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Peirce quincuncial projection

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Abstract. We present the essential theoretical basis and prove concrete practical formulas to compute the image of a point on the terrestrial sphere under Peirce quincuncial projection. We also develop a numerical method to implement such formulas in a digital computer and illustrate this method with examples. Then, we briefly discuss the criticism of Pierpont on the correctness of Peirce's formula for the projection. Finally, we draw some conclusions regarding the generalization of Peirce's original idea by means of Schwarz-Christoffel transformations.
Keywords: Peirce quincuncial projection, elliptic functions, geographic maps, numerical conformal mapping, tessellations.
MSC2010: 30C30, 65E05, 01A55.

Proyección quincuncial de Peirce

Resumen. Presentamos los fundamentos teóricos esenciales y demostramos fórmulas concretas para calcular de manera práctica la imagen de un punto en la esfera terrestre bajo la proyección quincuncial de Peirce. Desarrollamos también un método numérico para implementar dicha proyección en un computador digital, el cual ilustramos con ejemplos. Luego discutimos brevemente las objeciones de Pierpont sobre la validez de la fórmula de Peirce. Por último, esbozamos algunas conclusiones sobre la generalización de la idea de Peirce por medio de transformaciones de Schwarz-Christoffel.
Palabras clave: Proyección quincuncial de Peirce, funciones elípticas, cartas geográficas, métodos numéricos, aplicaciones conformes, teselados.

1. Introduction

The word "map" derives from the medieval Latin *mappa mundi*, meaning napkin or cloth of the world. Here, a map projection is a smooth transformation of a sphere into a plane. There are many types of map projections and they are usually constructed to preserve

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some metric properties of parts of the sphere. These properties comprise area, shape, distance, among others. In 1879, the American scientist, geodesist and philosopher Charles Sanders Peirce suggested an angle-preserving map projection useful "for meteorological, magnetological and other purposes". The map also shows the connection of all parts on the Earth's surface. Peirce quincuncial projection presents the sphere as a square and has been given the name of "quincuncial". Certainly, the Latin noun *quincunx* denotes the pattern of five points on the corresponding face of a die, or on the volume of a Byzantine Church.

Peirce's original paper [10] is extremely laconic. Perhaps because of this, some mathematicians tried, years later, to understand and explain his elegant idea. In particular, Pierpont [11] detected an error in Peirce's formula and found a correct expression for the projection. Other remarks were given shortly after by Frischauf [5].

The essential ingredient in Peirce's construction is a Jacobi elliptic function, i.e., a meromorphic complex function of one complex variable with two linearly independent periods. The theory of elliptic functions can be addressed in several ways. The contemporary approach to the study of these functions is due to Weierstrass and its modern notation shows indeed advantages in regard to elegance and symmetry. However, the present paper is concerned with numerical computing and so, for our purposes, it is more convenient to use the older Jacobian notation. Because of this, our primary reference on elliptic functions are Jacobi's original work [7] and the modern account by Solanilla [13]. In addition, we are interested in some specific results due to Richelot [12] and Durège [4].

By the way, the application of elliptic functions to conformal map projections constitutes an active research field, both in pure an applied mathematics. We refer the reader to Lee [8] for a more comprehensive (and very agreeable) treatment of these matters.

In Section 2 we define Peirce quincuncial projection and derive formally practical formulas to compute it. In Section 3 we discuss some symmetries arising from the double periodicity of the elliptic function involved and use them to clarify some mathematical considerations left over in the previous sections. Section 4 is devoted to the numerical calculation of the formulas. We briefly describe a way to program a computer in order to implement the mathematical expressions found before. Initially, we graph circles of latitude and lines of longitude. Then, we present a version of a map of the World. We also exchange views on the only, and rather enigmatic, formula given by Peirce [10]. Lastly, we draw some concluding remarks and announce generalized Peirce-like map projections.

2. Representation of the sphere

The quincuncial projection results from the composition of the famous stereographic projection with the "inverse" of a Jacobi elliptic function.

2.1. Stereographic projection

We model the globe as a sphere S^2 of unit radius. To each point P in this sphere, we associate its geographical coordinates $\theta \in (0, 2\pi)$ and $l \in (-\pi/2, \pi/2)$, longitude and latitude, respectively. Instead of l, it is sometimes convenient to use the parameter $p = \pi/2 + l \in (0, \pi)$. Thus, we consider the function that projects the sphere without the

