Revista Integración
ISSN: 0120-419X
integracion@matematicas.uis.edu.co
Universidad Industrial de Santander
Colombia

García Camacho, Gonzalo; Posada Vera, Liliana
Lq estimates of functions in the kernel of an elliptic operator and applications
Revista Integración, vol. 34, núm. 1, 2016, pp. 1-21
Universidad Industrial de Santander
Bucaramanga, Colombia

Revista Integración
Escuela de Matemáticas
Universidad Industrial de Santander
Vol. 34, No. 1, 2016, pág. 1–21

DOI: http://dx.doi.org/10.18273/revint.v34n1-2016001

L^q estimates of functions in the kernel of an elliptic operator and applications

Gonzalo García Camacho, Liliana Posada Vera*

Universidad del Valle, Departamento de Matemáticas, Cali, Colombia.

Abstract. In this work, we will find a family of small functions η_y in the Kernel of an operator defined in the intersection of the Sobolev space $H^{2,q}(S^n)$ with the orthogonal complement in $H^{1,2}(S^n)$ of the first eigenspace of the laplacian on S^n, parameterized with a variable y belonging to a small ball contained in B^{n+1}. We will find L^q estimates of these functions and we will use those estimates to find a subcritical solution to the scalar curvature problem on S^n, and a solution $u_{y_1} = \alpha_{F_{y_1}^{-1}}(1+\eta_{y_1}) = |F'_{y_1}|^{\frac{n-2}{2}}(1+\eta_{y_1})\circ F_{y_1}$ of a nonlinear elliptical problem related to that problem, where $F_{y_1} : S^n \to S^n$ is a centered dilation.

Keywords: Sobolev spaces, conformal deformations, elliptic equations.
MSC2010: 53C21, 58J32, 46E35, 58E11.

Estimativos L^q de funciones en el núcleo de un operador elíptico y aplicaciones

Resumen. En este trabajo, vamos a encontrar una familia de pequeñas funciones η_y en el kernel de un operador definido en la intersección del espacio de Sóbolev $H^{2,q}(S^n)$ con el complemento ortogonal en $H^{1,2}(S^n)$ del primer espacio propio del laplaciano sobre S^n, parametrizado con una variable y que pertenece a una pequeña bola contenida en B^{n+1}. Encontraremos estimativos L^q de estas funciones, las cuales utilizaremos para encontrar una solución subcrítica al problema de curvatura escalar sobre S^n y una solución $u_{y_1} = \alpha_{F_{y_1}^{-1}}(1 + \eta_{y_1}) = |F'_{y_1}|^{\frac{n-2}{2}}(1 + \eta_{y_1}) \circ F_{y_1}$ de un problema elíptico no lineal relacionado con este problema, donde $F_{y_1} : S^n \to S^n$ es una dilatación centrada.

Palabras clave: Espacios de Sóbolev, deformaciones conformes, ecuaciones elípticas.

* E-mail: liliana.posada@correounivalle.edu.co
Received: 24 October 2015, Accepted: 21 January 2016.
To cite this article: G. García Camacho, L. Posada Vera, L^q estimates of functions in the kernel of an elliptic operator and applications, *Rev. Integr. Temas Mat.* 34 (2016), No. 1, 1–21.

1. Introduction

Let (S^n, δ_{ij}) be the unitary sphere with the standard metric. A natural question in Riemannian geometry is: given a function $K : S^n \to \mathbb{R}$, is there a metric g conformally related to the standard metric δ_{ij} such that K is the scalar curvature of S^n with respect to the metric g? This is equivalent to the problem of finding a positive smooth function $u : S^n \to \mathbb{R}$ which satisfies the equation

$$\Delta u - \frac{n(n-2)}{4}u + \frac{n-2}{4(n-1)}Ku^{\frac{n+2}{n-2}} = 0. \qquad (1)$$

If we set $g = u^{\frac{4}{n-2}}\delta_{ij}$, where u is a solution of this problem, then the function K is the scalar curvature of S^n with respect to the metric g.

The problem of conformal deformation of metrics in S^n have been extensively studied by many authors (for example, see [1], [2], [3], [5], [6], [7], [8], [9] and the references therein). An important feature of this problem is that it is a conformal invariant one. More precisely, if u is a solution of equation (1) then for any conformal map $F : S^n \to S^n$ the function $\alpha_F(u) = |(F^{-1})'|^{\frac{n-2}{2}} u \circ F^{-1}$ is a solution to problem (1) with scalar curvature $K \circ F$.

The problem of conformal deformation of metrics in S^n can be approached using the so called Yamabe method, which consists in studying first the subcritical problem in the equation (1):

$$\Delta u_p - \frac{n(n-2)}{4}u_p + \frac{n-2}{4(n-1)}Ku_p^p = 0, \qquad (2)$$

with $p \in \left(1, \frac{n+2}{n-2}\right)$, and then consider the limit of the solutions when $p \uparrow \frac{n+2}{n-2}$.

Let $E(u)$ be the energy norm associated with the linear part of (2), and let $\mathcal{S}$ be the set of non-negative functions $u \in W^{2,q}(S^n)$, $(q > \frac{n}{2})$ such that $E(u) = E(1)$. Let us consider the open unit ball B^{n+1} and the map $\Phi : B^{n+1} \to \mathcal{S}$ defined by

$$\Phi(y) = \alpha_y := \alpha_{F_y}(1) = |(F_y^{-1})'|^{\frac{n-2}{2}},$$

where $F_y : S^n \to S^n$ is the restriction to S^n of a special conformal map $F_y : \overline{B^{n+1}} \to \overline{B^{n+1}}$ that satisfies $F_y(0) = y$ and fix the points $\pm\frac{y}{|y|}$; this function maps 0 to y and commutes with rotations about the line joining the origin and the point y. This map is referred to as a centered dilation.

For $p \in \left(1, \frac{n+2}{n-2}\right)$ and $u \in \mathcal{S}$, let $J_p(u)$ defined by $J_p(u) = \int_{S^n} Ku^{p+1}d\sigma$. If u is a critical point of $J_p(\cdot)$ on $\mathcal{S}$, then a multiple of u satisfies problem (2). Let us define the function $\overline{J}_p = J_p \circ \Phi$. In this paper, we will consider the equation

$$Lu + \frac{n(n-2)}{4}vol(S^n)(\overline{J}_p(y))^{-1}Ku^p = 0, \qquad (3)$$

where $K : S^n \to \mathbb{R}$ is a nondegenerate function (Morse function) with $\Delta K \neq 0$ in its critical points, and $Lu = \Delta u - \frac{n(n-2)}{4}u$.

Let $F : S^n \to S^n$ be a conformal transformation and $v = \alpha_F(u) : |(F^{-1})'|^{\frac{n-2}{2}} u \circ F^{-1}$. A straightforward calculation shows that u is solution of (3) if and only if the function $\eta = v - 1$ is a solution of an equation of the form

$$\mathcal{L}(\eta) + \mathcal{Q}(\eta) = \frac{(n-2)n}{4}(1-a)(1+\eta)^{\frac{n+2}{n-2}}, \tag{4}$$

where $a = vol(S^n)(\overline{J}_p(y))^{-1} K \circ F^{-1} |(F^{-1})'|^{\frac{n-2}{2}\delta}(1+\eta)^{-\delta}$, $\mathcal{L}(\eta) = \Delta\eta + n\eta$, $\mathcal{Q}(\eta)$ is a term which is quadratically small in η, and $\delta = \frac{n+2}{n-2} - p$. The linear operator $\mathcal{L}$ has an $(n+1)$ dimensional kernel consisting of the first order spherical harmonics. This obstruction to invert the linear operator $\mathcal{L}$ may be removed by replacing equation (4) by the projected equation $T(y, \eta) = 0$, where

$$T(y,\eta) = \mathcal{L}(\eta) + \mathbf{P}(\mathcal{Q}(\eta)) - \mathbf{P}\left(\frac{(n-2)n}{4}(1-a)(1+\eta)^{\frac{n+2}{n-2}}\right), \tag{5}$$

and $\mathbf{P}$ denotes the $\mathbb{L}^2$-orthogonal projection onto the orthogonal complement W of the first eigenspace of the laplacian on S^n.

This work is motivated by the work of Schoen and Zhang in [8] on the prescribed scalar curvature problem on the n-dimensional sphere, $n \geq 3$, and by the work of Escobar and García in [3] on the prescribed mean curvature on the n-dimensional unit ball, $n \geq 3$. In fact our method parallels those of [8] and [3]. In this paper we will find in Section 3, using the inverse function Theorem, small solutions η_y of equation (5), where y is close to a critical point of $\overline{J}_p$. In Section 4, we will find L^q and integral estimates of η_y and its first two derivatives.

In the last section, setting $u_y = \alpha_{F_y}(1 + \eta_y)$, we perturb the function u_y and consider the function $\widetilde{u}_y = \Lambda_y u_y$ in order to achieve that $E(\widetilde{u}_y) = E(1)$. Next we define the map $\widetilde{J}_p(y) = J_p(\widetilde{u}_y)$ and we show that the functions $\overline{J}_p(y)$ and $\widetilde{J}_p(y)$ are close in the C^2 norm, using the estimates of the functions η_y. The fact that the functions $\overline{J}_p(y)$ and $\widetilde{J}_p(y)$ are close implies that $\widetilde{J}_p(y)$ has a unique critical point y_1 close to the critical point y_0 of $\overline{J}_p(y)$. This implies that $\widetilde{u}_{y_1}$ is a solution of the equation

$$Lu + \frac{n(n-2)}{4} K vol(S^n)(J_p(u))^{-1} u^p = 0. \tag{6}$$

Multiplying the function $\widetilde{u}_{y_1}$ by suitable constants, we find a solution of problem (2) and prove that $u_{y_1} = \alpha_{F_{y_1}}(1 + \eta_{y_1})$ is a solution of problem (3), respectively.

