



Revista Integración

ISSN: 0120-419X

integracion@matematicas.uis.edu.co

Universidad Industrial de Santander

Colombia

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Revista Integración, vol. 34, núm. 2, 2016, pp. 147-168

Universidad Industrial de Santander

Bucaramanga, Colombia

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Armendariz property for skew PBW extensions and their classical ring of quotients

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Abstract. We consider a first approach to the notion of Armendariz ring for a skew Poincaré-Birkhoff-Witt (PBW for short) extension, and its classical ring of quotients. As an immediate application of this treatment, we study the properties Baer, quasi-Baer, p.p. and p.q.-Baer rings for these extensions. In this way, we generalize several results in the literature concerning Ore extensions and skew PBW extensions.

Keywords: Armendariz, Baer, quasi-Baer, p.p. and p.q.-rings, skew Poincaré-Birkhoff-Witt extensions.

MSC2010: 16D25, 16E50, 16S36.

Propiedad de Armendariz para las extensiones PBW torcidas y su anillo clásico de cocientes

Resumen. Consideramos un primer acercamiento a la noción de anillo de Armendariz para una extensión torcida de Poincaré-Birkhoff-Witt (PBW), y su anillo clásico de cocientes. Como una aplicación inmediata de este tratamiento, estudiamos las propiedades de Baer, quasi-Baer, p.p. y p.q.-Baer para estas extensiones. De esta manera, generalizamos varios resultados de la literatura para extensiones de Ore y extensiones PBW torcidas.

Palabras clave: Armendariz, Baer, quasi-Baer, p.p. y p.q.-anillos, extensiones torcidas de Poincaré-Birkhoff-Witt.

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Received: 15 June 2016, Accepted: 04 August 2016.

To cite this article: A. Reyes, H. Suárez, Armendariz property for skew PBW extensions and their classical ring of quotients, *Rev. Integr. Temas Mat.* 34 (2016), No. 2, 147–168.

1. Introduction

In commutative algebra, a ring B is called *Armendariz* (the term was introduced by Rege and Chhawchharia in [25]) if whenever polynomials $f(x) = a_0 + a_1x + \cdots + a_nx^n$, $g(x) = b_0 + b_1x + \cdots + b_mx^m \in B[x]$ satisfy $f(x)g(x) = 0$, then $a_ib_j = 0$ for every i, j . The interest of this notion lies in its natural and its useful role in understanding the relation between the annihilators of the ring B and the annihilators of the polynomial ring $B[x]$. In [2], Lemma 1, Armendariz showed that a *reduced* ring (i.e., a ring without nonzero nilpotent elements) always satisfies this condition (reduced rings are *Abelian*—that is, every idempotent is central—, and also semiprime, that is, its prime radical is trivial). With respect to the noncommutative algebra, more exactly the well-known Ore extensions, the notion of Armendariz has been also studied. In this way, commutative and noncommutative treatments have been investigated in several papers (see for example [1], [2], [12], [14], [17], [19], [23], [25] and others).

The examples of noncommutative (and commutative polynomial, of course) rings of interest for us in this article are the skew Poincaré-Birkhoff-Witt extensions introduced in [9]. These structures are more general than Ore extensions of injective type, and contain various well-known groups of algebras such as some types of Auslander-Gorenstein rings, some Calabi-Yau and skew Calabi-Yau algebras, some Artin-Schelter regular algebras, some Koszul algebras, quantum polynomials, some quantum universal enveloping algebras, etc. Indeed, it has been shown that skew PBW extensions contain several algebras which can not be expressed as Ore extensions (universal enveloping algebras of finite Lie algebras, diffusion algebras, and others (see [22], [26], [29], [28], or [30] for more details)).

Precisely, in [28], Remark 3.8, the first author announced one paper with a *skew* notion of Armendariz ring for skew PBW extensions. Well then, this is the first paper on that promise. Next, we describe the structure of this article. In Section 2 we establish some useful results about skew PBW extensions for the rest of the paper. In Section 3 we introduce two notions of Armendariz for these extensions: Σ -skew Armendariz (Definition 3.1) and a more general notion, the weak Σ -skew Armendariz (Definition 3.2). These definitions generalize the cases developed for both classical polynomial rings and Ore extensions of injective type (cf. [1], [2], [12], [14], [17], [19], [23], [24] and [25]). We show that every Σ -rigid ring ([28], Definition 3.2) is a Σ -skew Armendariz (Proposition 3.4), but the converse is false as Remark 3.5 shows. In other words, we have the relations Σ -rigid rings $\subsetneq \Sigma$ -skew Armendariz rings \subsetneq weak Σ -skew Armendariz rings. Nevertheless, in Theorem 3.6 we prove the following equivalences: for a skew PBW extension A of a ring R , R is reduced and Σ -skew Armendariz $\Leftrightarrow R$ is Σ -rigid $\Leftrightarrow A$ is reduced. In this way, our Theorem 3.6 generalizes [23], Theorem A, and [7], Theorem 1 and Corollary 3. In this section, we also present some key results with the aim of proving that if R is a weak Σ -skew Armendariz ring, then R and A are Abelian (Proposition 3.9 and Corollary 3.10, respectively). In Section 4, we characterize the weak Σ -skew Armendariz property over $Q(R)$ —the classical ring of quotients of R — in terms of the weak Σ -skew Armendariz property over R . This characterization is formulated in Theorem 4.2, and it turns out a generalization of [24], Theorem 2.3, and [17], Theorem 16. More generally than the characterization presented in Theorem 3.6, our Theorem 4.3 establishes the equivalence between the notions of Σ -rigid, Σ -skew Armendariz and weak Σ -skew Armendariz for the classical ring of quotients of a semiprime Goldie ring R , $Q(R)$, generalizing [24], Corollary 2.5.

Since Σ -rigid rings are contained strictly in weak Σ -skew Armendariz rings, in the last section of the paper, Section 5, we investigate the properties of being Baer, quasi-Baer, p.p. and p.q.-Baer (the interest of studying these properties comes from the question about annihilators of elements, as a key point to compute the uniform –also known as Goldie– dimension for these extensions (see [29]), for skew PBW extensions over weak Σ -skew Armendariz rings). In this way, our Theorems 5.1 and 5.3 generalizes [28], Theorems 3.9 and 3.12, respectively, and our Corollary 5.5 generalizes [28], Theorems 3.10 and 3.13. The techniques used here follow some ideas presentes in several articles on Ore extensions, and the results presented are new for skew PBW extensions and generalize others existing in the literature concerning Ore extensions of injective type and skew PBW extensions.

2. Definitions and elementary properties

In this section we recall the definition of skew PBW extension and present some key properties of these rings.

Definition 2.1 ([9], Definition 1). Let R and A be rings. We say that A is a skew PBW extension of R (also called a σ -PBW extension of R) if the following conditions hold:

- (i) $R \subseteq A$.
- (ii) there exist elements $x_1, \dots, x_n \in A$ such that A is a left free R -module, with basis the basic elements $\text{Mon}(A) := \{x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mid \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n\}$, and $x_1^0 \cdots x_n^0 := 1 \in \text{Mon}(A)$.
- (iii) For each $1 \leq i \leq n$ and any $r \in R \setminus \{0\}$, there exists an element $c_{i,r} \in R \setminus \{0\}$ such that $x_i r - c_{i,r} x_i \in R$.
- (iv) For any elements $1 \leq i, j \leq n$ there exists $c_{i,j} \in R \setminus \{0\}$ such that $x_j x_i - c_{i,j} x_i x_j \in R + R x_1 + \cdots + R x_n$.

Under these conditions we will write $A := \sigma(R)\langle x_1, \dots, x_n \rangle$.

Proposition 2.2 ([9], Proposition 3). Let A be a skew PBW extension of R . For each $1 \leq i \leq n$, there exists an injective endomorphism $\sigma_i : R \rightarrow R$ and a σ_i -derivation $\delta_i : R \rightarrow R$ such that $x_i r = \sigma_i(r) x_i + \delta_i(r)$, for each $r \in R$. We write $\Sigma := \{\sigma_1, \dots, \sigma_n\}$, and $\Delta := \{\delta_1, \dots, \delta_n\}$, that is, Δ is the family of Σ -derivations in A .