North Pole N by prolonging the straight line joining N with P until it reaches the plane containing the equatorial line. By virtue of the Inscribed Angle Theorem, in Figure 1 it holds $\angle NSP = (\pi - p)/2$. Let $\zeta = \xi + iv \in \mathbb{C}$ be the image of the sphere on the plane. Since $\triangle NO\zeta \sim \triangle NPS$,

$$\cot\left(\frac{\pi - p}{2}\right) = \tan\frac{p}{2} = |\zeta|$$

and the stereographic projection takes the form

$$\zeta = \tan\frac{p}{2} \times \exp(i\theta).$$

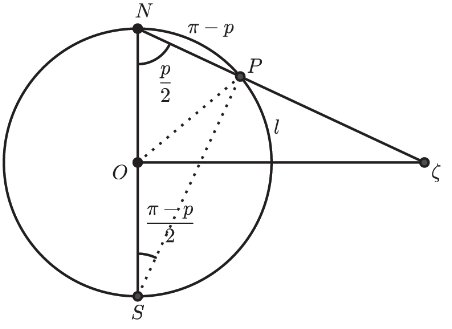


Figure 1. The stereographic projection.

2.2. *Cosine of the amplitude*

Peirce projection is the composite $(\theta, p) \mapsto \zeta \mapsto z = x + iy$, where z is given implicitly by

$$\zeta = \mathrm{cn}\left(z, \frac{1}{\sqrt{2}}\right).$$

The function cn on the right is known as Jacobi *cosinus amplitudinis* –cosine of the amplitude– with modulus $1/\sqrt{2}$. Inasmuch as the most remarkable features of the quincuncial projection rely on the geometric properties of this function, it is convenient to review some basic facts on elliptic functions.

2.3. *Jacobi elliptic functions*

Elliptic functions emerged historically from the inverse function of Legendre's elliptic integral of the first kind

$$F(\phi) = \int_0^\phi \frac{d\varphi}{\sqrt{1 - k^2 \sin^2\varphi}}.$$

As we only need here $k = 1/\sqrt{2}$, no further reference will be made to the general modulus k. The inverse function of F is called the amplitude am. Jacobi defined elliptic functions by composing the amplitude with some well-known functions. In this paper we will only make use of

$$\sin \mathrm{am} z = \mathrm{sn} z,\ \cos \mathrm{am} z = \mathrm{cn} z \text{ and } \mathrm{dn} z = \sqrt{1 - \frac{1}{2}\mathrm{sn}^2 z}.$$

The main properties of these functions arise out of their addition formulas. In particular,

$$\mathrm{cn}(z+w) = \frac{\mathrm{cn} z\ \mathrm{cn} w - \mathrm{sn} z\ \mathrm{sn} w\ \mathrm{dn} z\ \mathrm{dn} w}{1 - \frac{1}{2}\mathrm{sn}^2 z\ \mathrm{sn}^2 w}.$$

We refer the reader to Jacobi [7], or Solanilla [13], for details concerning the proof of the identities we might need. In order to properly understand Peirce projection, we need first the following relation.

Proposition 2.1. *Let $z = x + iy$, $cnz = \rho \exp(i\theta)$, $snz = \sigma \exp(i\lambda)$ and $dnz = \tau \exp(i\mu)$. Then,*

$$cn2x = \frac{\rho^2 - \sigma^2\tau^2}{1 - \frac{1}{2}\sigma^4} \text{ and } cn2y = \frac{1 - \frac{1}{2}\sigma^4}{\rho^2 + \sigma^2\tau^2}.$$

Proof. From the complex conjugate $\bar{z} = x - iy$, we get $2x = z + \bar{z}$ and $2yi = z - \bar{z}$. Therefore, the addition formula for cn implies

$$\mathrm{cn} 2x = \frac{\mathrm{cn} z\ \mathrm{cn}\bar{z} - \mathrm{sn} z\ \mathrm{sn}\bar{z}\ \mathrm{dn} z\ \mathrm{dn}\bar{z}}{1 - \frac{1}{2}\mathrm{sn}^2 z\ \mathrm{sn}^2\bar{z}}.$$

Similarly, since sn is odd and cn, dn are even,

$$\mathrm{cn} 2yi = \frac{\mathrm{cn} z\ \mathrm{cn}\bar{z} + \mathrm{sn} z\ \mathrm{sn}\bar{z}\ \mathrm{dn} z\ \mathrm{dn}\bar{z}}{1 - \frac{1}{2}\mathrm{sn}^2 z\ \mathrm{sn}^2\bar{z}}.$$

Now we use $\mathrm{cn}\bar{z} = \overline{\mathrm{cn} z} = \rho \exp(-i\theta)$, $\mathrm{sn}\bar{z} = \overline{\mathrm{sn} z} = \sigma \exp(-i\lambda)$ and $\mathrm{dn}\bar{z} = \overline{\mathrm{dn} z} = \tau \exp(-i\mu)$. The second statement follows from the identity $\mathrm{cn} iv = 1/\mathrm{cn} v$, $v \in \mathbb{R}$. ☑

2.4. Turning back to the sphere

Now we should relate the coordinates or parameters θ, p with $z = x + iy$. The following result brings us closer to the desired relation.