2. Preliminaries

Let $y \in B^{n+1}$. Up to a rotation we will assume that $y = (0, \dots, 0, y_{n+1})$, $y_{n+1} \geq 0$. In this case the centered dilation function F_y is given by $F_y(x) = \Sigma^{-1} \circ D_\mu \circ \Sigma(x)$, where the function

$$\Sigma(x) = \frac{2\overline{x}}{1 + x_{n+1}}$$

is the stereographic projection from the south pole of the sphere, the function

$$\Sigma^{-1}(\overline{x}) = \left(\frac{4\overline{x}}{|\overline{x}|^2 + 4}, \frac{4 - |\overline{x}|^2}{|\overline{x}|^2 + 4}\right)$$

is the inverse of the stereographic projection, and the function $D_\mu : \mathbb{R}^n \to \mathbb{R}^n$ is defined by $D_\mu(\overline{x}) = \mu\overline{x}$, where $x = (\overline{x}, x_{n+1}) \in S^n$ with $\overline{x} = (x_1, \dots, x_n)$ and $\mu = \frac{1-|y|}{1+|y|}$.

Since $F_y = \Sigma^{-1} \circ D_\mu \circ \Sigma$, then $F_y(x) = B^{-1}(4\mu A\overline{x}, (A^2 - 4\mu^2|\overline{x}|^2)$ and $F_y(0) = y$, where

$$A = 2(1 + x_{n+1}) \quad \text{and} \quad B = 4\mu^2|\overline{x}|^2 + 4(1 + x_{n+1})^2.$$

Note that $F_y^{-1} = F_{-y}$.

If $y \in B_{\beta(1-|y_0|)}(y_0)$ for some $0 < \beta < 1$, then we have

$$(1-\beta)(1-|y_0|) \leq 1 - |y| \leq (1+\beta)(1-|y_0|). \tag{7}$$

The number μ satisfies the inequalities

$$\mu \leq C(1-|y_0|) \tag{8}$$

and

$$\frac{1}{\mu} \leq \frac{C}{1-|y_0|}. \tag{9}$$

Consider the map $\Phi : B^{n+1} \to \mathcal{S}$ defined by $\Phi(y) = \alpha_y := \alpha_{F_y}(1) = |(F_y^{-1})'|^{\frac{n-2}{2}}$, where $F_y : S^n \to S^n$ is the conformal map that satisfies $F_y(0) = y$, and fix the points $\pm\frac{y}{|y|}$. For $p \in \left(1, \frac{n+2}{n-2}\right]$ and $u \in \mathcal{S}$, let $J_p(u)$ be defined by

$$J_p(u) = \int_{S^n} K u^{p+1} d\sigma.$$

If u is a critical point of $J_p(\cdot)$ on $\mathcal{S}$, $p \in \left(1, \frac{n+2}{n-2}\right)$, then a multiple of u satisfies problem (2). Let us define $\overline{J}_p = J_p \circ \Phi$. The functions $\overline{J}_p$ are eigenfunctions of the laplacian on B^{n+1} with the hyperbolic metric. In fact,

$$\triangle\overline{J}_p + \lambda_p\overline{J}_p = 0; \quad \lambda_p = \left(\frac{n-2}{2}\right)^2 (p+1)\delta,$$

where $\delta = \frac{n+2}{n-2} - p$.

Let us define the function $v_p(y) = \int_{S^n}(\alpha_y(\xi))^{p+1} d\sigma(\xi)$, so that $v_p(y) = vol(S^n)$ for $p = \frac{n+2}{n-2}$. The function v_p is positive and radially symmetric. Let us define the function $\widehat{J}_P = v_p^{-1}\overline{J}_p$. For $n \geq 3$ the functions $\widehat{J}_P$ are uniformly bounded in the $C^2(B^{n+1})$ norm and they agree with K on S^n. Using that all critical points of the function K are non-degenerate and $\triangle K \neq 0$ at each critical point, the following facts are proven in Proposition 2.1 in [8]. Since $\widehat{J}_P$ is C^2 in the closed ball, then $\frac{\partial \widehat{J}_P}{\partial r} = 0$ in the boundary of the ball. From here it can be seen that the critical points of $\widehat{J}_P$ near ∂B^{n+1} actually lie on ∂B^{n+1} and are the critical points of K. If y_0 is a critical point of the function $\overline{J}_p$ near ∂B^{n+1}, then $|\frac{\partial v_p}{\partial r}(y_0)| \leq C v_p(y_0)(1-|y_0|)$. It is also proven that there exist constants $C_1, C_2 > 0$ such that

$$C_1\delta \leq (1-|y_0|)^2 \leq C_2\delta, \tag{10}$$

and consequently,

$$C_1\delta \leq \mu^2 \leq C_2\delta. \tag{11}$$

The estimates of the following proposition (see [4]) are very useful in this work.

Proposition 2.1. *Let y_0 be a point near ∂B^{n+1} which is the critical point of the function $\overline{J}_p$ and let $y \in B_{\beta(1-|y_0|)}(y_0)$. Then,*

1. $\left|\nabla K\left(\frac{y_0}{|y_0|}\right)\right| \leq C\mu^{1-w}$, *where w is any small positive number less than one.*
2. *If* $f = \boldsymbol{P}\left(K - K\left(\frac{y}{|y|}\right)\right)$, $\|f \circ F_y\|_{0,q} \leq C\mu^{2-w}$, *with* $0 < w < 1$.
3. *If* $\frac{n}{2} < q < n$, $\|\nabla_y(K \circ F_y)\|_{0,q} \leq C\mu^{1-w}$, *where* $0 < w < 1$.
4. *For* $\frac{n}{2} < q < n$ *and* $1 - \frac{n}{2q} < r < \frac{1}{2}$, $\|\nabla_y\nabla_y(K \circ F_y)\|_{0,q} \leq \mu^{-2r}$.

The following propositions, which are useful to find a solution of problem (2), are respectively the Corollary 2.2 and Lemma 2.3 in [8].

Proposition 2.2. *There is a number $\beta < 1$ such that, if we denote by y_0 one of the critical points of $\overline{J}_p$ near ∂B^{n+1}, then the following bound holds for $y \in B_{\beta(1-|y_0|)}(y_0)$:*

$$(1 - |y_0|)^{-1}\|\nabla\overline{J}_p\| + \|\nabla\nabla\overline{J}_p\| \leq c, \qquad |\det(\mathrm{Hess}(\overline{J}_p)| \geq c^{-1}.$$

For $y \in B_{\beta(1-|y_0|)}(y_0)$ we have $\|\nabla\overline{J}_p\| \geq c^{-1}(1 - |y_0|)$.

Proposition 2.3. *Suppose f, g are $\mathcal{C}^2$ functions in the closed unit ball $\overline{B}^{n+1}$ in $\mathbb{R}^{n+1}$. Suppose there is a positive constant c such that*

$$\|\nabla f\| + \|\nabla\nabla f\| \leq c, \qquad |\det(\mathrm{Hess}(f)| \geq c^{-1} \quad \text{and} \quad \inf_{\partial B_1} \|\nabla f\| \geq c^{-1}.$$

Assume f has a unique critical point y_0 in B^{n+1}, and g is close to f in the sense that

$$\|\nabla(f - g)\| + \|\nabla\nabla(f - g)\| \leq \epsilon.$$

If ϵ is sufficiently small, then g has a unique critical point y_1 in B^{n+1}.

3. The projected equation

To begin with, we will do several transformations of equation (2). One of those transformations involves the definition of an operator

$$\mathcal{T} : \mathcal{B}^{2,q} \to \mathcal{B}^{0,q}, \quad \text{where} \quad \mathcal{B}^{j,q} = C^2(B_{\beta(1-|y_0|)}(y_0), H^{j,q}(S^n) \cap W), \quad j = 0, 2,$$

by setting $\mathcal{T}(\eta)(y) = T(y, \eta)$; this operator and the inverse function Theorem allow us to find a solution to problem (5).

After multiplying a solution u of equation (2) by a suitable constant, we can rewrite that equation as

$$Lu + \frac{n(n-2)}{4}K vol(S^n)(J_p(u))^{-1}u^p = 0, \tag{12}$$

where $Lu = \Delta u - \frac{n(n-2)}{4}u$. Let y_0 be a critical point of $\overline{J}_p$ which is one of the critical points of $\overline{J}_p$ near ∂B^{n+1} given by Proposition 2.1 in [8]. Let $y \in B_{\beta(1-|y_0|)}(y_0)$, with $0 < \beta < 1$. To find a solution of equation (12), we will consider first the equation

$$Lu + \frac{n(n-2)}{4}vol(S^n)(\overline{J}_p(y))^{-1}Ku^p = 0, \tag{13}$$

where we have replaced $J_p(u)$ by $\overline{J}_p(y)$.

A straightforward calculation shows that if u is solution of (13), $F : S^n \to S^n$ is a conformal transformation and $v = \alpha_F(u) : |(F^{-1})'|^{\frac{n-2}{2}} u \circ F^{-1}$, then v is a solution of the problem

$$Lv + \frac{(n-2)n}{4} vol(S^n)(\overline{J}_p(y))^{-1} K \circ F^{-1} |(F^{-1})'|^{\frac{n-2}{2}\delta} v^p = 0. \tag{14}$$

Setting $v = 1 + \eta$, and defining $\mathcal{L}(\eta) = \Delta\eta + n\eta$, $\mathcal{Q}(\eta) = \frac{n(n-2)}{4}\left((1+\eta)^{\frac{n+2}{n-2}} - 1 - \frac{n+2}{n-2}\eta\right)$, and $a = vol(S^n)(\overline{J}_p(y))^{-1} K \circ F^{-1} |(F^{-1})'|^{\frac{n-2}{2}\delta}(1+\eta)^{-\delta}$, if v is a solution of equation (14), then η is a solution of problem

$$\mathcal{L}(\eta) + \mathcal{Q}(\eta) = \frac{(n-2)n}{4}(1-a)(1+\eta)^{\frac{n+2}{n-2}}. \tag{15}$$

Let $\{\xi_1, \xi_2, \dots \xi_{n+1}\}$ a generator set of the first eigenfunctions of the laplacian of S^n, that is,

$$\mathcal{L}(\xi_i) = \Delta\xi_i + n\xi_i = 0, \quad i = 1, 2 \dots, n+1.$$

This obstruction to invert the linear operator $\mathcal{L}$ may be removed by replacing equation (15) by the projected equation $T(y, \eta) = 0$, where

$$T(y,\eta) = \mathcal{L}(\eta) + \mathbf{P}(\mathcal{Q}(\eta)) - \mathbf{P}\left(\frac{(n-2)n}{4}(1-a)(1+\eta)^{\frac{n+2}{n-2}}\right), \tag{16}$$

and $\mathbf{P}$ denotes the $\mathbb{L}^2$-orthogonal projection onto the orthogonal complement W of the first eigenspace of S^n, where we have used that $(\mathcal{L}(\eta), \xi_i) = 0$ implies $\mathbf{P}(\mathcal{L}(\eta)) = \mathcal{L}(\eta)$.