Remark 2.3. With respect to the Definition 2.1 and the Proposition 2.2, we have the following facts:

- (i) Since $\text{Mon}(A)$ is a left R -basis of A , the elements $c_{i,r}$ and $c_{i,j}$ in Definition 2.1 are unique.
- (ii) In Definition 2.1 (iv), $c_{i,i} = 1$. This follows from $x_i^2 - c_{i,i} x_i^2 = s_0 + s_1 x_1 + \cdots + s_n x_n$, with $s_i \in R$, which implies $1 - c_{i,i} = 0 = s_i$.
- (iii) If $i < j$ and $d'_i, b'_j \in R$, then $d'_i x_i b'_j x_j = d'_i [\sigma_i(b'_j) x_i + \delta_i(b'_j)] x_j = d'_i \sigma_i(b'_j) x_i x_j + d'_i \delta_i(b'_j) x_j$. Since $x_j x_i = c_{i,j} x_i x_j + r^{(i,j)} + \sum_{k=1}^n r_k^{(i,j)} x_k$, then $d'_j x_j b'_i x_i =$

$d'_j[\sigma_j(b'_i)x_j + \delta_j(b'_i)]x_i = d'_j\sigma_j(b'_i)x_jx_i + d'_j\delta_j(b'_i)x_i = d'_j\sigma_j(b'_i)(c_{i,j}x_ix_j + r^{(i,j)} + \sum_{k=1}^n r_k^{(i,j)}x_k) + d'_j\delta_j(b'_i)x_i$, we obtain

$$\begin{aligned} d'_ix_ib'_jx_j + d'_jx_jb'_ix_i &= d'_i\sigma_i(b'_j)x_ix_j + d'_i\delta_i(b'_j)x_j \\ &\quad + d'_j\sigma_j(b'_i)\left(c_{i,j}x_ix_j + r^{(i,j)} + \sum_{k=1}^n r_k^{(i,j)}x_k\right) + d'_j\delta_j(b'_i)x_i \\ &= [d'_i\sigma_i(b'_j) + d'_j\sigma_j(b'_i)c_{i,j}]x_ix_j + d'_j\delta_j(b'_i)x_i \\ &\quad + d'_i\delta_i(b'_j)x_j + d'_j\sigma_j(b'_i)r^{(i,j)} + d'_j\sigma_j(b'_i)\sum_{k=1}^n r_k^{(i,j)}x_k \\ &= [d'_i\sigma_i(b'_j) + d'_j\sigma_j(b'_i)c_{i,j}]x_ix_j + [d'_j\delta_j(b'_i) + d'_j\sigma_j(b'_i)r_i^{(i,j)}]x_i \\ &\quad + [d'_i\delta_i(b'_j) + d'_j\sigma_j(b'_i)]x_j + d'_j\sigma_j(b'_i)r^{(i,j)} \\ &\quad + d'_j\sigma_j(b'_i)\sum_{k=1, k \neq i,j}^n r_k^{(i,j)}x_k. \end{aligned}$$

Definition 2.4. Let A be a skew PBW extension of R .

- (i) A is called *quasi-commutative* if the conditions (iii) and (iv) in Definition 2.1 are replaced by (iii'): for each $1 \leq i \leq n$ and all $r \in R \setminus \{0\}$ there exists $c_{i,r} \in R \setminus \{0\}$ such that $x_ir = c_{i,r}x_i$; (iv)': for any $1 \leq i, j \leq n$ there exists $c_{i,j} \in R \setminus \{0\}$ such that $x_jx_i = c_{i,j}x_ix_j$ ([9], Definition 4).
- (ii) A is called *bijective* if σ_i is bijective for each $1 \leq i \leq n$, and $c_{i,j}$ is invertible for any $1 \leq i < j \leq n$ ([9], Definition 4).

Example 2.5. The class of skew PBW extensions contains various well-known groups of algebras such as some types of Auslander-Gorenstein rings, some Calabi-Yau and skew Calabi-Yau algebras, some Artin-Schelter regular algebras, some Koszul algebras (cf. [30]), quantum polynomials, some quantum universal enveloping algebras, etc. A detailed list of examples of skew PBW extensions is presented in [22], [26] or [27].

Definition 2.6 ([9], Definition 6). Let A be a skew PBW extension of R with endomorphisms σ_i , $1 \leq i \leq n$, as in Proposition 2.2.

- (i) For $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, $\sigma^\alpha := \sigma_1^{\alpha_1} \cdots \sigma_n^{\alpha_n}$, $|\alpha| := \alpha_1 + \cdots + \alpha_n$. If $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{N}^n$, then $\alpha + \beta := (\alpha_1 + \beta_1, \dots, \alpha_n + \beta_n)$.
- (ii) For $X = x^\alpha \in \text{Mon}(A)$, $\exp(X) := \alpha$, $\deg(X) := |\alpha|$, and $X_0 := 1$. The symbol \succeq will denote a total order defined on $\text{Mon}(A)$ (a total order on \mathbb{N}^n). For an element $x^\alpha \in \text{Mon}(A)$, $\exp(x^\alpha) := \alpha \in \mathbb{N}^n$. If $x^\alpha \succeq x^\beta$ but $x^\alpha \neq x^\beta$, we write $x^\alpha \succ x^\beta$. Every element $f \in A$ can be expressed uniquely as $f = a_0 + a_1X_1 + \cdots + a_mX_m$, with $a_i \in R$, and $X_m \succ \cdots \succ X_1$. With this notation, we define $\text{lm}(f) := X_m$, the *leading monomial* of f ; $\text{lc}(f) := a_m$, the *leading coefficient* of f ; $\text{lt}(f) := a_mX_m$, the *leading term* of f ; $\exp(f) := \exp(X_m)$, the *order* of f ; and $E(f) := \{\exp(X_i) \mid 1 \leq i \leq t\}$. Note that $\deg(f) := \max\{\deg(X_i)\}_{i=1}^t$. Finally, if $f = 0$, then $\text{lm}(0) := 0$, $\text{lc}(0) := 0$, $\text{lt}(0) := 0$. We also consider $X \succ 0$ for any $X \in \text{Mon}(A)$. For a detailed description of monomial orders in skew PBW extensions, see [9], Section 3.

Proposition 2.7 ([9], Theorem 7). *Let A be a polynomial ring with coefficients in R and the set of variables $\{x_1, \dots, x_n\}$. A is a skew PBW extension of R if and only if the following conditions are satisfied:*

- (i) *for each $x^\alpha \in \text{Mon}(A)$ and every $0 \neq r \in R$, there exist unique elements $r_\alpha := \sigma^\alpha(r) \in R \setminus \{0\}$, $p_{\alpha,r} \in A$, such that $x^\alpha r = r_\alpha x^\alpha + p_{\alpha,r}$, where $p_{\alpha,r} = 0$ or $\deg(p_{\alpha,r}) < |\alpha|$, if $p_{\alpha,r} \neq 0$. If r is left invertible, so is r_α .*
- (ii) *For each $x^\alpha, x^\beta \in \text{Mon}(A)$ there exist unique elements $c_{\alpha,\beta} \in R$ and $p_{\alpha,\beta} \in A$ such that $x^\alpha x^\beta = c_{\alpha,\beta} x^{\alpha+\beta} + p_{\alpha,\beta}$, where $c_{\alpha,\beta}$ is left invertible, $p_{\alpha,\beta} = 0$ or $\deg(p_{\alpha,\beta}) < |\alpha + \beta|$, if $p_{\alpha,\beta} \neq 0$.*

Remark 2.8. With respect to the Proposition 2.7, we have two observations:

- (i) ([28], Proposition 2.9) If $\alpha := (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ and $r \in R$, then

$$\begin{aligned} x^\alpha r &= x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_{n-1}^{\alpha_{n-1}} x_n^{\alpha_n} r = x_1^{\alpha_1} \cdots x_{n-1}^{\alpha_{n-1}} \left(\sum_{j=1}^{\alpha_n} x_n^{\alpha_n-j} \delta_n(\sigma_n^{j-1}(r)) x_n^{j-1} \right) \\ &+ x_1^{\alpha_1} \cdots x_{n-2}^{\alpha_{n-2}} \left(\sum_{j=1}^{\alpha_{n-1}} x_{n-1}^{\alpha_{n-1}-j} \delta_{n-1}(\sigma_{n-1}^{j-1}(\sigma_n^{\alpha_n}(r))) x_{n-1}^{j-1} \right) x_n^{\alpha_n} \\ &+ x_1^{\alpha_1} \cdots x_{n-3}^{\alpha_{n-3}} \left(\sum_{j=1}^{\alpha_{n-2}} x_{n-2}^{\alpha_{n-2}-j} \delta_{n-2}(\sigma_{n-2}^{j-1}(\sigma_{n-1}^{\alpha_{n-1}}(\sigma_n^{\alpha_n}(r)))) x_{n-2}^{j-1} \right) x_{n-1}^{\alpha_{n-1}} x_n^{\alpha_n} \\ &+ \cdots + x_1^{\alpha_1} \left(\sum_{j=1}^{\alpha_2} x_2^{\alpha_2-j} \delta_2(\sigma_2^{j-1}(\sigma_3^{\alpha_3}(\sigma_4^{\alpha_4}(\cdots(\sigma_n^{\alpha_n}(r))))) x_2^{j-1} \right) x_3^{\alpha_3} x_4^{\alpha_4} \cdots x_{n-1}^{\alpha_{n-1}} x_n^{\alpha_n} \\ &+ \sigma_1^{\alpha_1}(\sigma_2^{\alpha_2}(\cdots(\sigma_n^{\alpha_n}(r)))) x_1^{\alpha_1} \cdots x_n^{\alpha_n}, \quad \sigma_j^0 := \text{id}_R \text{ for } 1 \leq j \leq n. \end{aligned}$$

- (ii) ([28], Remark 2.10) Using (i), it follows that for the product $a_i X_i b_j Y_j$, if $X_i := x_1^{\alpha_{i1}} \cdots x_n^{\alpha_{in}}$ and $Y_j := x_1^{\beta_{j1}} \cdots x_n^{\beta_{jn}}$, then

$$\begin{aligned} a_i X_i b_j Y_j &= a_i \sigma^{\alpha_i}(b_j) x^{\alpha_i} x^{\beta_j} + a_i p_{\alpha_i, \sigma^{\alpha_i}(b_j)} x_2^{\alpha_{i2}} \cdots x_n^{\alpha_{in}} x^{\beta_j} \\ &+ a_i x_1^{\alpha_{i1}} p_{\alpha_{i2}, \sigma^{\alpha_i}(b_j)} x_3^{\alpha_{i3}} \cdots x_n^{\alpha_{in}} x^{\beta_j} \\ &+ a_i x_1^{\alpha_{i1}} x_2^{\alpha_{i2}} p_{\alpha_{i3}, \sigma^{\alpha_i}(b_j)} x_4^{\alpha_{i4}} \cdots x_n^{\alpha_{in}} x^{\beta_j} \\ &+ \cdots + a_i x_1^{\alpha_{i1}} x_2^{\alpha_{i2}} \cdots x_{i(n-2)}^{\alpha_{i(n-2)}} p_{\alpha_{i(n-1)}, \sigma^{\alpha_i}(b_j)} x_n^{\alpha_{in}} x^{\beta_j} \\ &+ a_i x_1^{\alpha_{i1}} \cdots x_{i(n-1)}^{\alpha_{i(n-1)}} p_{\alpha_{in}, b_j} x^{\beta_j}. \end{aligned}$$

In this way, when we compute every summand of $a_i X_i b_j Y_j$ we obtain products of the coefficient a_i with several evaluations of b_j in σ 's and δ 's, depending of the coordinates of α_i .