Proposition 2.2. *With the notations in Proposition 2.1, if $\rho = \tan(p/2)$, then*

$$\sigma^4 = 1 - 2\tan^2\frac{p}{2}\cos 2\theta + \tan^4\frac{p}{2},$$
$$\tau^4 = \frac{1}{4} \times \left(1 + 2\tan^2\frac{p}{2}\cos 2\theta + \tan^4\frac{p}{2}\right).$$

Proof. We depart from the Pythagorean identity $\mathrm{cn}^2 z + \mathrm{sn}^2 z = 1$, i.e.,

$$\rho^2 \exp(2\theta i) + \sigma^2 \exp(2\lambda i) = 1.$$

After equating the imaginary parts[1],

$$\rho^2 \sin 2\theta = -\sigma^2 \sqrt{1-\cos^2 2\lambda} \text{ iff } \cos^2 2\lambda = 1 - \frac{\rho^4}{\sigma^4}\sin^2 2\theta.$$

After equating the real parts,

$$\sigma^2 \cos 2\lambda = \sigma^2 \sqrt{1 - \frac{\rho^4}{\sigma^4}\sin^2 2\theta} = 1 - \rho^2 \cos 2\theta.$$

Therefore $\sigma^4 - \rho^4 \sin^2 2\theta = 1 - 2\rho^2 \cos 2\theta + \rho^4 \cos^2 2\theta$, that is to say,

$$\sigma^4 = 1 - 2\rho^2 \cos 2\theta + \rho^4 = 1 - 2\tan^2 \frac{p}{2} \cos 2\theta + \tan^4 \frac{p}{2}.$$

Correspondingly, the elliptic identity $\frac{1}{2}\left(\mathrm{cn}^2 z + 1\right) = \mathrm{dn} z$ or

$$\frac{1}{2}\rho^2 \exp(2\theta i) + \frac{1}{2} = \tau^2 \exp(2\mu i)$$

yields, by taking the imaginary part,

$$\rho^2 \sin 2\theta = 2\tau^2 \sqrt{1-\cos^2 2\mu} \text{ iff } \cos^2 2\mu = 1 - \frac{\rho^4}{4\tau^4}\sin^2 2\theta.$$

So, the real part furnishes

$$2\tau^2 \cos 2\mu = 2\tau^2 \sqrt{1 - \frac{\rho^4}{4\tau^4}\sin^2 2\theta} = 1 + \rho^2 \cos 2\theta.$$

In consequence, we have $4\tau^4 - \rho^4 \sin^2 2\theta = 1 + 2\rho^2 \cos 2\theta + \rho^4 \cos^2 2\theta$. In other words,

$$4\tau^4 = 1 + 2\tan^2 \frac{p}{2} \cos 2\theta + \tan^4 \frac{p}{2}.$$

☑

Corollary 2.3. *With the previous notations,*

$$\sigma^2\tau^2 = \frac{1}{2} \times \sqrt{\left(1 + \tan^4 \frac{p}{2}\right)^2 - 4\tan^4 \frac{p}{2} \cos^2 2\theta}.$$

We hereby achieve our primary objective.

Theorem 2.4. *Peirce quincuncial projection $(\theta, p) \mapsto x + iy$ is given by*

$$x = \frac{1}{2} F\left(\arccos \frac{2\tan^2 \frac{p}{2} - \sqrt{\left(1+\tan^4 \frac{p}{2}\right)^2 - 4\tan^4 \frac{p}{2}\cos^2 2\theta}}{1 + 2\tan^2 \frac{p}{2}\cos 2\theta - \tan^4 \frac{p}{2}}\right),$$

$$y = \frac{1}{2} F\left(\arccos \frac{1 + 2\tan^2 \frac{p}{2}\cos 2\theta - \tan^4 \frac{p}{2}}{2\tan^2 \frac{p}{2} + \sqrt{\left(1+\tan^4 \frac{p}{2}\right)^2 - 4\tan^4 \frac{p}{2}\cos^2 2\theta}}\right).$$

[1] Without loss of generality –as it will be clear below–, $\mathrm{sn} z$ (and so the angle λ) lies in the first quadrant.

Proof. Putting together the whole argument, Proposition 2.2 implies

$$\cos \mathrm{am} 2x = \frac{\tan^2 \frac{p}{2} - \frac{1}{2} \times \sqrt{\left(1+\tan^4 \frac{p}{2}\right)^2 - 4\tan^4 \frac{p}{2}\cos^2 2\theta}}{1 - \frac{1}{2}(1 - 2\tan^2 \frac{p}{2}\cos 2\theta + \tan^4 \frac{p}{2})},$$

$$\cos \mathrm{am} 2y = \frac{1 - \frac{1}{2}(1 - 2\tan^2 \frac{p}{2}\cos 2\theta + \tan^4 \frac{p}{2})}{\tan^2 \frac{p}{2} + \frac{1}{2} \times \sqrt{\left(1+\tan^4 \frac{p}{2}\right)^2 - 4\tan^4 \frac{p}{2}\cos^2 2\theta}}.$$

We recall that $\mathrm{am}^{-1} = F$. ☑

We observe that the functions in this theorem are real-valued of one real variable. The function F is well-known and is already implemented in most standard mathematical computer programs. A comprehensive explanation on the incomplete integral F can be found in Bellachi [2] and Solanilla et al. [14].