In order to keep track of the dependence on y, as in [8], we define a map

$$\mathcal{T} : \mathcal{B}^{2,q} \to \mathcal{B}^{0,q}, \quad \text{where} \quad \mathcal{B}^{j,q} = C^2(B_{\beta(1-|y_0|)}(y_0), H^{j,q}(S^n) \cap W) \quad j = 0, 2,$$

by setting $\mathcal{T}(\eta)(y) = T(y, \eta)$, where η is the map $\eta(y) = \eta_y$. We choose a norm on $\mathcal{B}^{j,q}$ which reflects the scales which appear in the problem:

$$||\eta||_{\mathcal{B}^{j,q}} = \sup_y \{||\eta_y||_{j,q} + (1-|y_0|)||\nabla_y \eta_y||_{j,q} + (1-|y_0|)^2 ||\nabla_y \nabla_y \eta_y||_{j,q}\}, \quad j = 0, 2.$$

Hence,

$$||\mathcal{T}(\eta)||_{\mathcal{B}^{0,q}} = \sup_y \{||T(y,\eta)||_{0,q} + (1-|y_0|)||\nabla_y T(y,\eta)||_{0,q} + (1-|y_0|)^2 ||\nabla_y \nabla_y T(y,\eta)||_{0,q}\}.$$

One of the main objectives of this work is to prove the existence of solutions of the projected equation (16). To reach it we will prove a similar result to Lemma 2.5 in [8].

Theorem 3.1. *For $p \to \frac{n+2}{n-2}$ and $q \in (n/2, n)$, the function $\mathcal{T}$ is C^1 and satisfies the following bounds:*

1. *$||\mathcal{T}(0)|| \leq C\epsilon(p)\mu^\sigma$, where $\epsilon(p) \to 0$ when $p \to \frac{n+2}{n-2}$ and $\sigma < 2$.*
2. *$||\mathcal{T}'(0)|| \leq C$.*

3. $||\mathcal{T}'(\eta_1) - \mathcal{T}'(\eta_0)|| \le C||\eta_1 - \eta_0||$, $\|\eta_1\| \le \frac{1}{4}$, $\|\eta_0\| \le \frac{1}{4}$.

Moreover, $||(\mathcal{T}'(0))^{-1}|| \le C$, *where the constant* C *is independent on* p. *There exists* $\eta \in \mathcal{B}^{2,q}$ *with* $\|\eta\| \le C\epsilon(p)\mu^{\sigma}$ *and* $\mathcal{T}(\eta) = 0$. *Furthermore* η *is the unique small solution of* $\mathcal{T}(\eta) = 0$.

Proof. The bound for

$$||\mathcal{T}(0)||_{\mathcal{B}^{0,q}} = \{\sup_y ||T(y,0)||_{0,q} + (1-|y_0|)||\nabla_y T(y,0)||_{0,q} + (1-|y_0|)^2||\nabla_y\nabla_y T(y,0)||_{0,q}\}$$

follows from the following three lemmas.

Lemma 3.2. *For any* $q \in \left(\frac{n}{2}, n\right)$, $\|T(y,0)\|_{0,q} \le C\mu^{2-2w}$, *where* $0 < w < 1$.

Proof. For $\eta = 0$ we have that

$$\begin{aligned} T(y,0) &= -\mathbf{P}\left(\frac{(n-2)n}{4}(1-a_0)\right) \\ &= \frac{(n-2)n}{4} vol(S^n)(\overline{J_p}(y))^{-1}\mathbf{P}\left(K\circ F_y|F_y'|^{\frac{n-2}{2}\delta} - (vol(S^n))^{-1}\overline{J_p}(y)\right), \end{aligned}$$

where $a_0 = a(\xi,y,0) = vol(S^n)(\overline{J_p}(y))^{-1}K\circ F_y|F_y'|^{\frac{n-2}{2}\delta}$, and $|F_y'| = \frac{1-|y|^2}{|y+\xi|^2}$, $\xi \in S^n$.
It is easy to see that

$$|T(y,0)| \le C\left[\left||F_y'|^{\frac{n-2}{2}\delta} - 1\right| + \left|(vol(S^n))^{-1}\overline{J_p}(y) - K\left(\frac{y}{|y|}\right)\right| + \left|K\circ F_y - K\left(\frac{y}{|y|}\right)\right|\right].$$

To finish the lemma, in the following claims we will show that the terms in the right hand side of the previous inequality have the required bound.

Claim 1. $||F_y'|^{\frac{n-2}{2}\delta} - 1| \le C\mu^{2-2w}$, with $0 < w < 1$ and $y \in B_{\beta(1-|y_0|)}(y_0)$.

Proof. Let us observe that $|F_y'|^{\frac{n-2}{2}\delta}$ is of the form δ^δ. Taking $0 < w < 1$ and using the L'Hôpital rule we get

$$\lim_{\delta\to 0}\frac{\delta^\delta - 1}{\delta^{1-w}} = 0.$$

Then, for δ small enough, $|\delta^\delta - 1| \le C\delta^{1-w} \le C\mu^{2-2w}$, and consequently,

$$||F_y'|^{\frac{n-2}{2}\delta} - 1| \le C\mu^{2-2w}.$$ ☑

Claim 2. $\left|(vol(S^n))^{-1}\overline{J_p}(y) - K\left(\frac{y}{|y|}\right)\right| \le C\mu^{2-2w}$, where $0 < w < 1$.

Proof. First observe that

$$\left|(vol(S^n))^{-1}\overline{J_p}(y) - K\left(\frac{y}{|y|}\right)\right| \le \left|\frac{\overline{J_p}(y)}{vol(S^n)} - \frac{\overline{J_p}(y)}{v_p(y)}\right| + \left|\frac{\overline{J_p}(y)}{v_p(y)} - K\left(\frac{y}{|y|}\right)\right|.$$

Using Claim (1), we get

$$\left|\frac{\overline{J_p}(y)}{vol(S^n)}-\frac{\overline{J_p}(y)}{v_p(y)}\right|\leq C_1\left|\frac{v_p(y)-vol(S^n)}{v_p(y)vol(S^n)}\right|\leq M_1\mu^{2-2w}.$$

To find the bound of the second term in the right hand side, we consider the function $\widehat{J_p}=\frac{\overline{J_p}}{v_p}$. By Taylor's Theorem, there exists ζ between y and $\frac{y}{|y|}$ such that

$$\widehat{J_p}(y)=\widehat{J_p}\left(\frac{y}{|y|}\right)+\frac{\partial\widehat{J_p}}{\partial r}\left(\frac{y}{|y|}\right)\left(y-\frac{y}{|y|}\right)+\frac{\partial^2\widehat{J_p}}{\partial r^2}(\zeta)\left(y-\frac{y}{|y|}\right)^2.$$

Since $\frac{\partial\widehat{J_p}}{\partial r}\left(\frac{y}{|y|}\right)=0$ and $\widehat{J_p}|_{S^n}=K$, then

$$\left|\frac{\overline{J_p}(y)}{v_p(y)}-K\left(\frac{y}{|y|}\right)\right|=\left|\widehat{J_p}(y)-\widehat{J_p}\left(\frac{y}{|y|}\right)\right|\leq\left|\frac{\partial^2\widehat{J_p}}{\partial r^2}(\zeta)\right|\left|y-\frac{y}{|y|}\right|^2\leq C\mu^2.$$

Therefore,

$$\left|(vol(S^n))^{-1}\overline{J_p}(y)-K\left(\frac{y}{|y|}\right)\right|\leq C\mu^{2-2w}+C\mu^2\leq C\mu^{2-2w}.\qquad ☑$$

The inequality $|T(y,0)|\leq C\mu^{2-2w}$ follows from Claims 1 and 2 and Proposition 2.1. Consequently,

$$||T(y,0)||_{0,q}=\left(\int_{S^n}|T(y,0)|^q d\sigma_g\right)^{1/q}\leq C\mu^{2-2w}.\qquad ☑$$

Now, we will do the estimates of the first derivative of $T(y,0)$ in the y variable.

Lemma 3.3. *For any* $q\in(\frac{n}{2},n)$, $\|\nabla_y T(y,0)\|_{0,q}\leq C\mu^{1-w}$, *with* $0<w<1$.

Proof. A calculation shows that

$$\left|\frac{\partial T(y,0)}{\partial y_i}\right|\leq C\left[\left|\frac{\partial|F_y'|^{\frac{n-2}{2}\delta}}{\partial y_i}\right|+\left|\frac{\partial K\circ F_y}{\partial y_i}\right|+\left|\frac{\partial(\overline{J_p}(y))^{-1}}{\partial y_i}\right|\right].$$

The proof of the following claims conclude the proof of the lemma.