3. Σ -Skew Armendariz and weak Σ -skew Armendariz

In [12], p. 104, it was introduced the notion of Armendariz for Ore extensions of endomorphism type (i.e., the σ -derivation δ is the zero mapping) in the following way: let σ

be an endomorphism of a ring B . B is called a skew Armendariz ring with the endomorphism σ (simply, a σ -skew Armendariz ring) if for $p = \sum_{i=0}^m a_i x^i$, and $q = \sum_{j=0}^n b_j x^j$ in $B[x; \sigma]$, $pq = 0$ implies $a_i \sigma^i(b_j) = 0$ for all $0 \leq i \leq m$, and $0 \leq j \leq n$ (note that there exists an endomorphism σ of an Armendariz ring B such that B is not σ -skew Armendariz; see [12], Example 2). In [24], Definition 2.1, it was adapted this definition for general Ore extensions $B[x; \sigma, \delta]$ with δ not necessarily the zero mapping. With this in mind, and in the search of a first notion of Armendariz for skew PBW extensions, we introduce the following definition which extends both definitions in [12] and [24].

Definition 3.1. Let A be a skew PBW extension of a ring R . R is called a *skew Armendariz ring* with the family Σ (or a Σ -skew Armendariz ring) if for elements $f = \sum_{i=0}^m a_i X_i$ and $g = \sum_{j=0}^t b_j Y_j$ in A , the equality $fg = 0$ implies $a_i \sigma^{\alpha_i}(b_j) = 0$ for all $0 \leq i \leq m$ and $0 \leq j \leq t$, where $\alpha_i = \exp(X_i)$.

Next we define the weak Σ -skew Armendariz as a more general class of rings than Σ -skew Armendariz. Our Definition 3.2 extends [24], Definition 2.2.

Definition 3.2. Let A be a skew PBW extension of a ring R . R is called a *weak skew Armendariz ring* with the family Σ (or a weak Σ -skew Armendariz ring) if for elements $f = \sum_{i=0}^n a_i x_i$ and $g = \sum_{j=0}^n b_j x_j$ in A ($x_0 := 1$), the equality $fg = 0$ implies $a_i \sigma_i(b_j) = 0$ for all $0 \leq i, j \leq n$ ($\sigma_0 := \text{id}_R$).

We have the following preliminary result.

Lemma 3.3. *If R is a weak skew Armendariz ring, the equality $ab = 0$ implies $\sigma^\alpha(a)\delta^\alpha(b) = \delta^\alpha(a)b = 0$ for every $a, b \in R$ and $\alpha \in \mathbb{N}^n$.*

Proof. It is sufficient to prove the case $\sigma_i(a)\delta_i(b) = \delta_i(a)b = 0$ for $i = 1, \dots, n$. Since $ab = 0$, then $0 = \delta_i(ab) = \sigma_i(a)\delta_i(b) + \delta_i(a)b$, or equivalently, $\delta_i(a)b = -\sigma_i(a)\delta_i(b)$. Let $f, g \in A$ given by $f = \delta_i(a) + 0x_1 + \dots + 0x_{i-1} + \sigma_i(a)x_i + 0x_{i+1} + \dots + 0x_n$, and $g = b + bx_1 + \dots + bx_n$, respectively. Note that $fg = 0$;

$$\begin{aligned} fg &= \delta_i(a)b + \delta_i(a)bx_1 + \dots + \delta_i(a)bx_n + \sigma_i(a)x_i b + \sigma_i(a)x_i bx_1 + \dots + \sigma_i(a)x_i bx_n \\ &= \delta_i(a)b + \delta_i(a)bx_1 + \dots + \delta_i(a)bx_n + \sigma_i(a)[\sigma_i(b)x_i + \delta_i(b)] + \sigma_i(a)[\sigma_i(b)x_i + \delta_i(b)]x_1 \\ &\quad + \dots + \sigma_i(a)[\sigma_i(b)x_i + \delta_i(b)]x_n \\ &= \delta_i(a)b + \delta_i(a)bx_1 + \dots + \delta_i(a)bx_n + \sigma_i(a)\sigma_i(b)x_i + \sigma_i(a)\delta_i(b) + \sigma_i(a)\sigma_i(b)x_i x_1 \\ &\quad + \sigma_i(a)\delta_i(b)x_1 + \dots + \sigma_i(a)\sigma_i(b)x_i x_n + \sigma_i(a)\delta_i(b)x_n \\ &= 0, \end{aligned}$$

since $\delta_i(a)b = -\sigma_i(a)\delta_i(b)$ and $\sigma_i(a)\sigma_i(b) = \sigma_i(ab) = \sigma_i(0) = 0$. By the weak skew Armendariz condition on R , $\delta_i(a)b = 0$, and hence, $\sigma_i(a)\delta_i(b) = 0$ for every i . \square

For a ring B with a ring endomorphism $\sigma : B \rightarrow B$, and a σ -derivation $\delta : B \rightarrow B$, Krempa in [18] considered the Ore extension $B[x; \sigma, \delta]$ and defined σ as a *rigid endomorphism* if $b\sigma(b) = 0$ implies $b = 0$ for $b \in B$. Krempa called B σ -*rigid* if there exists a rigid endomorphism σ of B . Since Ore extensions of injective type are particular examples of skew PBW extensions, in [28], Definition 3.2, the first author introduced the following

definition with the purpose of studying the notion of *rigidness* for these extensions: if B is a ring and Σ is a family of endomorphisms of B , then Σ is called a *rigid endomorphisms family* if $r\sigma^\alpha(r) = 0$ implies $r = 0$ for every $r \in B$ and $\alpha \in \mathbb{N}^n$. B is called to be Σ -*rigid* if there exists a rigid endomorphisms family Σ of B .

The next proposition shows that every Σ -rigid ring is an Σ -skew Armendariz. From now on, we suppose that the elements $c_{i,j}$ in Definition 2.1 (iv) are in the center of R .

Proposition 3.4. *Let A be a skew PBW extension of a ring R . If R is Σ -rigid, then R is Σ -skew Armendariz.*

Proof. By [28], Proposition 3.6, we know that if $f = \sum_{i=0}^m a_i X_i$, $g = \sum_{j=0}^t b_j Y_j$ are elements of A , then $fg = 0$ if and only if $a_i b_j = 0$ for every $0 \leq i \leq m$ and $0 \leq j \leq t$. Now, [28], Lemma 3.3 (i) guarantees that $a_i \sigma^{\alpha_i}(b_j) = 0$ for every j , where $\alpha_i = \exp(X_i)$, that is, R is Σ -skew Armendariz. \square

Remark 3.5. The converse of Proposition 3.4 is false as the following examples show:

- Consider the ring

$$B = \left\{ \begin{pmatrix} a & t \\ 0 & a \end{pmatrix} \mid a \in \mathbb{Z}, t \in \mathbb{Q} \right\}.$$

Then B is a commutative ring. Let σ be the automorphism of R given by $\sigma\left(\begin{pmatrix} a & t \\ 0 & a \end{pmatrix}\right) = \begin{pmatrix} a & t/2 \\ 0 & a \end{pmatrix}$. In [12], Example 1, it was shown that R is σ -skew Armendariz and is not a σ -rigid. Since Σ -rigid and Σ -skew Armendariz are generalizations of σ -rigid and σ -skew Armendariz, respectively, this example shows that the converse of Proposition 3.4 is false.

- Let $B = \mathbb{Z}_2[x]$ be the commutative polynomial ring over \mathbb{Z}_2 , and σ the endomorphism of $B = \mathbb{Z}_2[x]$ defined by $\sigma(f(x)) = f(0)$. Then $B = \mathbb{Z}_2[x]$ is σ -skew Armendariz and is not σ -rigid ([12], Example 5).

Therefore we have the relations

$$\Sigma\text{-rigid rings} \subsetneq \Sigma\text{-skew Armendariz rings} \subsetneq \text{weak } \Sigma\text{-skew Armendariz rings.} \quad (1)$$

In [7] and [23], both authors of those papers give a positive answer to the following question formulated in [12], p. 115: Let σ be a monomorphism (or automorphism) of a (commutative) reduced ring B and B be a σ -skew Armendariz. Is B σ -rigid? The content of Theorem 3.6 is the generalization of this answer to skew PBW extensions. Again, we suppose that the elements $c_{i,j}$ in Definition 2.1 (iv) commute with every element of R .

