilbarg and Trudinger 1983) we obtain $u \in C^{2,\alpha}(\Omega')$. Repeating the same procedure we conclude that $u \in C^\infty(\Omega')$. Thus, $\widetilde{M}$ is C^∞. So, $\widetilde{M}$ is a regular compact closed surface embedded in $\mathbb{R}^3$ with constant mean curvature. By the Theorem 5.2 (Chapter V, (Hopf 1983)) we conclude that $\widetilde{M}$ is a (round) sphere and therefore M is a hemisphere.

ACKNOWLEDGMENTS

This work is part of my Doctoral Thesis at the Universidade Federal do Ceará – UFC. I want to thank my advisor, Professor J. Lucas Barbosa, and also Professor A. G. Colares, by the encouragement and many helpful conversations. The research was supported by a scholarship of CAPES.

RESUMO

Consideramos superfícies compactas com curvatura média constante e não nula as quais têm como bordo uma curva de Jordan plana convexa. Provamos que, se uma tal superfície é ortogonal ao plano do bordo então é um hemisfério.

Palavras-chave: superfícies com bordo, curvatura média constante, equações diferenciais parciais elípticas.

REFERENCES

Brito F and Earp RS. 1991. Geometric Configurations of Constant Mean Curvature Surfaces with Planar Boundary. An Acad bras Ci 63: 5-19.

Brito F, Earp RS, Meeks W and Rosenberg H. 1991. Structure Theorems for Constant Mean Curvature Surfaces Bounded by a Planar Curve. Indiana Univ Math J 40(1): 333-343.

Gilbarg D and Trudinger NS. 1983. Elliptic Partial Differential Equations of Second Order. 2nd edition, Springer-Verlag, Berlin.

Hopf H. 1983. Differential Geometry in the Large. Lectures Notes in Mathematics, 1000, Springer-Verlag, Berlin.