3. Symmetries

Once we pick a domain for cn making it into a one-to-one function, the expressions in Theorem 2.4 are well-defined. Let us see.

3.1. Fundamental parallelogram

By definition, $\mathrm{cn} z$ is a meromorphic doubly periodic function. Then, it suffices to study its behavior in the fundamental parallelogram Π, that is, the parallelogram defined by the origin and the periods –among other choices–. These can be determined with the help of the so-called complete integral of the first kind

$$K = F\left(\frac{\pi}{2}\right) = \int_0^{\frac{\pi}{2}} \frac{d\varphi}{\sqrt{1 - \frac{1}{2}\sin^2\varphi}} \approx 1.854074677.$$

Rightly, since Jacobi [7], it is well-known that the periods of our function are $4K$ and $2K(1+i)$ (as a consequence of the addition formula). The zeros in Π are located at $K, 3K$; the poles, at $K(2+i), K(4+i)$. Figure 2 illustrates the situation.

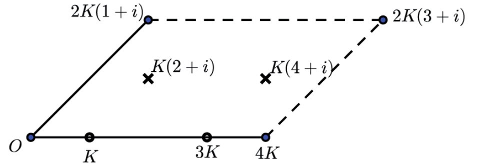


Figure 2. Zeros ($\circ$) and poles ($\times$) of $\mathrm{cn} z$ in Π.

Unfortunately, the function $\mathrm{cn} : \Pi \to \mathbb{C}$ is not yet one-to-one. Nevertheless, the closure of the fundamental parallelogram can be represented as

$$\overline{\Pi} = \left\{4Ks + 2K(1+i)t : (s,t) \in [0,1]^2\right\},$$

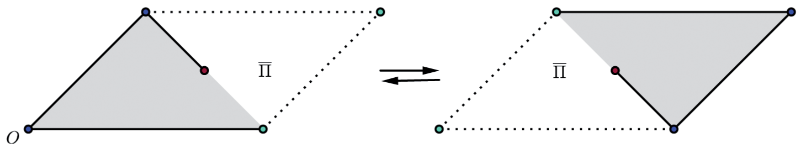


Figure 3. Involution of $\overline{\Pi}$, choices of Σ.

and the map $4Ks+2K(1+i)t \mapsto 4K(1-s)+2K(1+i)(1-t)$ is an involution of $\overline{\Pi}$ which works as displayed in Figure 3. $2K + K(1+i)$ is the fixed point of this map. In plain English, $\overline{\Pi}$ is split into two parts. Moreover, if $z \in \overline{\Pi}$ has image (under the involution) $w \in \overline{\Pi}$, then

$$z + w = 4K + 2K(1+i) \equiv 0,$$

modulo $4K, 2K(1+i)$. This fact constitutes one of the most important discoveries of Liouville [9] in his researches on elliptic functions. As cn is even, $\mathrm{cn}(z) = \mathrm{cn}(w)$. Therefore, if Σ denotes one of the the shaded areas in Figure 3, the function $\mathrm{cn} : \Sigma \to \mathbb{C}$ becomes at last one-to-one.

3.2. Fundamental square

Additional underlying symmetries can be exploited to obtain a nicer fundamental region for cn. In the first place, the translations of triangle Δ by $2K(1+i)$ and triangle Δ' by $-2K(1+i)$ transform the fundamental parallelogram into a square, as it is sketched in Figure 4. This will be the fundamental square T of cn. We also notice that the zeros ($\circ$) of the elliptic function are now symmetrically located with respect to the center of T. The inverse function of $\mathrm{cn} : T \to \mathbb{C}$ produces the above-mentioned representation of

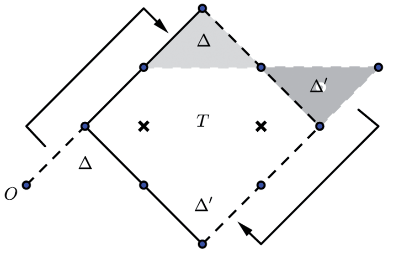


Figure 4. Construction of the fundamental square T.

S^2 as a square. The center and vertices of T constitute, for that reason, a *quincunx*.