Theorem 3.6. *If A is a skew PBW extension of a ring R , then the following statements are equivalent:*

- (i) R is reduced and Σ -skew Armendariz;
- (ii) R is Σ -rigid;
- (iii) A is reduced.

Proof. (ii) \Leftrightarrow (iii). This equivalence follows from [28], Proposition 3.5. (ii) \Rightarrow (i) From [28] we know that a Σ -rigid ring is reduced, and as we saw above, every Σ -rigid ring is also Σ -skew Armendariz. Let us see (i) \Rightarrow (ii). Suppose that R is reduced, Σ -skew Armendariz and is not Σ -rigid. Then there exists $\beta \in \mathbb{N}^n$ with $a\sigma^\beta(a) = 0$ and $a \neq 0$. Note that $\sigma^\beta(a)\sigma^\beta(\sigma^\beta(a)) = \sigma^\beta(a\sigma^\beta(a)) = 0$. Using that R is reduced, the equality $(\sigma^\beta(a)a)^2 = \sigma^\beta(a)a\sigma^\beta(a)a = 0$ implies $\sigma^\beta(a)a = 0$. Equivalently, since $a \neq 0$, σ^β is injective, and R is reduced, then $\sigma^\beta(a) \neq 0$ and $(\sigma^\beta(a))^2 \neq 0$. With this in mind, consider the elements $f = \sigma^\beta(a) + \sigma^\beta(a)x^\beta$, $g = a - \sigma^\beta(a)x^\beta$. Then

$$\begin{aligned} fg &= (\sigma^\beta(a) + \sigma^\beta(a)x^\beta)(a - \sigma^\beta(a)x^\beta) \\ &= \sigma^\beta(a)a - (\sigma^\beta(a))^2x^\beta + \sigma^\beta(a)x^\beta a - \sigma^\beta(a)x^\beta \sigma^\beta(a)x^\beta \\ &= -(\sigma^\beta(a))^2x^\beta + \sigma^\beta(a)[\sigma^\beta(a)x^\beta + p_{\beta,a}] - \sigma^\beta(a)[\sigma^\beta(\sigma^\beta(a))x^\beta + q_{\beta,\sigma^\beta(a)}]x^\beta \\ &= \sigma^\beta(a)p_{\beta,a} - \sigma^\beta(a\sigma^\beta(a))x^\beta x^\beta - \sigma^\beta(a)q_{\beta,\sigma^\beta(a)}x^\beta \\ &= \sigma^\beta(a)p_{\beta,a} - \sigma^\beta(a)q_{\beta,\sigma^\beta(a)}x^\beta, \end{aligned}$$

where $p_{\beta,a} = 0$ or $\deg(p_{\beta,a}) < |\beta|$, if $p_{\beta,r} \neq 0$, and $q_{\beta,\sigma^\beta(a)} = 0$ or $\deg(q_{\beta,\sigma^\beta(a)}) < |\beta|$, if $q_{\beta,\sigma^\beta(a)} \neq 0$. Since $a\sigma^\beta(a) = \sigma^\beta(a)a = 0$, Remark 2.8 and Lemma 3.3 guarantee that $\sigma^\beta(a)p_{\beta,a} = \sigma^\beta(a)q_{\beta,\sigma^\beta(a)}x^\beta = 0$, so $fg = 0$. By assumption, R is Σ -skew Armendariz, that is, $-(\sigma^\beta(a))^2 = 0$, but $-(\sigma^\beta(a))^2 \neq 0$, i.e., we have obtained a contradiction. Hence, R is Σ -rigid. \square

Next we present others key results about Σ -skew Armendariz and weak Σ -skew Armendariz rings.

Proposition 3.7. *If A is a skew PBW extension of a Σ -skew Armendariz ring R , $e^2 = e \in A$, with $e = \sum_{i=0}^m e_i X_i$, then $e = e_0$.*

Proof. Using the equalities $e(1-e) = 0 = (1-e)e$, i.e., $(\sum_{i=0}^m e_i X_i)((1-e_0) - \sum_{i=1}^m X_i) = 0$ and $((1-e_0) - \sum_{i=1}^m e_i X_i)(\sum_{i=0}^m e_i X_i) = 0$, and the assumption on R , then $e_0(1-e_0) = 0$, $e_0 e_i = 0$ and $(1-e_0)e_i = 0$ for $1 \leq i \leq m$. Therefore, $e_i = 0$ for $1 \leq i \leq m$, that is, $e = e_0 = e_0^2$. \square

Proposition 3.8. *If R is a weak Σ -skew Armendariz ring, and e is an idempotent element of R , then $\sigma_i(e) = e$ and $\delta_i(e) = 0$, for every $i = 1, \dots, n$.*

Proof. Consider an idempotent element e of R . Then $\delta_i(e) = \sigma_i(e)\delta_i(e) + \delta_i(e)e$. Let $f, g \in A$ given by $f = \delta_i(e) + 0x_1 + \dots + 0x_{i-1} + \sigma_i(e)x_i + 0x_{i+1} + \dots + 0x_n$, and $g = e - 1 + (e-1)x_1 + \dots + (e-1)x_n$, respectively. Recall that $\delta_i(1) = 0$ for every i . Let us show that $fg = 0$:

$$\begin{aligned} fg &= \delta_i(e)(e-1) + \sum_{j=1}^n \delta_i(e)(e-1)x_j + \sigma_i(e)x_i(e-1) + \sum_{j=1}^n \sigma_i(e)x_i(e-1)x_j \\ &= \delta_i(e)(e-1) + \sum_{j=1}^n \delta_i(e)(e-1)x_j + \sigma_i(e)[\sigma_i(e-1)x_i + \delta_i(e-1)] \\ &\quad + \sum_{j=1}^n \sigma_i(e)[\sigma_i(e-1)x_i + \delta_i(e-1)]x_j. \end{aligned}$$

Equivalently,

$$\begin{aligned}
fg &= \delta_i(e)(e-1) + \sum_{j=1}^n \delta_i(e)(e-1)x_j + \sigma_i(e)[(\sigma_i(e) - \sigma_i(1))x_i + \delta_i(e)] \\
&\quad + \sum_{j=1}^n \sigma_i(e)[(\sigma_i(e) - \sigma_i(1))x_i + \delta_i(e)]x_j \\
&= \delta_i(e)(e-1) + \sum_{j=1}^n \delta_i(e)(e-1)x_j + \sigma_i(e)[\sigma_i(e)x_i - x_i + \delta_i(e)] \\
&\quad + \sum_{j=1}^n \sigma_i(e)[\sigma_i(e)x_i - x_i + \delta_i(e)]x_j \\
&= \delta_i(e)e - \delta_i(e) + \sum_{j=1}^n (\delta_i(e)e - \delta_i(e))x_j + \sigma_i(e)x_i - \sigma_i(e)x_i + \sigma_i(e)\delta_i(e) \\
&\quad + \sum_{j=1}^n (\sigma_i(e)x_i - \sigma_i(e)x_i + \sigma_i(e)\delta_i(e))x_j \\
&= \delta_i(e)e - \delta_i(e) + \sum_{j=1}^n \delta_i(e)ex_j - \sum_{j=1}^n \delta_i(e)x_j + \sigma_i(e)\delta_i(e) + \sum_{j=1}^n \sigma_i(e)\delta_i(e)x_j \\
&= \sigma_i(e)\delta_i(e) + \delta_i(e)e - \delta_i(e) + \left(\sum_{j=1}^n (\sigma_i(e)\delta_i(e) + \delta_i(e)e - \delta_i(e)) \right) x_j \\
&= 0.
\end{aligned}$$

Since R is weak Σ -skew Armendariz, we obtain $\delta_i(e)(e-1) = 0$, i.e., $\delta_i(e)e = \delta_i(e)$, and hence $\sigma_i(e)\delta_i(e) = 0$.

Now, consider the elements s and t of A given by $s = \delta_i(e) - (1 - \sigma_i(e))x_i$ and $t = e + \sum_{j=1}^n ex_j$, respectively. Then $st = 0$. In fact:

$$\begin{aligned}
st &= \delta_i(e)e + \delta_i(e)e \sum_{j=1}^n x_j - (1 - \sigma_i(e))x_i e - (1 - \sigma_i(e))x_i e \sum_{j=1}^n x_j \\
&= \delta_i(e)e + \delta_i(e)e \sum_{j=1}^n x_j - x_i e + \sigma_i(e)x_i e - x_i e \sum_{j=1}^n x_j + \sigma_i(e)x_i e \sum_{j=1}^n x_j \\
&= \delta_i(e)e + \delta_i(e)e \sum_{j=1}^n x_j - (\sigma_i(e)x_i + \delta_i(e)) + \sigma_i(e)(\sigma_i(e)x_i + \delta_i(e)) \\
&\quad - (\sigma_i(e)x_i + \delta_i(e)) \sum_{j=1}^n x_j + \sigma_i(e)(\sigma_i(e)x_i + \delta_i(e)) \sum_{j=1}^n x_j \\
&= \delta_i(e)e + \delta_i(e)e \sum_{j=1}^n x_j - \sigma_i(e)x_i - \delta_i(e) + \sigma_i(e)x_i + \sigma_i(e)\delta_i(e) - \sigma_i(e)x_i \sum_{j=1}^n x_j \\
&\quad - \delta_i(e) \sum_{j=1}^n x_j + \sigma_i(e)x_i \sum_{j=1}^n x_j + \sigma_i(e)\delta_i(e) \sum_{j=1}^n x_j.
\end{aligned}$$

Since $\delta_i(e) = \delta_i(e)e$ and $\sigma_i(e)\delta_i(e) = 0$, then $st = 0$. By Armendariz condition we know that $\delta_i(e)e = 0$, which shows that $\delta_i(e) = 0$.