Claim 3.

$$\left\|\frac{\partial(\overline{J_p}(y))^{-1}}{\partial y_i}\right\|_{0,q}\leq C\mu.$$

Proof. Since $\frac{\partial\hat{J}_P}{\partial r}=0$ in ∂B^{n+1}, the mean value Theorem implies $\left|\frac{\partial\hat{J}_p}{\partial r}(y)\right|\leq C(1-|y_0|)$. Hence, $\left|\frac{\partial\hat{J}_p}{\partial y_i}\right|\leq C(1-|y_0|)$. From $\hat{J}_p(y)=\frac{\overline{J_p}(y)}{v_p(y)}$ and $\frac{\partial(\overline{J_p}(y))}{\partial y_i}=v_p(y)\frac{\partial(\hat{J}_p(y))}{\partial y_i}+\hat{J}_p(y)\frac{\partial(v_p(y))}{\partial y_i}$, we get

$$\left|\frac{\partial(\overline{J_p}(y))}{\partial y_i}\right|\leq C\left|\frac{\partial(\hat{J}_p(y))}{\partial y_i}\right|+C\left|\frac{\partial(v_p(y))}{\partial y_i}\right|\leq C(1-|y_0|).$$

Therefore

$$\left|\frac{\partial(\overline{J_p}(y))^{-1}}{\partial y_i}\right| \le C\left|\frac{\partial(\overline{J_p}(y))}{\partial y_i}\right| \le C(1-|y_0|) \le C\mu. \qquad \boxtimes$$

Claim 4.

$$\|\nabla_y |F_y'|^{\frac{n-2}{2}\delta}\|_{0,q} \le C\mu.$$

Proof. Since $|F_y'|(\xi) = \frac{1-|y|^2}{|y+\xi|^2}$, a straightforward calculation shows that

$\frac{\partial |F_y'|^{\frac{n-2}{2}\delta}}{\partial y_i} = -(n-2)\delta |F_y'|^{\frac{n-2}{2}\delta}\left(\frac{y_i}{1-|y|^2} + \frac{y_i+\xi_i}{|y+\xi|^2}\right)$, and therefore $\left|\frac{\partial |F_y'|^{\frac{n-2}{2}\delta}}{\partial y_i}\right| \le C\mu.$ ☑

Proposition 2.1 and Claims 3 and 4 yields to $|\nabla_y T(y,0)| \le C\mu^{1-w}$, and therefore,

$$||\nabla_y T(y,0)||_{0,q} = \left(\int_{S^n} |\nabla_y T(y,0)|^q d\sigma\right)^{1/q} \le C\mu^{1-w},$$

where w is a positive number less than one. ☑

Now, we will estimate the second derivatives of $T(y,0)$ with respect to the y variable.

Lemma 3.4. *For any* $q \in (\frac{n}{2}, n)$ *and* $1-\frac{n}{2q} < r < \frac{1}{2}$, *we have* $\|\nabla_y\nabla_y T(y,0)\|_{0,q} \le C\mu^{-2r}$.

Proof. Differentiating $T(y,0)$ twice with respect to the y variable we get

$\frac{\partial^2 T(y,0)}{\partial y_j \partial y_i} = vol(S^n)\frac{n(n-2)}{4}\frac{\partial}{\partial y_j}\mathbf{P}\left[A+B+D\right],$ where $A = (\overline{J_p}(y))^{-1} K\circ F_y \frac{\partial |F_y'|^{\frac{n-2}{2}\delta}}{\partial y_i},$

$$B = (\overline{J_p}(y))^{-1}|F_y'|^{\frac{n-2}{2}\delta}\frac{\partial K\circ F_y}{\partial y_i} \quad \text{and} \quad D = K\circ F_y |F_y'|^{\frac{n-2}{2}\delta}\frac{\partial(\overline{J_p}(y))^{-1}}{\partial y_i}.$$

Let us estimate the first derivatives of A, B and D. Since

$$\frac{\partial A}{\partial y_j} = K\circ F_y \frac{\partial |F_y'|^{\frac{n-2}{2}\delta}}{\partial y_i}\frac{\partial(\overline{J_p}(y))^{-1}}{\partial y_j} + (\overline{J_p}(y))^{-1}\frac{\partial |F_y'|^{\frac{n-2}{2}\delta}}{\partial y_i}\frac{\partial K\circ F_y}{\partial y_j} + (\overline{J_p}(y))^{-1}K\circ F_y\frac{\partial^2 |F_y'|^{\frac{n-2}{2}\delta}}{\partial y_j\partial y_i},$$

and

$$\frac{\partial}{\partial y_j}\left(\frac{\partial |F_y'|^{\frac{n-2}{2}\delta}}{\partial y_i}\right) = -\frac{(n-2)^2}{2}\delta^2 |F_y'|^{\frac{n-2}{2}\delta}\left(\frac{|y+\xi|^2}{1-|y|^2}\right)\frac{\partial}{\partial y_j}\left(\frac{1-|y|^2}{|y+\xi|^2}\right)\left(\frac{y_i}{1-|y|^2} + \frac{y_i+\xi_i}{|y+\xi|^2}\right),$$

Claims 3 and 4 and Proposition 2.1 yield to $\left\|\frac{\partial A}{\partial y_i}\right\|_{0,q} \le C.$

Now,

$$\begin{aligned}\frac{\partial B}{\partial y_j} &= \frac{\partial}{\partial y_j}\left((\overline{J_p}(y))^{-1}|F_y'|^{\frac{n-2}{2}\delta}\frac{\partial K\circ F_y}{\partial y_i}\right) = |F_y'|^{\frac{n-2}{2}\delta}\frac{\partial K\circ F_y}{\partial y_i}\frac{\partial(\overline{J_p}(y))^{-1}}{\partial y_j} \\ &\quad + (\overline{J_p}(y))^{-1}\frac{\partial K\circ F_y}{\partial y_i}\frac{\partial |F_y'|^{\frac{n-2}{2}\delta}}{\partial y_j} + (\overline{J_p}(y))^{-1}|F_y'|^{\frac{n-2}{2}\delta}\frac{\partial^2 K\circ F_y}{\partial y_j\partial y_i}.\end{aligned}$$

Hence, the inequality $\left\|\frac{\partial B}{\partial y_i}\right\|_{0,q} \leq C\mu^{-2r}$ follows from the inequalities in Proposition 2.1 and Lemma 3.3. Finally, since

$$\begin{aligned}\frac{\partial D}{\partial y_j} &= \frac{\partial}{\partial y_j}\left(K\circ F_y|F_y'|^{\frac{n-2}{2}\delta}\frac{\partial(\overline{J_p}(y))^{-1}}{\partial y_i}\right) = |F_y'|^{\frac{n-2}{2}\delta}\frac{\partial(\overline{J_p}(y))^{-1}}{\partial y_i}\frac{\partial K\circ F_y}{\partial y_j}\\ &\quad + K\circ F_y\frac{\partial(\overline{J_p}(y))^{-1}}{\partial y_i}\frac{\partial|F_y'|^{\frac{n-2}{2}\delta}}{\partial y_j} + K\circ F_y|F_y'|^{\frac{n-2}{2}\delta}\frac{\partial^2(\overline{J_p}(y))^{-1}}{\partial y_j\partial y_i},\end{aligned}$$

from Claims 3 and 4 and Proposition 2.1, we get $\left\|\frac{\partial D}{\partial y_i}\right\|_{0,q} \leq C$. The previous inequalities yield $||\nabla_y\nabla_y T(y,0)||_{0,q} \leq C\mu^{-2r}$, as desired. ☑

Using the previous lemmas, we reach the bound

$$\begin{aligned}||\mathcal{T}(0)||_{\mathcal{B}^{0,q}} &= \sup_y\{||T(y,0)||_{0,q} + (1-|y_0|)||\nabla_y T(y,0)||_{0,q} + (1-|y_0|)^2||\nabla_y\nabla_y T(y,0)||_{0,q}\}\\ &\leq C\mu^{2-2w} + C\mu^{2-2r} \leq C\epsilon(p)\mu^{\sigma},\end{aligned}$$

where $\sigma < 2$ and $\epsilon(p) = \mu^{\sigma'}$, with σ' a small positive number.

Now we will estimate $||\mathcal{T}'(0)|| = \sup_{||\phi||_{B^{2,q}}\leq 1} ||\mathcal{T}'(0)\phi||_{0,q}$, where $||\mathcal{T}'(0)\phi||_{0,q}$ is given by

$$\sup_y\{||T'(y,0)(\phi)||_{0,q} + (1-|y_0|)||\nabla_y T'(y,0)(\phi)||_{0,q} + (1-|y_0|)^2||\nabla_y\nabla_y T'(y,0)(\phi)||_{0,q}\}.$$

For this, consider $\phi \in B^{2,q}$ satisfying $||\phi||_{B^{2,q}} \leq 1$. Let $y \in B_{\alpha(1-|y_0|)}(y_0)$. Since

$$\begin{aligned}T(y,\eta) &= \mathcal{L}(\eta) + \mathbf{P}(Q(\eta))\\ &\quad -\mathbf{P}\Big(\frac{n(n-2)}{4}(1-vol(S^n)(\overline{J_p}(y))^{-1}K\circ F_y|F_y'|^{\frac{n-2}{2}\delta}(1+\eta)^{-\delta})(1+\eta)^{\frac{n+2}{n-2}}\Big),\end{aligned}$$

we have that

$$T_y'(0)(\phi) = \mathcal{L}(\phi) - \mathbf{P}\left(\frac{n(n+2)}{4}\phi(1-a_0) + \frac{n(n-2)}{4}\delta\phi a_0\right),$$

where $a_0 = vol(S^n)(\overline{J_p}(y))^{-1}K\circ F_y|F_y'|^{\frac{n-2}{2}\delta}$. Since $q > \frac{n}{2}$, from the Sobolev embedding Theorem we get $\|\phi\|_{L^\infty} \leq C\|\phi\|_{2,q} \leq C\|\phi\|_{B^{2,q}} \leq C$. Therefore $|\mathcal{L}(\phi)| \leq C$.

From this inequality and the estimates of Lemma 3.2, we obtain $|T'(y,0)(\phi)| \leq C$, and $\|T'(y,0)(\phi)\|_{0,q} \leq C$. Working similarly, and using the fact that $\phi, \nabla_y\phi, \nabla_y\nabla_y\phi$ belong to $\mathcal{H}^{2,q}(S^n)$ for $q > \frac{n}{2}$, we get $||\mathcal{T}'(0)|| \leq C$.