On the other hand, the closure of T can be given by

$$\overline{T} = \left\{ K(s+t+1) + 2Ki(s-t+1) : (s,t) \in [0,2]^2 \right\}.$$

The map $K(s+t+1)+2Ki(s-t+1) \mapsto K(-s-t+2)+2Ki(-s+t)$ is the involution of $\overline{T}$ which sends each point to its antipodal point with respect to the center of the fundamental square. This center is the fixed point of the involution.

Now, the image of triangle Δ (*v.* Figure 4) under the involution of $\overline{\Pi}$ is triangle Δ' and *vice versa*. Then, the images of Δ, Δ' under the cosine of the amplitude coincide in the complex plane. This implies that the parts Ω where cn is one-to-one in the fundamental square are the shaded regions in Figure 5.

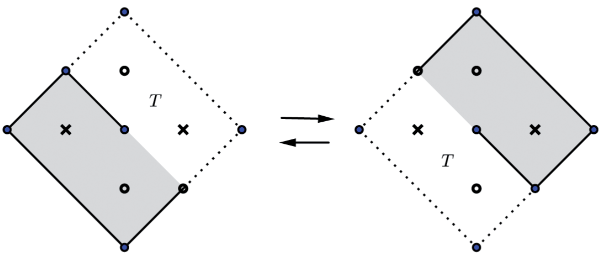


Figure 5. Involution of $\overline{T}$, choices of Ω.

3.3. Zero, circles of latitude and lines of longitude

When we substitute $p = 0$ (South Pole $S \in S^2$) in these formulas,

$$x = \frac{1}{2}F(\arccos(-1)) = \frac{1}{2}F(3\pi) = \frac{6K}{2} = 3K,$$
$$y = \frac{1}{2}F(\arccos(1)) = \frac{1}{2}F(2\pi) = \frac{4K}{2} = 2K.$$

In this way, we get the zero $K(3+2i) \in T$ of cn. The other choices of $\arccos(-1)$ give the remaining zeros of the function. The poles demand some extra, but straightforward, work.

Let $p > 0$ be a constant. Then, $|\zeta| = \rho = \tan(p/2)$ is also a constant and the formulas for x, y provide the following parametric form for the circles or parallels of latitude:

$$x(\theta) = \frac{1}{2}F\left(\arccos \frac{2\rho^2 - \sqrt{(1+\rho^4)^2 - 4\rho^4\cos^2 2\theta}}{1 + 2\rho^2\cos 2\theta - \rho^4}\right),$$
$$y(\theta) = \frac{1}{2}F\left(\arccos \frac{1 + 2\rho^2\cos 2\theta - \rho^4}{2\rho^2 + \sqrt{(1+\rho^4)^2 - 4\rho^4\cos^2 2\theta}}\right);$$

$\theta \in [0, 2\pi)$ is the parameter. Similarly, if we fix θ and write $\chi = \cos 2\theta$, we obtain the following for the lines of longitude:

$$x(p) = \frac{1}{2}F\left(\arccos \frac{2\tan^2\frac{p}{2} - \sqrt{\left(1+\tan^4\frac{p}{2}\right)^2 - 4\chi^2\tan^4\frac{p}{2}}}{1 + 2\chi\tan^2\frac{p}{2} - \tan^4\frac{p}{2}}\right),$$
$$y(p) = \frac{1}{2}F\left(\arccos \frac{1 + 2\chi\tan^2\frac{p}{2} - \tan^4\frac{p}{2}}{2\tan^2\frac{p}{2} + \sqrt{\left(1+\tan^4\frac{p}{2}\right)^2 - 4\chi^2\tan^4\frac{p}{2}}}\right).$$

The parameter is now $p \in [0, \pi)$.

3.4. Conformality and magnification

Our version of Peirce projection is the composite function of the stereographic projection $S^2 - \{N\} \to \mathbb{R}^2, (\theta, p) \mapsto \zeta$, and the restricted inverse elliptic function $\mathbb{R}^2 \to \Omega, \zeta \mapsto z$. Our choice of Ω is the right region in Figure 5. We notice the pole (marked by $\times$) in Ω coincides with the image of the missing North Pole. Therefore, after taking care of minor details concerning a few points in Ω and disregarding the equatorial line, this composite projection is in general conformal, i.e., angle-preserving. This implies the images of the circles of latitude and the lines of longitude cut each other, in general, at right angles.