Consider the elements $u, v \in A$ given by $u = 1 - e + (1 - e)\sigma_i(e)x_i$ and $v = e + (e - 1)\sigma_i(e)x_i$. Then

$$\begin{aligned}
 uv &= e + (e - 1)\sigma_i(e)x_i - e^2 - e(e - 1)\sigma_i(e)x_i + (1 - e)\sigma_i(e)x_ie \\
 &\quad + (1 - e)\sigma_i(e)x_i(e - 1)\sigma_i(e)x_i \\
 &= e\sigma_i(e)x_i - \sigma_i(e)x_i - e\sigma_i(e)x_i + e\sigma_i(e)x_i + (1 - e)\sigma_i(e)(\sigma_i(e)x_i + \delta_i(e)) \\
 &\quad + (1 - e)\sigma_i(e)(\sigma_i(e)x_i - x_i + \delta_i(e))\sigma_i(e)x_i \\
 &= -\sigma_i(e)x_i + e\sigma_i(e)x_i + \sigma_i(e)x_i + \sigma_i(e)\delta_i(e) - e\sigma_i(e)x_i - e\sigma_i(e)\delta_i(e) \\
 &\quad + [\sigma_i(e)x_i - \sigma_i(e)x_i + \sigma_i(e)\delta_i(e) - e\sigma_i(e)x_i + e\sigma_i(e)x_i - e\sigma_i(e)\delta_i(e)]\sigma_i(e)x_i \\
 &= 0.
 \end{aligned}$$

Hence, by the Armendariz condition, $(1 - e)(e - 1)\sigma_i(e) = 0$, i.e., $e\sigma_i(e) = \sigma_i(e)$.

Now, let $w = e + e(1 - \sigma_i(e))x_i$, $z = 1 - e - e(1 - \sigma_i(e))x_i$ be elements of A . Then

$$\begin{aligned}
 wz &= e - e^2 - e^2(1 - \sigma_i(e))x_i + e(1 - \sigma_i(e))x_i - e(1 - \sigma_i(e))x_ie \\
 &\quad - e(1 - \sigma_i(e))x_ie(1 - \sigma_i(e))x_i \\
 &= -e(1 - \sigma_i(e))x_ie[1 + (1 - \sigma_i(e))x_i] \\
 &= -e(1 - \sigma_i(e))(\sigma_i(e)x_i + \delta_i(e))[1 + (1 - \sigma_i(e))x_i] \\
 &= (-e\sigma_i(e)x_i + e\sigma_i(e)x_i)[1 + (1 - \sigma_i(e))x_i] \\
 &= 0,
 \end{aligned}$$

since $\delta_i(e) = 0$ and $\sigma_i(e)\sigma_i(e) = \sigma_i(e)$. Then, Armendariz condition implies $e(-e(1 - \sigma_i(e))) = 0$, which shows that $e\sigma_i(e) = e$, and so $\sigma_i(e) = e$. Therefore, we conclude the proof. \square

Next, we show that every weak Σ -skew Armendariz ring is Abelian.

Proposition 3.9. *If R is a weak Σ -skew Armendariz ring, then R is Abelian.*

Proof. Let $e^2 = e$, $a \in R$. Consider the elements f, g of A given by $f = e - \sum_{i=1}^n ea(1 - e)x_i$, and $g = 1 - e + \sum_{i=1}^n ea(1 - e)x_i$. Since

$$\begin{aligned}
 fg &= e - e^2 + e^2a(1 - e) \sum_{i=1}^n x_i - \sum_{i=1}^n ea(1 - e)x_i + \sum_{i=1}^n ea(1 - e)x_ie \\
 &\quad - \left(\sum_{i=1}^n ea(1 - e)x_iea(1 - e) \right) \left(\sum_{i=1}^n x_i \right) \\
 &= \sum_{i=1}^n ea(1 - e)(\sigma_i(e)x_i + \delta_i(e)) - \left(\sum_{i=1}^n ea(1 - e)x_i(ea - eae) \right) \left(\sum_{i=1}^n x_i \right).
 \end{aligned}$$

Equivalently,

$$\begin{aligned}
 fg &= (ea - eae) \sum_{i=1}^n \sigma_i(e)x_i \\
 &\quad - \left(\sum_{i=1}^n (ea - eae)(\sigma_i(ea - eae)x_i + \delta_i(ea - eae)) \right) \left(\sum_{i=1}^n x_i \right) \\
 &= (ea - eae) \sum_{i=1}^n \sigma_i(e)x_i \\
 &\quad - \left(\sum_{i=1}^n (ea - eae)((e\sigma_i(a) - e\sigma_i(a)e)x_i + \delta_i(ea) - \delta_i(eae)) \right) \left(\sum_{i=1}^n x_i \right) \\
 &= ea \sum_{i=1}^n \sigma_i(e)x_i - eae \sum_{i=1}^n \sigma_i(e)x_i \\
 &\quad - \left(\sum_{i=1}^n (ea - eae)(e\sigma_i(a)x_i - e\sigma_i(a)ex_i + \sigma_i(e)\delta_i(a) \right. \\
 &\quad \left. - \sigma_i(ea)\delta_i(e) - \delta_i(ea)e) \right) \left(\sum_{i=1}^n x_i \right),
 \end{aligned}$$

or what is the same,

$$\begin{aligned}
 fg &= ea \sum_{i=1}^n \sigma_i(e)x_i - eae \sum_{i=1}^n \sigma_i(e)x_i \\
 &\quad - \left(\sum_{i=1}^n (ea - eae)(e\sigma_i(a)x_i - e\sigma_i(a)ex_i + e\delta_i(a) - e\delta_i(a)e) \right) \left(\sum_{i=1}^n x_i \right) \\
 &= ea \sum_{i=1}^n \sigma_i(e)x_i - eae \sum_{i=1}^n \sigma_i(e)x_i - \left(\sum_{i=1}^n \{eae\sigma_i(a)x_i - eae\sigma_i(a)ex_i + eae\delta_i(a) \right. \\
 &\quad \left. - eae\delta_i(a)e - eae\sigma_i(a)x_i + eae\sigma_i(a)ex_i - eae\delta_i(a) + eae\delta_i(a)e\} \right) \left(\sum_{i=1}^n x_i \right) \\
 &= 0.
 \end{aligned}$$

Since R is weak Σ -skew Armendariz, $eea(1-e) = 0$, that is, $ea = eae$. Now, consider the elements p, q of A given by $p = 1 - e - \sum_{i=1}^n (1-e)aex_i$ and $q = e + \sum_{i=1}^n (1-e)aex_i$. Then $pq = 0$. More exactly,

$$\begin{aligned}
 pq &= e + \sum_{i=1}^n (1-e)aex_i - e^2 - e \sum_{i=1}^n (1-e)aex_i - \sum_{i=1}^n (1-e)aex_i e \\
 &\quad - \left(\sum_{i=1}^n (1-e)aex_i \right) \left(\sum_{i=1}^n (1-e)aex_i \right),
 \end{aligned}$$

or what is the same,

$$\begin{aligned}
 pq &= \sum_{i=1}^n (1-e)ae x_i - \sum_{i=1}^n (1-e)ae(\sigma_i(e)x_i + \delta_i(e)) \\
 &\quad - \left(\sum_{i=1}^n (1-e)ae x_i (1-e)ae \right) \left(\sum_{i=1}^n x_i \right) \\
 &= - \left(\sum_{i=1}^n (ae - eae)(\sigma_i(ae - eae)x_i + \delta_i(ae - eae)) \right) \left(\sum_{i=1}^n x_i \right) \\
 &= - \left(\sum_{i=1}^n (ae - eae)(\sigma_i(a)ex_i - e\sigma_i(a)ex_i + \delta_i(a)e - \delta_i(ea)e) \right) \left(\sum_{i=1}^n x_i \right) \\
 &= - \left(\sum_{i=1}^n \{ae\sigma_i(a)ex_i - ae\sigma_i(a)ex_i + ae\delta_i(a)e - ae\sigma_i(e)\delta_i(a)e - eae\sigma_i(a)ex_i \right. \\
 &\quad \left. + eae\sigma_i(a)ex_i - eae\delta_i(a)e + eae\sigma_i(e)\delta_i(a)e \} \right) \left(\sum_{i=1}^n x_i \right) \\
 &= 0,
 \end{aligned}$$

since $e\sigma_i(e) = \sigma_i(e) = e$. By the weak Σ -skew Armendariz condition on R , we know that $(1-e)(1-e)ae = 0$, or equivalently, $ae = eae$. Now, as it was shown above, $ea = eae$, which means that $ae = ea$, i.e., R is Abelian. \square

Propositions 3.7, 3.8, and 3.9 imply the following corollary.

Corollary 3.10. *If R is a Σ -skew Armendariz ring R , then A is an Abelian ring.*

4. Classical ring of quotients

In this section we will characterize the Armendariz property for the classical ring of quotients of a bijective skew PBW extension. The key concept is the weak Σ -skew Armendariz property. Theorem 4.2 extends [24], Theorem 2.3, and [17], Theorem 16, and Theorem 4.3 generalizes [24], Corollary 2.5.