Now, we will show that the derivative of $\mathcal{T}'$ is Lipschitz; that is,

$$||\mathcal{T}'(\eta_1) - \mathcal{T}'(\eta_0)|| \leq C||\eta_1 - \eta_0||, \quad \|\eta_1\|, \|\eta_0\| \leq \frac{1}{4}.$$

For this, taking $\phi \in \mathcal{B}^{2,p}$ such that $\|\phi\|_{\mathcal{B}^{2,p}} \leq 1$, we get

$$\begin{aligned}\mathcal{T}'(\eta).\phi &= \mathcal{L}(\phi) + \frac{n(n+2)}{4}\mathbf{P}\left[(1+\eta)^{\frac{4}{n-2}}\phi - \phi\right]\\ &\quad -\mathbf{P}\left[\frac{n(n+2)}{4}(1-a_\eta)(1+\eta)^{\frac{4}{n-2}}\phi - \delta\frac{n(n-2)}{4}a_\eta(1+\eta)^{\frac{4}{n-2}}\phi\right],\end{aligned}$$

where $a_\eta = a_0(1+\eta)^{-\delta}$. Since

$$\begin{aligned}(\mathcal{T}_y'(\eta_1)-\mathcal{T}_y'(\eta_0))\phi = &\mathbf{P}\left(\left[(1+\eta_1)^{\frac{4}{n-2}}-(1+\eta_0)^{\frac{4}{n-2}}\right]\phi\right)\\ &-\mathbf{P}\left[\Big(\frac{n(n+2)}{4}+\delta\frac{n(n-2)}{4}\Big)(a_{\eta_0}-a_{\eta_1})(1+\eta_1)^{\frac{4}{n-2}}\phi\right]\\ &-\mathbf{P}\left[\Big(\tfrac{n(n+2)}{4}(a_{\eta_0}-1)+\delta\tfrac{n(n-2)}{4}a_{\eta_0}\Big)[(1+\eta_1)^{\frac{4}{n-2}}-(1+\eta_0)^{\frac{4}{n-2}}]\phi\right],\end{aligned}$$

using that $\|\eta_1\|, \|\eta_0\| \le \frac{1}{4}$ and the mean value Theorem, we get

$$\begin{aligned}|(\mathcal{T}_y'(\eta_1)-\mathcal{T}_y'(\eta_0))\phi| \le\, &C|\eta_1-\eta_0||\phi| + C|a_0|\delta|\eta_1-\eta_0||\phi|\\ &+C(|a_{\eta_0}-1|+|a_{\eta_0}|)|\eta_1-\eta_0||\phi|,\end{aligned}$$

and therefore

$$\|(\mathcal{T}_y'(\eta_1)-\mathcal{T}_y'(\eta_0))\phi\|_{0,q} \le C\|\eta_1-\eta_0\|_{0,q}.$$

To finish the proof of Theorem 1, we need to show that $\mathcal{T}'(0)$ has a bounded inverse. Let $\phi \in \mathcal{B}^{2,q}(S^n)$ and $\Psi \in \mathcal{B}^{0,q}(S^n)$. Consider the problem $\mathcal{T}'(0)\phi = \Psi$. Let us recall that

$$\|\phi\|_{\mathcal{B}^{2,q}(S^n)} = \sup_y\{\|\phi\|_{2,q} + (1-|y_0|)\|\nabla_y\phi\|_{2,q} + (1-|y_0|)^2\|\nabla_y\nabla_y\phi\|_{2,q}\}.$$

Elliptic estimates shows that $\|\phi\|_{2,q} \le C\|\mathcal{L}(\phi)\|_{0,q}$. Since

$$\Psi = T_y'(0)(\phi) = \mathcal{L}(\phi) - \mathbf{P}\Big(\frac{n(n+2)}{4}\phi(1-a_0) + \frac{n(n-2)}{4}\delta\phi a_0\Big),$$

from the estimates of Lemma 3.2 we get

$$\begin{aligned}\Big\|\mathbf{P}\Big(\frac{n(n+2)}{4}\phi(1-a_0)+\frac{n(n-2)}{4}\delta\phi a_0\Big)\Big\|_{0,q} &\le C\epsilon(p)\mu^\sigma\|\phi\|_{0,q}\\ &\le C\epsilon(p)\mu^\sigma\|\phi\|_{2,q};\end{aligned}$$

then,

$$\|\phi\|_{2,q} \le C\|\mathcal{L}(\phi)\|_{0,q} \le k(\|\Psi\|_{0,q} + C\epsilon(p)\mu^\sigma\|\phi\|_{2,q}).$$

Taking $\mu^\sigma\epsilon(p)$ small we get that $1 - kC\epsilon(p)\mu^\sigma > 0$ and $\|\phi\|_{2,q} \le C\|\Psi\|_{0,q}$. Working analogously, we have that

$$\|\nabla_y\phi\|_{2,q} \le L\|\nabla_y\Psi\|_{0,q} + L_1\mu^{1-w}\|\Psi\|_{0,q}$$

and

$$\|\nabla_y\nabla_y\phi\|_{2,q} \le C_1\|\nabla_y\nabla_y\Psi\|_{0,q} + C_2\mu^{1-w}\|\nabla_y\Psi\|_{0,q} + (C_3\mu^{-2r} + C_4\mu^{2-2w})\|\Psi\|_{0,q}.$$

Therefore,

$$\|\phi\|_{\mathcal{B}^{2,q}(S^n)} \le C\sup_y\{\|\Psi\|_{0,q}+(1-|y_0|)\|\nabla_y\Psi\|_{0,q}+(1-|y_0|)^2\|\nabla_y\nabla_y\Psi\|_{0,q}\} \le C\|\Psi\|_{\mathcal{B}^{0,q}(S^n)}.$$

The rest of the proof follows from the inverse function Theorem. ☑

4. Integral and L^q estimates of the function η_y

In this section, given the solution η_y, $y \in B_{\beta(1-|y_0|)}$, of the projected equation, we will find L^q estimates not only of the function η_y, but also of its first and second y derivatives; in addition, we will do also integral estimates of $\nabla_y \eta_y$ and $\nabla_y \nabla_y \eta_y$.

Lemma 4.1. *For $q \in \left(\frac{n}{2}, n\right)$, $\|\eta_y\|_{0,q} \leq C\epsilon(p)\mu^\sigma$, with $\sigma < 2$, where $\epsilon(p) \to 0$ as $p \to \frac{n+2}{n-2}$.*

Proof. From Theorem 3.1, $T(y, \eta_y) = 0$. Then,

$$\mathcal{L}(\eta_y) = -\frac{n(n-2)}{4}\mathbf{P}\left((1+\eta_y)^{\frac{n+2}{n-2}} - 1 - \frac{n+2}{n-2}\eta_y\right) + \frac{n(n-2)}{4}\mathbf{P}\left((1-a)(1+\eta_y)^{\frac{n+2}{n-2}}\right).$$

Setting $a = a_0 D$, where $D = (1+\eta_y)^{-\delta}$, we have

$$|1-a| = |a-1| = |a_0 D - 1| = |a_0(D-1) + (a_0 - 1)| \leq |a_0||D-1| + |a_0 - 1|.$$

From the mean value Theorem it follows that

$$|\mathcal{L}(\eta_y)| \leq C|\eta_y|^2 + C\delta|a_0||\eta_y| + C|a_0 - 1|.$$

Using Hölder's inequality,the estimates of Lemma 1, Theorem 1, $q > \frac{n}{2}$ and the Sobolev embedding Theorem, we have

$$\begin{aligned} \|\mathcal{L}(\eta_y)\|_{0,q,S^n} &\leq C\|\eta_y\|_\infty \|\eta_y\|_{0,q,S^n} + C\mu^2\|\eta_y\|_{0,q,S^n} + C\epsilon(p)\mu^\sigma \\ &\leq C\epsilon(p)\mu^\sigma\|\eta_y\|_{2,q,S^n} + C\mu^2\|\eta_y\|_{2,q,S^n} + C\epsilon(p)\mu^\sigma. \end{aligned}$$

Since $\|\eta_y\|_{2,q,S^n} \leq C\|\mathcal{L}(\eta_y)\|_{0,q,S^n}$, then $\|\eta_y\|_{0,q,S^n} \leq \|\eta_y\|_{2,q,S^n} \leq C\epsilon(p)\mu^\sigma$, as desired. ☑

Lemma 4.2. *For $q \in \left(\frac{n}{2}, n\right)$, $\|\nabla_y \eta_y\|_{0,q} \leq C\mu^{1-w}$, with $0 < w < 1$.*

Proof. Differentiating the equation

$$0 = T(y, \eta_y) = \mathcal{L}(\eta_y) + \mathbf{P}(Q(\eta_y)) - \mathbf{P}\left(\frac{n(n-2)}{4}(1-a)(1+\eta_y)^{\frac{n+2}{n-2}}\right),$$

we find that the terms of its derivative satisfy the inequalities

$$|\nabla_y a| \leq C(|\nabla_y a_0| + \mu^2|\eta_y'|), \quad \text{where} \quad a = a_0(1+\eta_y)^{-\delta},$$

$$\begin{aligned} \left|\frac{\partial}{\partial y_i}\mathbf{P}\left(\frac{n(n-2)}{4}(1-a)(1+\eta_y)^{\frac{n+2}{n-2}}\right)\right| &\leq C|1-a||\eta_y'| + C|\nabla_y a| \\ &\leq (C\delta|a_0||\eta_y| + C_2|a_0 - 1|)|\eta_y'| + C_3|\nabla_y a| \\ &\leq C(\mu^2|a_0||\eta_y| + |a_0 - 1| + \mu^2)|\eta_y'| + C|\nabla_y a_0|, \end{aligned}$$

and

$$\left|\frac{n(n-2)}{4}\frac{\partial}{\partial y_i}\mathbf{P}\left((1+\eta_y)^{\frac{n+2}{n-2}} - 1 - \frac{n+2}{n-2}\eta_y\right)\right| \leq C|\eta_y'||\eta_y|,$$

where we have used the estimates of Theorem 3.1 and $\delta = C\mu^2$.

Hence,

$|\mathcal{L}(\eta_y')| \le C\mu^2|a_0||\eta_y||\eta_y'| + C_2|a_0 - 1||\eta_y'| + C_3|\nabla_y a_0| + C\mu^2|\eta_y'| + C|\eta_y'||\eta_y|$.