The magnification, wherever it makes sense, can be computed by noticing with Pierpont [11] that the symmetries involved imply that $m(\theta, p) = |dz/dp|$. By virtue of the chain rule,

$$m(\theta, p) = \left|\frac{dz}{d\zeta}\right| \left|\frac{d\zeta}{dp}\right|,$$

where

$$\left|\frac{d\zeta}{dp}\right| = \frac{1}{2\cos^2(p/2)} \text{ and } \left|\frac{dz}{d\zeta}\right| = \left|\frac{1}{d\mathrm{cn}/dz}\right| = \frac{1}{|\sigma\tau|}.$$

In the last step we have used $d\mathrm{cn}(z)/dz = -\mathrm{sn}(z)\mathrm{dn}(z)$. This yields, by Corollary 2.3,

$$m(\theta, p) = \frac{1}{\sqrt{2}\cos^2\frac{p}{2}\sqrt[4]{\left(1+\tan^4\frac{p}{2}\right)^2 - 4\tan^4\frac{p}{2}\cos^2 2\theta}}.$$

Clearly, m blows up at the North Pole and at the Equator for $\theta \in \{0, \pi/2, 3\pi/2, 2\pi\}$. In general, along a parallel of latitude, m attains its minimum for $\theta = \{\pi/4, 3\pi/4, 5\pi/4, 7\pi/4\}$ and a maximum for $\theta \in \{0, \pi/2, 3\pi/2, 2\pi\}$. By the North and South Pole, the forms are nearly circles. As p tends to zero, that is, when we approach the Equator, this behavior results in more and more square-like forms with rounded corners.

4. Computer implementation

4.1. An algorithm

The actions we have used to compute the image of the projection are outlined in the flowchart shown below.

Firstly, the program reads the geographical coordinates θ, l of a point on the sphere. The longitude θ consists of

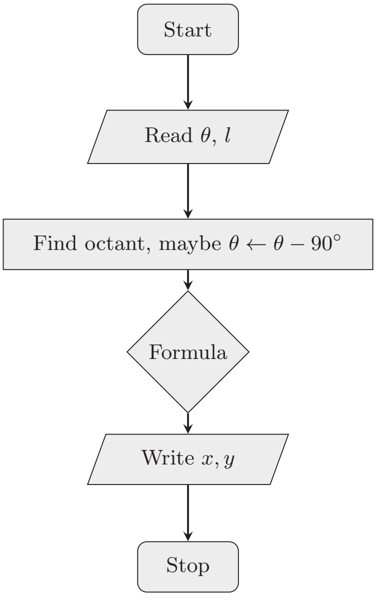
- An angle position in degrees or meridian measured from $0°$ (Greenwich in England) to $180°$ and
- An indication of direction, namely E (East) or W (West).

The latitude l is given by

- An angle position in degrees from 0° (Equator) to 90° and
- An indication of direction, namely N (North) or S (South).

With this information, the program determines

- One of the octants marked $I, II, \cdots, VIII$ in Figure 6 and
- A possible shift of the angle of longitude (also denoted by θ) by 90°.



The arrangement of the geographical coordinates and the octant is shown in Table 1.

Then, there is a decision step. The program chooses and computes the right formula according to the octant. Let x, y be the formulas in Theorem 2.4. They might need the shift described in Table 2. Besides field operations and trigonometric functions, the computation of an elliptic integral of the first kind is required. A command like `InverseJacobiAM(`ϕ, k`)` in Maple can easily accomplish this task.

Finally, the program writes the image x, y of the spherical point θ, l on a Cartesian plane.

4.2. A map of the world

As a first example of the application of the algorithm, we portrait in Figure 7 the images of some circles of latitude and some meridians.

Figure 8 shows our first sketch of the whole world, obtained with the algorithm described above. In just Europe, we have used more than 120 points.

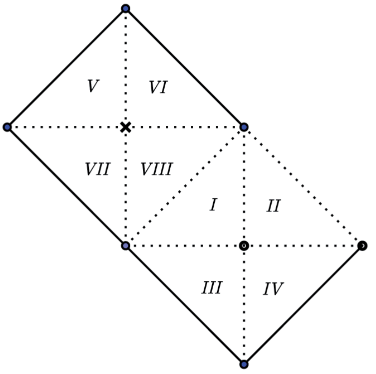


Figure 6. Octants.

Latitude	Longitude	Octant	Shift
S	$0^\circ \leq \theta < 90^\circ\ E$	I	no
	$0^\circ \leq \theta < 90^\circ\ W$	II	no
	$90^\circ \leq \theta < 180^\circ\ E$	III	$\theta \leftarrow \theta - 90^\circ$
	$90^\circ \leq \theta < 180^\circ\ W$	IV	$\theta \leftarrow \theta - 90^\circ$
N	$0^\circ \leq \theta < 90^\circ\ E$	V	no
	$0^\circ \leq \theta < 90^\circ\ W$	VI	no
	$90^\circ \leq \theta < 180^\circ\ E$	VII	$\theta \leftarrow \theta - 90^\circ$
	$90^\circ \leq \theta < 180^\circ\ W$	$VIII$	$\theta \leftarrow \theta - 90^\circ$

Table 1. Octants and shifts of longitude.