Let us recall the key facts about noncommutative localization. If B is a ring and S is a multiplicative subset of B ($1 \in S$, $0 \notin S$, $ss' \in S$ for every $s, s' \in S$), then the left ring of fractions of B exists if and only if two conditions hold: (i) given $a \in B$ and $s \in S$ with $as = 0$, there exists $s' \in S$ such that $s'a = 0$; (ii) (left Ore condition) given $a \in B$ and $s \in S$, there exist $s' \in S$ and $a' \in B$ with $s'a = a's$. If these conditions hold, then the left ring of fractions of B with respect to S is denoted by $S^{-1}B$, and its elements are classes denoted using fractions. More exactly, $\frac{a}{s} := \frac{b}{t}$ are equal if and only if there exist $c, d \in B$ such that $ca = db$, $cs = dt \in S$; $\frac{a}{s} + \frac{b}{t} := \frac{ca+db}{u}$, where $u := cs = dt \in S$, for some $c, d \in B$; $\frac{a}{s} \frac{b}{t} := \frac{cb}{us}$, where $ua = ct$, for some $u \in S$ and $c \in B$. Similarly, it is defined the right Ore condition, and hence the ring of fractions of B . The nonzero divisors elements of B are called *regular* and the set of regular elements of B is denoted by $S_0(B)$. Recall that if B is both left and right Ore, then its classical left ring of quotients $Q_{cl}^l(B)$ and its

classical right ring of quotients $Q_{cl}^r(B)$ coincide, and it is denoted by $Q(B)$. A key result about the classical ring of quotients of B is the common denominator property: if B is a ring, $S \subset B$ is a multiplicative subset and $S^{-1}B$ exists, then any finite set $\{q_1, \dots, q_n\}$ of elements of $S^{-1}B$ posses a common denominator, i.e., there exist $r_1, \dots, r_n \in B$ and $s \in S$ such that $q_i = \frac{r_i}{s}$ for every i (see [15] for a detailed treatment of localization in noncommutative rings).

Proposition 4.1 ([21], Lemma 2.6). *Let A be a bijective skew PBW extension of a ring R . If $S \subseteq S_0(R)$ is a multiplicative subset of R with $\sigma_i(S) = S$ for every $i = 1, \dots, n$, then*

- (a) *If $S^{-1}R$ exists, then $S^{-1}A$ exists and it is a bijective skew PBW extension of $S^{-1}R$, denoted $S^{-1}A = \sigma(S^{-1}R)\langle x'_1, \dots, x'_n \rangle$, where $x'_i := \frac{x_i}{1}$, and the systems of constants of $S^{-1}R$ is given by $c'_{i,j} = \frac{c_{i,j}}{1}$, $c'_{i,\frac{r}{s}} := \frac{\sigma_i(r)}{\sigma_i(s)}$, for $1 \leq i, j \leq n$. The automorphisms $\overline{\sigma}_i$ of $S^{-1}R$ and the $\overline{\sigma}_i$ -derivations $\overline{\delta}_i$ ($1 \leq i \leq n$), are defined by $\overline{\sigma}_i(\frac{a}{s}) := \frac{\sigma_i(a)}{\sigma_i(s)}$, and $\overline{\delta}_i(\frac{a}{s}) := -\frac{\delta_i(s)}{\sigma_i(s)} \frac{a}{s} + \frac{\delta_i(a)}{\sigma_i(s)}$. Let $\overline{\Sigma} := \{\overline{\sigma}_1, \dots, \overline{\sigma}_n\}$ and $\overline{\Delta} := \{\overline{\delta}_1, \dots, \overline{\delta}_n\}$.*
- (b) *If RS^{-1} exists, then AS^{-1} exists and it is a bijective skew PBW extension of RS^{-1} , denoted $AS^{-1} = \sigma(RS^{-1})\langle x''_1, \dots, x''_n \rangle$, where $x''_i := \frac{x_i}{1}$, and the systems of constants of $S^{-1}R$ is given by $c''_{i,j} = \frac{c_{i,j}}{1}$, $c''_{i,\frac{r}{s}} := \frac{\sigma_i(r)}{\sigma_i(s)}$, for $1 \leq i, j \leq n$. The automorphisms $\overline{\sigma}_i$ of $S^{-1}R$ and the $\overline{\sigma}_i$ -derivations $\overline{\delta}_i$ ($1 \leq i \leq n$), are defined by $\overline{\sigma}_i(\frac{a}{s}) := \frac{\sigma_i(a)}{\sigma_i(s)}$, and $\overline{\delta}_i(\frac{a}{s}) := -\frac{\sigma_i(a)}{\sigma_i(s)} \frac{\delta_i(s)}{s} + \frac{\delta_i(a)}{s}$. Let $\overline{\Sigma} := \{\overline{\sigma}_1, \dots, \overline{\sigma}_n\}$ and $\overline{\Delta} := \{\overline{\delta}_1, \dots, \overline{\delta}_n\}$.*

If no confusion arises, we simply denote x'_i and x''_i by x_i for $1 \leq i \leq n$. Now, analogously to the definitions of Σ -rigid, Σ -skew Armendariz and weak Σ -skew Armendariz, we consider these notions, i.e., $\overline{\Sigma}$ -rigid, $\overline{\Sigma}$ -skew Armendariz and weak $\overline{\Sigma}$ -skew Armendariz, for the classical quotient ring $Q(R)$ of R .

Next, Theorem 4.2 generalizes [24], Theorem 2.3, for Ore extensions of automorphism type, and hence, [17], Theorem 16, for reduced rings.

Theorem 4.2. *Let A be a bijective skew PBW extension of a ring R . If the classical ring of quotients $Q(R)$ of R exists, then R is weak Σ -skew Armendariz if and only if $Q(R)$ is weak $\overline{\Sigma}$ -skew Armendariz.*

Proof. It is clear that if $Q(R)$ is weak $\overline{\Sigma}$ -skew Armendariz, then R is weak Σ -skew Armendariz.

Conversely, consider $f = c_0^{-1}a_0 + \sum_{i=1}^n c_i^{-1}a_i x_i$ and $g = s_0^{-1}b_0 + \sum_{j=1}^n s_j^{-1}b_j x_j$ elements of $S^{-1}A$ such that $fg = 0$. Let us prove that $c_i^{-1}a_i \overline{\sigma}_i(s_j^{-1}b_j) = 0$ for $0 \leq i, j \leq n$, where $\overline{\sigma}_0 = \text{id}_{S^{-1}A}$.

We know that there exist $a'_i, b'_j \in R$ and $c, s \in S_0(R)$ satisfying $c_i^{-1}a_i = c^{-1}a'_i$ and

$s_j^{-1}b_j = s^{-1}b'_j$ for $0 \leq i, j \leq n$. In this way, we can write

$$\begin{aligned}
0 &= \left(c^{-1}a'_0 + \sum_{i=1}^n c^{-1}a'_i x_i \right) \left(s^{-1}b'_0 + \sum_{j=1}^n s^{-1}b'_j x_j \right) \\
&= \left(a'_0 + \sum_{i=1}^n a'_i x_i \right) s^{-1} \left(b'_0 + \sum_{j=1}^n b'_j x_j \right) \\
&= \left(a'_0 s^{-1} + \sum_{i=1}^n a'_i [\overline{\sigma_i}(s^{-1})x_i + \overline{\delta_i}(s^{-1})] \right) \left(b'_0 + \sum_{j=1}^n b'_j x_j \right) \\
&= \left(a'_0 s^{-1} + \sum_{i=1}^n (a'_i \sigma_i(s)^{-1}x_i - a'_i \sigma_i(s)^{-1}\delta_i(s)s^{-1}) \right) \left(b'_0 + \sum_{j=1}^n b'_j x_j \right).
\end{aligned}$$

There exist $d_i \in R$ ($1 \leq i \leq n$) and $s_2 \in S_0(R)$ such that $\delta_i(s)s^{-1} = s_2^{-1}d_i$, which shows that

$$0 = \left(a'_0 s^{-1} + \sum_{i=1}^n (a'_i \sigma_i(s)^{-1}x_i - a'_i \sigma_i(s)^{-1}s_2^{-1}d_i) \right) \left(b'_0 + \sum_{j=1}^n b'_j x_j \right).$$

Since there exist $a''_0, a''_i, a'''_i \in R$ ($1 \leq i \leq n$) and $s_3, s_4, s_5 \in S_0(R)$ with $a'_0 s^{-1} = s_3^{-1}a''_0$, $a'_i \sigma_i(s)^{-1} = s_4^{-1}a''_i$, and $a'_i \sigma_i(s)^{-1}s_2^{-1} = s_5^{-1}a'''_i$, we have

$$0 = \left(s_3^{-1}a''_0 + \sum_{i=1}^n (s_4^{-1}a''_i x_i - s_5^{-1}a'''_i d_i) \right) \left(b'_0 + \sum_{j=1}^n b'_j x_j \right).$$