Using Hölder's inequality, the estimates in Theorem 3.1 and Lemma 4.1, we arrive to

$$\begin{aligned}\|\eta_y'\|_{2,q,S^n} \le& \|\mathcal{L}(\eta_y')\|_{0,q,S^n} \le C_2\epsilon(p)\mu^{\sigma+2}\|\eta_y'\|_{0,q,S^n} + C_3\|\nabla_y a_0\|_{0,q,S^n}\\ &+ C\mu^2\|\eta_y'\|_{0,q,S^n} + C\|\eta_y'\|_{0,q,S^n}\|\eta_y\|_\infty + C_2\|a_0-1\|_{0,q}\|\eta_y'\|_\infty\\ \le& C_2\epsilon(p)\mu^{\sigma+2}\|\eta_y'\|_{0,q,S^n} + C_3\mu^{1-w} + C\mu^2\|\eta_y'\|_{0,q,S^n} + C\epsilon(p)\mu^\sigma\|\eta_y'\|_{0,q,S^n}\\ &+ C\epsilon(p)\mu^\sigma\|\eta_y'\|_{2,q,S^n},\end{aligned}$$

and therefore $\|\eta_y'\|_{2,q,S^n} \le C\mu^{1-w}$ for $0 < w < 1$. ☑

Differentiating twice the equation $T(y,\eta) = 0$ and working as in Lemma 4.2, we get

Lemma 4.3. *For* $q \in \left(\frac{n}{2}, n\right)$, $\|\nabla_y\nabla_y\eta_y\|_{0,q} \le C\mu^{-2r}$, *with* $1 - \frac{n}{2q} < r < \frac{1}{2}$.

In what follows, we will estimate the integral of the function η_y', $y \in B_{\beta(1-y_0)}(y_0)$.

Lemma 4.4. *For* $q \in \left(\frac{n}{2}, n\right)$ *and* $y \in B_{\beta(1-y_0)}(y_0)$, $\left|\int_{S^n}\nabla_y\eta_y d\sigma\right| \le C\epsilon(p)\mu^\sigma$, *with* $\sigma < 2$.

Proof. From $\mathcal{L}(\eta_y) + \mathbf{P}(Q(\eta_y)) - \mathbf{P}\left(\frac{n(n-2)}{4}(1-a)(1+\eta_y)^{\frac{n+2}{n-2}}\right) = 0$, and $\int_{S^n}\mathbf{P}(f)d\sigma = \int_{S^n} f d\sigma$, $f \in L^2(S^n)$, we have

$$0 = \int_{S^n} T(y,\eta_y)d\sigma = \int_{S^n}\mathcal{L}(\eta_y)d\sigma + \int_{S^n} Q(\eta_y)d\sigma - \int_{S^n}\left(\frac{n(n-2)}{4}(1-a)(1+\eta_y)^{\frac{n+2}{n-2}}\right)d\sigma.$$

Using that $\mathcal{L}(\eta_y) = \Delta\eta_y + n\eta_y$, we obtain

$$\int_{S^n} n\eta_y d\sigma = -\int_{S^n} Q(\eta_y)d\sigma + \int_{S^n}\left(\frac{n(n-2)}{4}(1-a)(1+\eta_y)^{\frac{n+2}{n-2}}\right)d\sigma.$$

Setting $A = Vol(S^n)\overline{J_p}^{-1}(y)K\circ F_y|F_y'|^{\frac{n-2}{2}\delta}$, $D = (1+\eta_y)^{-\delta}$ and $E = (1+\eta_y)^{\frac{n+2}{n-2}}$, we get

$$\int_{S^n}\left(\frac{n(n-2)}{4}(1-a)(1+\eta_y)^{\frac{n+2}{n-2}}\right)d\sigma = \frac{n(n-2)}{4}\int_{S^n}(1-AD)Ed\sigma.$$

Hence,

$$\int_{S^n} n\eta_y d\sigma = -\int_{S^n} Q(\eta_y)d\sigma + \frac{n(n-2)}{4}\int_{S^n}(1-AD)Ed\sigma,$$

and therefore,

$$\int_{S^n}\eta_y d\sigma = -\frac{1}{n}\int_{S^n} Q(\eta_y)d\sigma - \frac{n-2}{4}\int_{S^n}(AD-1)Ed\sigma.$$

Writing $(AD-1)E = (AD-1)(E-1)+A(D-1)+A-1$, and observing that $\int_{S^n} Ad\sigma = cte$, we have

$$\frac{\partial}{\partial y_i}\int_{S^n}[(AD-1)E]d\sigma = \int_{S^n}[(A'D+AD')(E-1)+(AD-1)E']d\sigma + \int_{S^n}[A'(D-1)+AD']d\sigma.$$

On the other hand,

$$\frac{\partial Q(\eta_y)}{\partial y_i} = \frac{n(n+2)}{4}\eta_y'[(1+\eta_y)^{\frac{4}{n-2}}-1].$$

Then,

$$\int_{S^n}\frac{\partial \eta_y}{\partial y_i}d\sigma = \mathcal{A}+\mathcal{B}+\mathcal{C}, \tag{17}$$

where $\mathcal{A} = -\frac{1}{n}\int_{S^n}\left[\frac{n(n+2)}{4}\eta_y'[(1+\eta_y)^{\frac{4}{n-2}}-1]\right]d\sigma$, $\mathcal{C} = -\frac{n-2}{4}\int_{S^n}[A'(D-1)+AD']d\sigma$ and $\mathcal{B} = -\frac{n-2}{4}\int_{S^n}[(A'D+AD')(E-1)+(AD-1)E']d\sigma$.

Using the estimates on η_y,η_y', the mean value Theorem and Hölder's inequality, we arrive to

$$\left|\int_{S^n}\left((1+\eta_y)^{\frac{4}{n-2}}-1\right)\eta_y'd\sigma\right| \le C\int_{S^n}|\eta_y||\eta_y'|d\sigma \le C\|\eta_y\|_{0,s}\|\eta_y'\|_{0,s'} \le C\epsilon(p)\mu^{\sigma+1-w},$$

for s, s' such that $\frac{1}{s}+\frac{1}{s'} = 1$. Working similarly, we get

$$\begin{aligned}\left|\int_{S^n}(A'D+AD')(E-1)\right| &\le C\int_{S^n}|A'||\eta_y|d\sigma + C\delta\int_{S^n}|\eta_y|d\sigma\\ &\le \|\eta_y\|_{0,s'}\|A'\|_{0,s} + C\epsilon(p)\mu^{\sigma}\\ &\le C\epsilon(p)\mu^{\sigma+1-w} + C\epsilon(p)\mu^{\sigma}\\ &\le C\epsilon(p)\mu^{\sigma},\end{aligned}$$

where we have used the mean value Theorem, Proposition 2.1, Lemma 4.1 and the estimates of Theorem 3.1. Using Lemma 4.2 and proceeding as before, we get

$$\left|\int_{S^n}(AD-1)E'd\sigma\right| \le C\epsilon(p)\mu^{\sigma+1-w},$$

$$\left|\int_{S^n}A'(D-1)d\sigma\right| \le C\epsilon(p)\mu^{\sigma+3-w}$$

and

$$\left|\int_{S^n}AD'd\sigma\right| \le C\mu^{3-w}.$$

Consequently,

$$\left|\int_{S^n}\nabla_y\eta_y d\sigma\right| \le C\epsilon(p)\mu^{\sigma} + C\mu^{3-w} \le C\epsilon(p)\mu^{\sigma},$$

with $\sigma < 2$. ☑

Finally, we will estimate the integral of η''_y.

Lemma 4.5. *For* $q \in \left(\frac{n}{2}, n\right)$, $\left|\int_{S^n} \nabla_y \nabla_y \eta_y d\sigma\right| \leq C\epsilon\mu^{\sigma-2r}$, *with* $r < \frac{1}{2}$.

Proof. Denoting $\frac{\partial^2 \eta_y}{\partial y_j \partial y_i}$ by η''_y, and differentiating the terms on the right hand side of equation (17) with respect to y_j, we get

$$\int_{S^n} \eta''_y d\sigma = -\frac{n+2}{4}\int_{S^n} \eta''_y[(1+\eta_y)^{\frac{4}{n-2}} - 1]d\sigma - \frac{n+2}{n-2}\int_{S^n}(1+\eta_y)^{\frac{6-n}{n-2}}\eta'_{y_i}\eta'_{y_j}d\sigma$$
$$-\frac{n-2}{4}\int_{S^n}[(A''D + 2A'D' + AD'')(E-1) + (A'D + AD')E']d\sigma$$
$$-\frac{n-2}{4}\int_{S^n}[(A'D + AD')E' - (AD-1)E'' - A''(D-1) + 2A'D' + AD'']d\sigma.$$

In what follows we will estimate the terms in the right hand side of this equality. Using Hölder's inequality, Proposition 2.1 and the four previous lemmas, we have:

$$\left|\frac{n+2}{4}\int_{S^n}\eta_y''[(1+\eta_y)^{\frac{4}{n-2}} - 1]d\sigma\right| \leq C\int_{S^n}|\eta''_y||\eta_y|d\sigma \leq C\epsilon(p)\mu^{\sigma-2r};$$

$$\left|\frac{n+2}{n-2}\int_{S^n}(1+\eta_y)^{\frac{6-n}{n-2}}\eta'_{y_i}\eta'_{y_j}d\sigma\right| \leq C\int_{S^n}|\eta'_y|^2 d\sigma \leq C\mu^{2-2w};$$

$$\begin{aligned}\int_{S^n}|(A''D + 2A'D' + AD'')(E-1)|d\sigma &\leq C\int_{S^n}|(A''D + 2A'D' + AD'')||\eta_y|d\sigma\\ &\leq C\int_{S^n}|A''||\eta_y|d\sigma + C\delta\int_{S^n}|A'||\eta'_y||\eta_y|d\sigma\\ &+ C\int_{S^n}|A|(\delta(\delta+1)|\eta_y|^2 + \delta|\eta''_y|)|\eta_y|d\sigma\\ &\leq C\epsilon(p)\mu^{\sigma-2r};\end{aligned}$$

$$\left|\int_{S^n}(A'D + AD')E'd\sigma\right| \leq C\int_{S^n}|A'||\eta'_y|d\sigma + C\delta\int_{S^n}|A||\eta'_y|^2 d\sigma \leq C\mu^{2-2w};$$

$$\left|\int_{S^n}(AD-1)E''d\sigma\right| \leq C\int_{S^n}|AD-1|(|\eta'_y|^2 + |\eta''_y|)d\sigma \leq C\epsilon(p)\mu^{\sigma-2r};$$

$$\left|\int_{S^n}A''(D-1)d\sigma\right| \leq C\int_{S^n}|A''||\eta_y|d\sigma \leq C\epsilon(p)\mu^{\sigma-2r};$$

$$\left|\int_{S^n}2A'D'd\sigma\right| \leq C\delta\int_{S^n}|A'||\eta'_y|d\sigma \leq C\mu^{4-2w},$$

and

$$\left|\int_{S^n}AD''d\sigma\right| \leq C\delta(\delta+1)\int_{S^n}|\eta'_y|^2 d\sigma + C\delta\int_{S^n}|\eta''_y|d\sigma \leq C\mu^{2-2r}.$$

Putting together these inequalities, we obtain the desired bound for $\left|\int_{S^n}\nabla_y\nabla_y\eta_y d\sigma\right|$.