Octant	x	y
I	x	y
II	$x \leftarrow K - x$	y
III	x	$y \leftarrow -y$
IV	$x \leftarrow K - x$	$y \leftarrow -y$
V	$x \leftarrow x - K$	$y \leftarrow y + K$
VI	$x \leftarrow -x$	$y \leftarrow y + K$
VII	$x \leftarrow x - K$	$y \leftarrow -y + K$
VIII	$x \leftarrow -x$	$y \leftarrow -y + K$

Table 2. Octants and formulas.

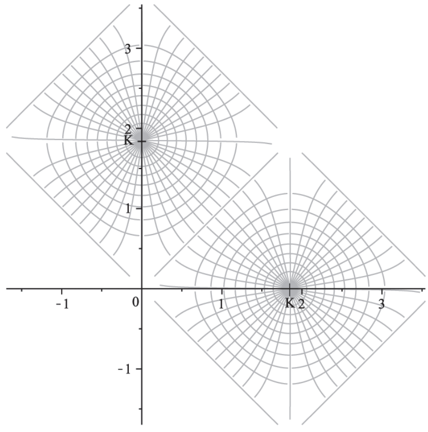


Figure 7. Images of parallels of latitude and meridians.

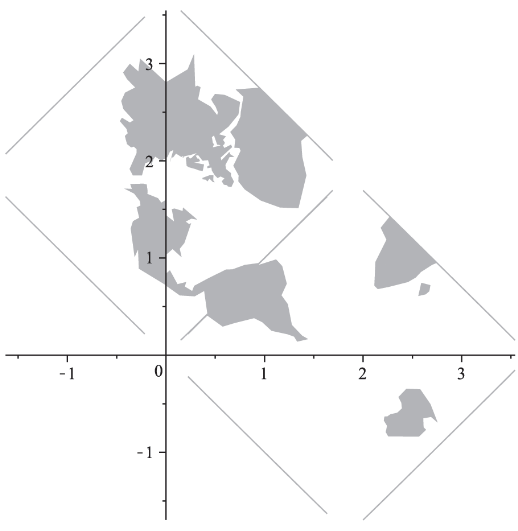


Figure 8. The whole world.

4.3. Tessellation

This drawing can be used as the motif –distinctive recurring form– of a tessellation or tiling of the plane. Actually, the Latin word *tessella* denotes a small cube of clay, stone or glass used to make mosaics.

By the way, it is no coincidence that Peirce [10] presented his quincuncial projection in the tessellated version of Figure 9. We notice that he did not utilize the Greenwich Meridian as prime meridian of longitude. Instead, he used something like the meridian through Cape Town, South Africa (approximately 18° E from Greenwich). This resolution could

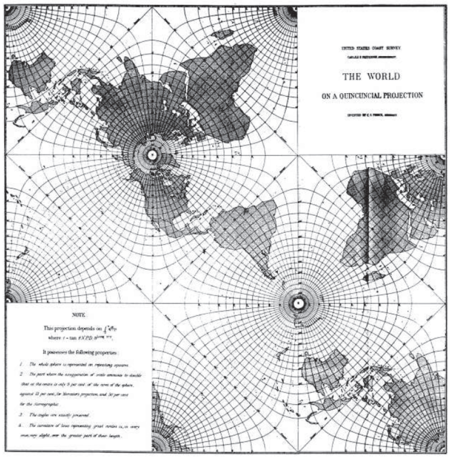


Figure 9. Tessellated version of Peirce projection.

be related to an interest to avoid points on dry land where the magnification of the quincuncial map becomes infinite, or maybe it just obeys to purely aesthetic reasons.

This tesselation is a distinctive feature of Peirce quincuncial projection. Unlike other projections, in order to use this World map for points or routes close to the borders it suffices to continue on the adjoining copy. At the onset of his paper Peirce remarks that this projection "shall show the connection of all parts of the surface".

4.4. Peirce's formula

Besides some explanatory remarks, Peirce's paper [10] consists just of the map shown in Figure 9, one formula and two tables of coordinates. His formula for the quincuncial projection has the form $(\theta, l) \mapsto x$,

$$x = \frac{1}{2} F \left(\arccos \sqrt{\frac{\sqrt{1 - \cos^2 l \cos^2 \theta} - \sin l}{1 + \sqrt{1 - \cos^2 l \cos^2 \theta}}} \right),$$

where x is "the value of one of the rectangular coördinates of the point in the new projection". For him, indeed, there was no need of giving an additional formula for y. For a given latitude, the longitude angle of y is equal to the complementary longitude angle of x. In other words, if θ_x, θ_y are respectively the longitudes of x, y, we must have $\theta_x + \theta_y = 90°$. Peirce's [10] original table gives at once the values of x and y corresponding to some fixed values of θ (longitude) and l (latitude).