Again, there exist $d_0, d'_i, d''_i \in R$ ($1 \leq i \leq n$) and $t \in S_0(R)$ with $s_3^{-1}a''_0 = t^{-1}d_0$, $s_4^{-1}a''_i = t^{-1}d'_i$, and $s_5^{-1}a'''_i = t^{-1}d''_i$, which guarantees that

$$\begin{aligned}
0 &= \left(t^{-1}d_0 + \sum_{i=1}^n (t^{-1}d'_i x_i - t^{-1}d''_i d_i) \right) \left(b'_0 + \sum_{j=1}^n b'_j x_j \right) \\
&= t^{-1} \left(d_0 + \sum_{i=1}^n (d'_i x_i - d''_i d_i) \right) \left(b'_0 + \sum_{j=1}^n b'_j x_j \right) \\
&= \left(d_0 - \sum_{i=1}^n d''_i d_i + \sum_{i=1}^n d'_i x_i \right) \left(b'_0 + \sum_{j=1}^n b'_j x_j \right). \tag{2}
\end{aligned}$$

By the Armendariz condition on R , from (2) we obtain the relations given by

$$\left(d_0 - \sum_{i=1}^n d''_i d_i \right) b'_j = 0 \quad (0 \leq j \leq n), \text{ and } d'_i \sigma_i(b'_j) = 0 \quad (1 \leq i \leq n) \quad (0 \leq j \leq n). \tag{3}$$

From the reasoning above we have the following equivalences for expressions in (3):

$$\begin{aligned}
 \left(d_0 - \sum_{i=1}^n d_i'' d_i\right) b_j' = 0 &\Leftrightarrow t^{-1} \left(d_0 - \sum_{i=1}^n d_i'' d_i\right) b_j' = 0 \\
 &\Leftrightarrow \left(t^{-1} d_0 - \sum_{i=1}^n t^{-1} d_i'' d_i\right) b_j' = 0 \\
 &\Leftrightarrow \left(s_3^{-1} a_0'' - \sum_{i=1}^n s_5^{-1} a_i''' d_i\right) b_j' = 0 \\
 &\Leftrightarrow \left(a_0' s^{-1} - \sum_{i=1}^n a_i' \sigma_i(s)^{-1} s_2^{-1} d_i\right) b_j' = 0 \\
 &\Leftrightarrow \left(a_0' s^{-1} - \sum_{i=1}^n a_i' \sigma_i(s)^{-1} \delta_i(s) s^{-1}\right) b_j' = 0 \\
 &\Leftrightarrow \left(a_0' s^{-1} + \sum_{i=1}^n a_i' \overline{\sigma}_i(s^{-1})\right) b_j' = 0 \quad (0 \leq j \leq n), \quad (4)
 \end{aligned}$$

and

$$\begin{aligned}
 d_i' \sigma_i(b_j') = 0 &\Leftrightarrow t^{-1} d_i' \sigma_i(b_j') = 0 \Leftrightarrow s_4^{-1} a_i'' \sigma_i(b_j') = 0 \\
 &\Leftrightarrow a_i' \sigma_i(s)^{-1} \sigma_i(b_j') = 0 \Leftrightarrow a_i' \overline{\sigma}_i(s^{-1}) \sigma_i(b_j') = 0 \\
 &\Leftrightarrow a_i' \overline{\sigma}_i(s^{-1}) \overline{\sigma}_i(b_j') = 0 \quad (1 \leq i \leq n) \quad (0 \leq j \leq n) \quad (5) \\
 &\Leftrightarrow c^{-1} a_i' \overline{\sigma}_i(s^{-1}) \overline{\sigma}_i(b_j') = 0 \Leftrightarrow c_i^{-1} a_i \overline{\sigma}_i(s^{-1}) \overline{\sigma}_i(b_j') = 0 \\
 &\Leftrightarrow c_i^{-1} a_i \overline{\sigma}_i(s^{-1} b_j') = 0 \Leftrightarrow c_i^{-1} a_i \overline{\sigma}_i(s_j^{-1} b_j) = 0 \quad (1 \leq i \leq n) \quad (0 \leq j \leq n). \quad (6)
 \end{aligned}$$

Now, from (2) we have

$$\begin{aligned}
 0 &= \left(d_0 - \sum_{i=1}^n d_i'' d_i\right) b_0' + \left(d_0 - \sum_{i=1}^n d_i'' d_i\right) \left(\sum_{j=1}^n b_j' x_j\right) \\
 &\quad + \left(\sum_{i=1}^n d_i' x_i\right) b_0' + \left(\sum_{i=1}^n d_i' x_i\right) \left(\sum_{j=1}^n b_j' x_j\right) \\
 &= \left(d_0 - \sum_{i=1}^n d_i'' d_i\right) b_0' + \left(d_0 - \sum_{i=1}^n d_i'' d_i\right) \left(\sum_{j=1}^n b_j' x_j\right) \\
 &\quad + \sum_{i=1}^n d_i' [\sigma_i(b_0') x_i + \delta_i(b_0')] + \left(\sum_{i=1}^n d_i' x_i\right) \left(\sum_{j=1}^n b_j' x_j\right) \\
 &= \left(d_0 - \sum_{i=1}^n d_i'' d_i\right) b_0' + \sum_{j=1}^n \left(d_0 - \sum_{i=1}^n d_i'' d_i\right) b_j' x_j + \sum_{i=1}^n d_i' \sigma_i(b_0') x_i + \sum_{i=1}^n d_i' \delta_i(b_0') \\
 &\quad + \sum_{i=1}^n d_i' x_i b_i' x_i + \sum_{i,j \in \{1, \dots, n\}, i \neq j} d_i' x_i b_j' x_j,
 \end{aligned}$$

or equivalently,

$$\begin{aligned}
0 &= \left(d_0 - \sum_{i=1}^n d_i'' d_i\right) b'_0 + \sum_{i=1}^n d_i' \delta_i(b'_0) + \sum_{j=1}^n \left\{ \left(d_0 - \sum_{i=1}^n d_i'' d_i\right) b'_j + d_j' \sigma_j(b'_0) \right\} x_j \\
&\quad + \sum_{i=1}^n d_i' [\sigma_i(b'_i) x_i + \delta_i(b'_i)] x_i + \sum_{i,j \in \{1, \dots, n\}, i \neq j} d_i' x_i b'_j x_j \\
&= \left(d_0 - \sum_{i=1}^n d_i'' d_i\right) b'_0 + \sum_{i=1}^n d_i' \delta_i(b'_0) + \sum_{j=1}^n \left\{ \left(d_0 - \sum_{i=1}^n d_i'' d_i\right) b'_j + d_j' \sigma_j(b'_0) \right\} x_j \\
&\quad + \sum_{i=1}^n d_i' \sigma_i(b'_i) x_i^2 + \sum_{i=1}^n d_i' \delta_i(b'_i) x_i + \sum_{i,j \in \{1, \dots, n\}, i \neq j} d_i' x_i b'_j x_j \\
&= \left(d_0 - \sum_{i=1}^n d_i'' d_i\right) b'_0 + \sum_{i=1}^n d_i' \delta_i(b'_0) + \sum_{j=1}^n \left\{ \left(d_0 - \sum_{i=1}^n d_i'' d_i\right) b'_j + d_j' \sigma_j(b'_0) + d_j' \delta_j(b'_j) \right\} x_j \\
&\quad + \sum_{i=1}^n d_i' \sigma_i(b'_i) x_i^2 + \sum_{i,j \in \{1, \dots, n\}, i \neq j} d_i' x_i b'_j x_j. \tag{7}
\end{aligned}$$

With the purpose of computing the last sum in (7), consider the Remark 2.3 (iii). Then

$$\begin{aligned}
0 &= \left(d_0 - \sum_{i=1}^n d_i'' d_i\right) b'_0 + \sum_{i=1}^n d_i' \delta_i(b'_0) \\
&\quad + \sum_{j=1}^n \left\{ \left(d_0 - \sum_{i=1}^n d_i'' d_i\right) b'_j + d_j' \sigma_j(b'_0) + d_j' \delta_j(b'_j) \right\} x_j \\
&\quad + \sum_{i=1}^n d_i' \sigma_i(b'_i) x_i^2 \\
&\quad + \sum_{i,j \in \{1, \dots, n\}, i \neq j} \left\{ [d_i' \sigma_i(b'_j) + d_j' \sigma_j(b'_i) c_{i,j}] x_i x_j + [d_j' \delta_j(b'_i) + d_j' \sigma_j(b'_i) r_i^{(i,j)}] x_i \right. \\
&\quad \left. + [d_i' \delta_i(b'_j) + d_j' \sigma_j(b'_i)] x_j + d_j' \sigma_j(b'_i) r^{(i,j)} + d_j' \sigma_j(b'_i) \sum_{k=1, k \neq i,j}^n r_k^{(i,j)} x_k \right\}. \tag{8}
\end{aligned}$$

By (3), the expression (8) takes the form

$$0 = \sum_{i=1}^n d_i' \delta_i(b'_0) + \sum_{j=1}^n d_j' \delta_j(b'_j) x_j + \sum_{i,j \in \{1, \dots, n\}, i \neq j} [d_j' \delta_j(b'_i) x_i + d_i' \delta_i(b'_j) x_j],$$

or equivalently,

$$0 = \sum_{i=1}^n d_i' \delta_i(b'_0) + \sum_{i,j \in \{1, \dots, n\}} d_i' \delta_i(b'_j) x_j.$$

By degree relations, we have necessarily the equalities

$$\sum_{i=1}^n d_i' \delta_i(b'_0) = \sum_{i=1}^n d_i' \delta_i(b'_j) = 0. \tag{9}$$