☑

5. Solutions of some nonlinear elliptic equations

In this section, using the estimates of Sections 3 and 4, we will prove that the functions $\widetilde{J}_p(y)$ and $\overline{J}_p(y)$ are close in the $\mathcal{C}^2$-norm. The fact this functions are close implies that $\widetilde{J}_p(y)$ has a unique critical point y_1 close to the critical point y_0 of $\overline{J}_p(y)$. This implies that $\widetilde{u}_{y_1}$ is a solution of equation (6).

Multiplying the function $\widetilde{u}_{y_1}$ by the constant $(J_p(\widetilde{u}_{y_1}))^{1-p}$ we will find that $u = (J_p(\widetilde{u}_{y_1}))^{1-p}\widetilde{u}_{y_1}$ is a solution of the subcritical problem (2). Recalling that η_y is a solution of the equation $T(y,\eta) = 0$, if we let $u_y = \alpha_{F_y^{-1}}(1+\eta_y)$ we will prove that $u_{y_1} = \alpha_{F_{y_1}^{-1}}(1+\eta_{y_1})$ is a solution of the perturbed equation (3).

Consider the quotient

$$(\Lambda_y)^{1-p} = \frac{\int_{S^n} K\alpha_y^{p+1}}{\int_{S^n} Ku_y^{p+1}},$$

and define the functions $\gamma_y = \Lambda_y(1+\eta_y)$ and $\widetilde{u}_y = \alpha_{F_y}(\gamma_y)$.

Recalling that $\mathcal{S}$ is the set of non-negative functions $u \in W^{2,q}(S^n)$, $(q > \frac{n}{2})$ such that $E(u) = E(1)$, we get the following proposition:

Proposition 5.1. *The function $\widetilde{u}_y$ belongs to the set $\mathcal{S}$.*

Proof. By Theorem 3.1, the function η_y satisfies the equation

$$\mathcal{L}(\eta) + \mathbf{P}(\mathcal{Q}(\eta)) - \frac{n(n-2)}{4}\mathbf{P}\left((1-a)(1+\eta)^{\frac{n+2}{n-2}}\right) = 0,$$

where

$$\mathcal{L}(\eta) = \Delta\eta + n\eta, \quad \mathcal{Q}(\eta) = \frac{n(n-2)}{4}\left((1+\eta)^{\frac{n+2}{n-2}} - 1 - \frac{n+2}{n-2}\eta\right)$$

and

$$a = vol(S^n)(\overline{J}_p(y))^{-1}K \circ F_y|F_y'|^{\frac{n-2}{2}\delta}(1+\eta)^{-\delta}.$$

Summing the constant $n - \frac{n(n+2)}{4}$ in both side of the equation $T(y,\eta) = 0$ and simplifying, we get

$$\mathcal{L}(1+\eta) - \mathbf{P}\left[\frac{n(n+2)}{4}(1+\eta)\right] + \mathbf{P}\left[\frac{n(n-2)}{4}\tilde{a}(1+\eta)^p\right] = 0,$$

where $\tilde{a} = a(1+\eta)^\delta$. Therefore,

$$\mathcal{L}(\gamma_y) - \mathbf{P}\left[\frac{n(n+2)}{4}\gamma_y\right] + \frac{1}{(\Lambda_y)^{p-1}}\mathbf{P}\left[\frac{n(n-2)}{4}\tilde{a}(\gamma_y)^p\right] = 0.$$

Since

$$(\Lambda_y)^{1-p} = \frac{\int_{S^n} K\alpha_y^{p+1}}{\int_{S^n} Ku_y^{p+1}},$$

we have

$$\mathcal{L}(\gamma_y) - \mathbf{P}\left[\frac{n(n+2)}{4}\gamma_y\right] + \frac{n(n-2)}{4}vol(S^n)\frac{1}{\int_{S^n} Ku_y^{p+1}dz}\mathbf{P}\left(K\circ F_y|F_y'|^{\frac{n-2}{2}\delta}\gamma_y^p\right) = 0.$$

Multiplying this equation by γ and integrating, we have

$$\int_{S^n}\left(\mathcal{L}(\gamma_y)\gamma_y - \frac{n(n+2)}{4}\gamma_y^2\right)d\zeta + \frac{n(n-2)}{4}vol(S^n) = 0,$$

where we have used that $\int_{S^n}\mathbf{P}(f) = \int_{S^n} f$ for every integrable function f, and $\int_{S^n} K\circ F_y|F_y'|^{\frac{n-2}{2}\delta}\gamma_y^{p+1}d\zeta = \int_{S^n} Ku_y^{p+1}dz$.
Consequently,

$$E(\gamma_y) = \int_{S^n}|\nabla\gamma_y|^2 d\zeta + \frac{n(n-2)}{4}\int_{S^n}\gamma_y^2 d\zeta = \frac{n(n-2)}{4}vol(S^n).$$

Since $\widetilde{u}_y = \alpha_{F_y}(\gamma_y)$, the conformal invariance of the energy E implies that the function $\widetilde{u}_y \in \mathcal{S}$, as desired. ☑

Let us define the function

$$\widetilde{J}_p(y) = \int_{S^n} K\widetilde{u}_y^{p+1}d\sigma.$$

Now, we will prove that the difference of the functions $\widetilde{J}_p(y)$ and $\overline{J_p}(y) = \int_{S^n} K\alpha_y^{p+1}$ are very close in C^2 norm.

Proposition 5.2. *Let y_0 be a critical point of the function $\overline{J_p}(y)$, and let $y \in B_{\beta(1-|y_0|)}(y_0)$. Then,*

$$|\widetilde{J}_p(y) - \overline{J}_p(y)| \le C\epsilon(p)\mu^\sigma,$$

$$\left|\nabla_y(\widetilde{J}_p(y) - \overline{J}_p(y))\right| \le C\mu^{1-w}$$

and

$$\left|\nabla_y\nabla_y(\widetilde{J}_p(y) - \overline{J}_p(y))\right| \le C\epsilon(p)\mu^{1-2r},$$

where $\sigma < 2$, $0 < w < 1$, $r < \frac{1}{2}$ and $\epsilon(p)$ goes to zero as p goes to $\frac{n+2}{n-2}$.

Proof. A change of variables yields

$$\begin{aligned}\widetilde{J}_p(y) - \overline{J}_p(y) &= \int_{S^n}\left(K\circ F_y\Lambda_y^{p+1}|(F_y)'|^{\frac{n-2}{2}\delta}[(1+\eta_y)]^{p+1}d\zeta - K\circ F_y|F_y'|^{\frac{n-2}{2}\delta}\right)\\ &= \int_{S^n} K\circ F_y|F_y'|^{\frac{n-2}{2}\delta}[[(1+\eta_y)]^{p+1}-1]d\zeta\\ &+ (\Lambda_y^{p+1}-1)\int_{S^n} K\circ F_y|F_y'|^{\frac{n-2}{2}\delta}[(1+\eta_y)]^{p+1}d\zeta.\end{aligned}$$

To estimate this difference, we will do it for the terms in the right hand side separately. The mean value Theorem and Theorem 3.1 implies

$$\begin{aligned}\left|\int_{S^n} K\circ F_y|F_y'|^{\frac{n-2}{2}\delta}[[(1+\eta_y)]^{p+1}-1]d\zeta\right| &\le C\int_{S^n}|\eta_y|d\zeta \le C\|\eta_y\|_\infty\\ &\le C\epsilon(p)\mu^\sigma,\end{aligned}$$

and

$$\left|\int_{S^n} K\circ F_y|F_y'|^{\frac{n-2}{2}\delta}[(1+\eta_y)]^{p+1}d\zeta\right| \le C.$$

To estimate $(\Lambda_y^{p+1}-1)$, we make a change of variables to get

$$\Lambda_y^2 = \frac{\int_{S^n} K\circ F_y|F_y'|^{\frac{n-2}{2}\delta}}{\int_{S^n} K\circ F_y|(F_y)'|^{\frac{n-2}{2}\delta}[(1+\eta_y)]^{p+1}d\zeta}.$$