Just like us, Peirce uses octants. However, they do not coincide with ours. Anyway, although his treatment of symmetries is right and time-saving, Pierpont [11] discovered that "there seems to be an error" in Peirce's formula for "special values of l and θ". For example, when $l = \theta = 0$, his formula produces

$$x = \frac{1}{2} F \left(\frac{\pi}{2} \right) = \frac{K}{2}.$$

By the way, Peirce's data [10] are normalized in the sense that he sets $K/2 = 1$. On the contrary, our formulas yield $x = y = (1/2)F(0) = 0$, since $p = \pi/2$.

Peirce's formula is –in general– not quite right and he certainly knew it. If we look carefully at his table of "rectangular coördinates", we would notice a couple of rectangular boxes at the bottom marked by relatively thick lines. He was aware that the data contained in these boxes were correct, although they had not been obtained by using his formula. Figure 10 shows one of these boxes. They correspond to latitudes $l = 0°, 5°, 10°, 15°$ and x-longitudes $\theta_x = 0°, 5°, 10°, 15°$.

20	.548	.545	.534	.518	.498
15	.609	.604	.589	.568	.544
10	.681	.672	.649	.620	.590
5	.775	.752	.713	.673	.635
0	1,000	.841	.774	.723	.679

Figure 10. One of the boxes marked by Peirce.

We must say that, in the rest of the table, Peirce's formula produces very good results. In order to detect the inaccuracies, we can sketch, in one octant, parallels of latitude and meridians according to his normalized formula. Figures 11 and 12 reveal the problems. Figure 11 displays the parallel of latitude corresponding to $p = 0.785$ using both Peirce's formula and ours (octant II). Our formula produces the graphic in green, whereas the red graphic has been obtained by using Peirce's formula. Our green curve is clearly more square-like than Peirce's. Something similar occurs with the meridians. In Figure 12 we have displayed the meridians for $\theta = 0.785$ and $\theta = 1.4827$ (octant II). As before, our formula yields the green graphics and Peirce's the red ones. The difference is clear: Peirce's red curves are not completely symmetric about the origin.

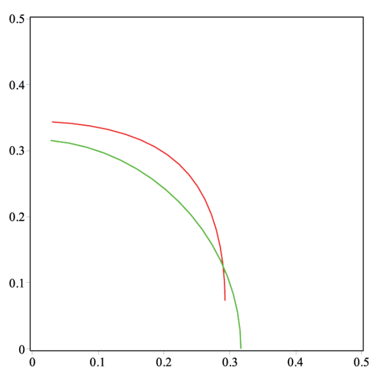


Figure 11. Peirce's parallel of latitude *versus* ours for $p = 0.785$.

So far, we have no clue about the procedure employed by Peirce to find his formula. Nor do we know anything about how he found the correct values inside the rectangles.

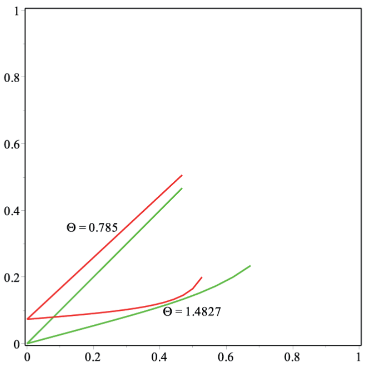


Figure 12. A pair of Peirce's meridians *versus* ours.

There are two other projections that are slight variations of Peirce quincuncial projection. They were subsequently introduced by Guyou [6] and Adams [1].

5. Concluding remarks

Let P be a simply connected domain in the complex plane enclosed by a closed polygonal curve Γ having vertices $w_1, \ldots, w_n$ and interior counterclockwise angles $\alpha_1\pi, \ldots, \alpha_n\pi$. The Riemann Mapping Theorem guarantees the existence of a conformal injective map $f : \mathbb{H} \to P$ of the upper half plane $\mathbb{H} = \{z = x + iy \in \mathbb{C} : y > 0\}$ onto P. It is known that, if we additionally require that $f(\infty) = w_n$, f can be written in the form

$$f(z) = a + b \int_{z_0}^{z} \prod_{k=1}^{n-1} (\zeta - z_k)^{\alpha_k - 1}\, d\zeta,$$

where $a, c \in \mathbb{C}$ are constants and $z_0 < z_1 < \cdots < z_n$ are real numbers such that $f(z_k) = w_k$, $k = 1, \ldots, n-1$. Conformal maps of this type are called Schwarz-Christoffel transformations (cf. Bergonio [3]). As it is evident from above, a convenient restriction of $\mathrm{cn}^{-1}(z, 1/\sqrt{2})$ gives an example of a Schwarz-Christoffel transformation.

Thus, Schwarz-Christoffel transformations provide a glimpse into the real possibility of generalizing Peirce quincuncial projection.

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