Since $|\Lambda_y|\le 1$ and $\Lambda_y^2-1=(\Lambda_y-1)(\Lambda_y+1)$, then

$$|\Lambda_y-1|\le C|\Lambda_y^2-1|\le C\left|\frac{I}{M}-1\right|\le C|M-I|,$$

where
$M=\int_{S^n} K\circ F_y|F_y'|^{\frac{n-2}{2}\delta}[(1+\eta_y)]^{p+1}d\zeta$, and $I=\int_{S^n} K\circ F_y|F_y'|^{\frac{n-2}{2}\delta}d\zeta$.
Then,

$$|\Lambda_y^{p+1}-1|\le C|M-I|\le C\epsilon(p)\mu^\sigma.$$

From the previous estimates we get

$$|\widetilde{J}_p(y)-\overline{J}_p(y)|\le C\epsilon(p)\mu^\sigma.$$

Now, we need to estimate the difference of the first derivatives:

$$\begin{aligned}\nabla_y\left(\widetilde{J}_p(y)-\overline{J}_p(y)\right) =&\nabla_y\left(\int_{S^n} K\circ F_y|F_y'|^{\frac{n-2}{2}\delta}[[(1+\eta_y)]^{p+1}-1]d\zeta\right)\\ &+\nabla_y(\Lambda_y^{p+1}-1)\int_{S^n} K\circ F_y|F_y'|^{\frac{n-2}{2}\delta}[(1+\eta_y)]^{p+1}d\zeta\\ &+(\Lambda_y^{p+1}-1)\nabla_y\left(\int_{S^n} K\circ F_y|F_y'|^{\frac{n-2}{2}\delta}[(1+\eta_y)]^{p+1}d\zeta\right).\end{aligned}$$

Let us write the first term in the right hand side as

$$\begin{aligned}&\left(\nabla_y\int_{S^n} K\circ F_y|F_y'|^{\frac{n-2}{2}\delta}[[(1+\eta_y)]^{p+1}-1]d\zeta\right)=\\ &\qquad=\int_{S^n}\nabla_y(K\circ F_y)|F_y'|^{\frac{n-2}{2}\delta}[(1+\eta_y)^{p+1}-1]d\zeta\\ &\qquad+\int_{S^n} K\circ F_y\nabla_y(|F_y'|^{\frac{n-2}{2}\delta})[(1+\eta_y)^{p+1}-1]d\zeta\\ &\qquad+\int_{S^n}\left[K\circ F_y|F_y'|^{\frac{n-2}{2}\delta}[(p+1)(1+\eta_y)^p\eta_y']\right]d\zeta,\end{aligned}$$

where,

$$\int_{S^n} K\circ F_y|F_y'|^{\frac{n-2}{2}\delta}(p+1)(1+\eta_y)^p\eta_y' d\zeta = \int_{S^n}(K\circ F_y-1)|F_y'|^{\frac{n-2}{2}\delta}[(p+1)(1+\eta_y)^p\eta_y']d\zeta + \int_{S^n}(|F_y'|^{\frac{n-2}{2}\delta}-1)(p+1)(1+\eta_y)^p\eta_y' d\zeta + \int_{S^n}[(p+1)[(1+\eta_y)^p-1]\eta_y' + (p+1)\eta_y']d\zeta,$$

Since K is a Morse function, from the proof of Proposition 1.1 in [8] we have that $\|1-K\circ F_y\|_{0,q}\le C\epsilon_0\mu$, where ϵ_0 can be chosen as small as we want. From this fact, the mean value Theorem, Hölder's inequality, Proposition 2.1, Theorem 3.1 and the integral and L^p estimates of the functions η_y and $\eta_y^{'}$, we arrive to

$$\left|\nabla_y\left(\int_{S^n} K\circ F_y|F_y'|^{\frac{n-2}{2}\delta}[[(1+\eta_y)]^{p+1}-1]d\zeta\right)\right| \le C\epsilon(p)\mu^{\sigma+1-w}.$$

Analogously,

$$\left|\nabla_y\left(\int_{S^n} K\circ F_y|F_y'|^{\frac{n-2}{2}\delta}(1+\eta_y)^{p+1}d\sigma\right)\right| \le C\mu^{1-w}.$$

A calculation shows that

$$|\nabla_y(\Lambda_y^{p+1}-1)|\le C|\nabla_y\Lambda_y|\le C_1|\nabla_y(M-I)|+C_2|M-I||\nabla_y M|,$$

and therefore

$$|\nabla_y(\Lambda_y^{p+1}-1)|\le C\epsilon(p)\mu^{\sigma+1-w}+C\epsilon(p)\mu^{\sigma}+C\mu^{1-w}\le C\mu^{1-w}.$$

Consequently,

$$\left|\nabla_y(\widetilde{J}_p(y)-\overline{J}_p(y))\right|\le C\epsilon(p)\mu^{\sigma+1-w}+C\mu^{1-w}\le C\mu^{1-w}.$$

Writing the difference of the second derivatives as

$$\begin{aligned}\nabla_y\nabla_y\left(\widetilde{J}_p(y)-\overline{J}_p(y)\right) =&\nabla_y\nabla_y\left(\int_{S^n} K\circ F_y|F_y'|^{\frac{n-2}{2}\delta}[[(1+\eta_y)]^{p+1}-1]d\zeta\right)\\ &+\nabla_y\nabla_y(\Lambda_y^{p+1}-1)\int_{S^n} K\circ F_y|F_y'|^{\frac{n-2}{2}\delta}[(1+\eta_y)]^{p+1}d\zeta\\ &+2\nabla_y(\Lambda_y^{p+1}-1)\nabla_y\left(\int_{S^n} K\circ F_y|F_y'|^{\frac{n-2}{2}\delta}[(1+\eta_y)]^{p+1}d\zeta\right)\\ &+(\Lambda_y^{p+1}-1)\nabla_y\nabla_y\left(\int_{S^n} K\circ F_y|F_y'|^{\frac{n-2}{2}\delta}[(1+\eta_y)]^{p+1}d\zeta\right),\end{aligned}$$

and working as before, we obtain the desired estimate. ☑

Proposition 5.3. *The function* $\widetilde{J}_p$ *has a unique critical point* y_1 *on* $B_{\beta(1-|y_0|)}(y_0)$.

Proof. The inequalities in Proposition 5.2 imply that there exists $\epsilon > 0$, sufficiently small, such that

$$(1-|y_0|)^{-1}\left|\nabla_y(\widetilde{J}_p(y)-\overline{J}_p(y))\right| + \left|\nabla_y\nabla_y(\widetilde{J}_p(y)-\overline{J}_p(y))\right| \leq \epsilon. \tag{18}$$

For $z \in B^{n+1}$ we define

$$f(z) = (1-|y_0|)^{-2}(\overline{J}_p(y_0+\beta(1-|y_0|)z)-\overline{J}_p(y_0)),$$

$$g(z) = (1-|y_0|)^{-2}(\widetilde{J}_p(y_0+\beta(1-|y_0|)z)-\widetilde{J}_p(y_0)).$$

On one hand, by Proposition 2.2 we have

$$|\nabla f| + |\nabla\nabla f| \leq \left(\frac{|\nabla\overline{J}_p(y_0+\beta(1-|y_0|)z)|}{(1-|y_0|)} - |\nabla\nabla\overline{J}_p(y_0+\beta(1-|y_0|)z)|\right) \leq c,$$

$$\inf_{\partial B^{n+1}} |\nabla f| \geq \frac{\beta}{(1-|y_0|)}\left(\inf_{y\in\partial B_{\beta(1-|y_0|)}(y_0)} |\nabla\overline{J}_p(y)|\right) \geq c^{-1},$$

and

$$|\det \mathrm{Hess} f| = \beta^{2(n+1)}|\det \mathrm{Hess}\overline{J}_p| \geq c^{-1}.$$

On the other hand, inequality (18) implies

$$\|\nabla(f-g)\| + \|\nabla\nabla(f-g)\| \leq \epsilon.$$

Proposition 2.3 implies Proposition 5.3. ☑

If we change, in the proof of Theorem 2.4 of [8], u_{y_1} for $\widetilde{u}_{y_1} = \Lambda_{y_1}u_{y_1}$, and we follow the arguments in there, we get

Proposition 5.4. *The critical point* $\widetilde{u}_{y_1}$ *of the function* $\widetilde{J}_p$ *in Proposition 5.3 is a solution of problem* (6).

Corollary 5.5. *The function* $u = (J_p(\widetilde{u}_{y_1}))^{1-p}\widetilde{u}_{y_1}$ *is a solution of the subcritical problem* (2) *and the function* $u_{y_1} = \Lambda_{y_1}^{-1}\widetilde{u}_{y_1} = \alpha_{F_{y_1}^{-1}}(1+\eta_{y_1})$ *is a solution of the perturbated equation* (3).

Acknowledgements

Gonzalo García and Liliana Posada thanks the Universidad del Valle for its support with the 7920 Project. The authors are very grateful to the reviewers for their suggestions and comments that really improved the paper.

References

[1] Bahri A. and Coron J.M., "The scalar-curvature problem on the standard three-dimensional sphere", *J. Funct. Anal.* 95 (1991), No. 1, 106–172.

[2] Chang Sun-Yung A., Gursky M.J. and Yang P.C., "The scalar curvature equation on the 2- and 3-sphere", *Calc. Var. Partial Differential Equations* 1 (1993), No. 2, 205–229.

[3] Escobar J.F. and García G., "Conformal metrics on the ball with zero scalar curvature and prescribed mean curvature on the boundary", *J. Funct. Anal.* 211 (2004), No. 1, 71–152.

[4] García G. and Posada V.L., "A priori estimates of the prescribed scalar curvature on the sphere", *Revista de Ciencias* 19 (2015), No. 1, 73–86.

[5] Han Z.C., "Asymptotic approach to singular solutions for nonlinear elliptic equations involving critical Sobolev exponent", *Ann. Inst. H. Poincaré Anal. Non Linéaire* 8 (1991), No. 2, 159–174.

[6] Kazdan J. and Warner F., "Existence and conformal deformation of metrics with prescribed Gaussian and scalar curvature", *Ann. of Math. (2)* 101 (1975), 317–331.

[7] Li Y.Y., "Prescribing scalar curvature on S^n and related problems. I", *J. Differential Equations* 120 (1995), No. 2, 319–410.

[8] Schoen R. and Zhang D., "Prescribed scalar curvature on the n-sphere", *Calc. Var. Partial Differential Equations* 4 (1996), No. 1, 1–25.

[9] Zhang D., "New results on geometric variational problems", Thesis (Ph.D), Stanford University, 1990, 85 p.