Indeed, for the first sum in (9) we note that

$$\begin{aligned}
 \sum_{i=1}^n d'_i \delta_i(b'_0) = 0 &\Leftrightarrow \sum_{i=1}^n t^{-1} d'_i \delta_i(b'_0) = 0 \Leftrightarrow \sum_{i=1}^n s_4^{-1} a''_i \delta_i(b'_0) = 0 \\
 &\Leftrightarrow \sum_{i=1}^n a'_i \sigma_i(s)^{-1} \delta_i(b'_0) = 0 \Leftrightarrow \sum_{i=1}^n c^{-1} a'_i \sigma_i(s)^{-1} \delta_i(b'_0) = 0 \\
 &\Leftrightarrow \sum_{i=1}^n c_i^{-1} a_i \sigma_i(s)^{-1} \delta_i(b'_0) = 0.
 \end{aligned} \tag{10}$$

Similarly, for the second sum in (9) we have

$$\begin{aligned}
 \sum_{i=1}^n d'_i \delta_i(b'_j) = 0 &\Leftrightarrow \sum_{i=1}^n t^{-1} d'_i \delta_i(b'_j) = 0 \Leftrightarrow \sum_{i=1}^n s_4^{-1} a''_i \delta_i(b'_j) = 0 \\
 &\Leftrightarrow \sum_{i=1}^n a_i \sigma_i(s)^{-1} \delta_i(b'_j) = 0 \Leftrightarrow \sum_{i=1}^n a_i \overline{\sigma}_i(s^{-1}) \overline{\delta}_i(b'_j) = 0 \quad (0 \leq j \leq n).
 \end{aligned} \tag{11}$$

Now, consider the elements $h = a'_0 + \sum_{i=1}^n a'_i x_i$, and $k = s^{-1} b'_j$ with $j = 1, \dots, n$. Then

$$\begin{aligned}
 hk &= a'_0 s^{-1} b'_j + \sum_{i=1}^n a'_i x_i s^{-1} b'_j \\
 &= a'_0 s^{-1} b'_j + \sum_{i=1}^n a'_i [\overline{\sigma}_i(s^{-1}) \overline{\sigma}_i(b'_j) x_i + \overline{\delta}_i(b'_j)] \\
 &= a_0 s^{-1} b'_j + \sum_{i=1}^n a'_i \overline{\sigma}_i(s^{-1}) \overline{\sigma}_i(b'_j) x_i + \sum_{i=1}^n a'_i \overline{\delta}_i(s^{-1} b'_j)
 \end{aligned} \tag{12}$$

$$= a_0 s^{-1} b'_j + \sum_{i=1}^n a'_i \overline{\sigma}_i(s^{-1}) \overline{\sigma}_i(b'_j) x_i + \sum_{i=1}^n a'_i \overline{\sigma}_i(s^{-1}) \overline{\delta}_i(b'_j) + \sum_{i=1}^n a'_i \overline{\delta}_i(s^{-1}) b'_j. \tag{13}$$

By (4), (5) and (11), we obtain

$$hk = a_0 s^{-1} b'_j + \sum_{i=1}^n a'_i \delta_i(s^{-1}) b'_j = \left(a'_0 s^{-1} + \sum_{i=1}^n a'_i \delta_i(s^{-1}) \right) b'_j = 0, \quad 1 \leq j \leq n.$$

Since there exist $m_j \in R$ for every j , and $n \in S_0(R)$ such that $s^{-1} b'_j = m_j n^{-1}$, then $(a'_0 + \sum_{i=1}^n a'_i x_i) m_j n^{-1} = 0$, whence $(a'_0 + \sum_{i=1}^n a'_i x_i) m_j = 0$. By the Armendariz condition on R , $a'_0 m_j = a'_i \sigma_i(m_j) = 0$ for every i and $0 \leq j \leq n$. Note that $a'_0 m_j = 0$ is equivalent to $a'_0 m_j n^{-1} = 0$, that is, $a'_0 s^{-1} b'_j = 0$ for each $1 \leq j \leq n$. Since we have the equivalences

$$a'_0 s^{-1} b'_j = 0 \Leftrightarrow c^{-1} a'_0 s_j^{-1} b_j = 0 \Leftrightarrow c_0^{-1} a_0 s_j^{-1} b_j = 0, \quad 0 \leq j \leq n, \tag{14}$$

expressions (6) and (14) show that the ring $Q(R)$ of R is weak Σ -skew Armendariz. \square

Note that every Ore domain is a Σ -skew Armendariz ring for every family of automorphisms Σ and every family of Σ -derivations Δ of R . Theorem 4.3 generalizes [24], Corollary 2.5.

Theorem 4.3. *If R is a semiprime Goldie ring and A is a bijective skew PBW extension of R , then the following statements are equivalent:*

- (i) R is Σ -rigid;
- (ii) R is Σ -skew Armendariz;
- (iii) R is weak Σ -skew Armendariz;
- (iv) $Q(R)$ is $\overline{\Sigma}$ -rigid;
- (v) $Q(R)$ is $\overline{\Sigma}$ -skew Armendariz;
- (vi) $Q(R)$ is weak $\overline{\Sigma}$ -skew Armendariz.

Proof. The implications (i) \Rightarrow (ii) \Rightarrow (iii) and (iv) \Rightarrow (v) \Rightarrow (vi) are clear. Equivalence (iii) \Leftrightarrow (vi) is the content of Theorem 4.2. (ii) \Rightarrow (i): If R is a Σ -skew Armendariz ring, then R is weak Σ -skew Armendariz, and by Theorem 4.2 we know that $Q(R)$ is weak $\overline{\Sigma}$ -skew Armendariz, that is, $Q(R)$ is Abelian by Proposition 3.9. In this way, $Q(R)$ is an Abelian semisimple ring, i.e., $Q(R)$ is reduced. If $a\overline{\sigma^\alpha}(a) = 0$ for an element $a \in Q(R)$, then $\overline{\sigma^\alpha}(a)a = 0$, since $Q(R)$ is reduced. Note that Lemma 3.3 implies $\overline{\sigma^\alpha}(a)\overline{\delta^\alpha}(\overline{\sigma^\alpha}(a)) = \overline{\delta^\alpha}(a)\overline{\sigma^\alpha}(a) = 0$.

Consider the elements $h = \overline{\sigma^\alpha}(a) - \overline{\sigma^\alpha}(a)x^\alpha$ and $k = a + \overline{\sigma^\alpha}(a)x^\alpha$ of $\sigma(Q(R))\langle x_1, \dots, x_n \rangle$. Then

$$\begin{aligned}
 hk &= \overline{\sigma^\alpha}(a)a + (\overline{\sigma^\alpha}(a))^2x^\alpha - \overline{\sigma^\alpha}(a)x^\alpha a - \overline{\sigma^\alpha}(a)x^\alpha\overline{\sigma^\alpha}(a)x^\alpha \\
 &= (\overline{\sigma^\alpha}(a))^2x^\alpha - \overline{\sigma^\alpha}(a)[\overline{\sigma^\alpha}(a)x^\alpha + p'_{\alpha,a}] - \overline{\sigma^\alpha}(a)[\overline{\sigma^\alpha}(\overline{\sigma^\alpha}(a))x^\alpha + q'_{\alpha,\overline{\sigma^\alpha}(a)}]x^\alpha \\
 &= -\overline{\sigma^\alpha}(a)p'_{\alpha,a} - \overline{\sigma^\alpha}(a\overline{\sigma^\alpha}(a))x^\alpha x^\alpha - \overline{\sigma^\alpha}(a)q'_{\alpha,\overline{\sigma^\alpha}(a)}x^\alpha \\
 &= -\overline{\sigma^\alpha}(a)p'_{\alpha,a} - \overline{\sigma^\alpha}(a)q'_{\alpha,\overline{\sigma^\alpha}(a)}x^\alpha,
 \end{aligned} \tag{15}$$

where the polynomials $p'_{\alpha,a}$ and $q'_{\alpha,\overline{\sigma^\alpha}(a)}$ have the properties established in Proposition 2.7. Since $a\overline{\sigma^\alpha}(a) = \overline{\sigma^\alpha}(a)a = 0$, Remark 2.8 (ii) and Lemma 3.3 show that $\overline{\sigma^\alpha}(a)p'_{\alpha,a} = \overline{\sigma^\alpha}(a)q'_{\alpha,\overline{\sigma^\alpha}(a)} = 0$, whence $hk = 0$. Since $Q(R)$ is weak $\overline{\Sigma}$ -skew Armendariz, we have $\overline{\sigma^\alpha}(a)\overline{\sigma^\alpha}(a) = 0$, and using the fact that $Q(R)$ is reduced and $\overline{\sigma^\alpha}$ is a monomorphism, we obtain $a = 0$, that is, $Q(R)$ is $\overline{\Sigma}$ -rigid, and so R is Σ -rigid. With this reasoning we have proved also (vi) \Rightarrow (iv) and (iii) \Rightarrow (i). \square

5. Baer, quasi-Baer, p.p. and p.q.-rings

Kaplansky in [16] defined a ring B as a *Baer* (resp. *quasi-Baer*, which was defined by Clark in [8]) ring if the right annihilator of every nonempty subset (resp. ideal) of B is generated by an idempotent. Another generalization of Baer rings are the p.p.-rings. A ring B is called *right* (resp. *left*) *p.p.* if the right (resp. *left*) annihilator of each element

of B is generated by an idempotent (or equivalently, rings in which each principal right (resp. left) ideal is projective). Birkenmeier et al. in [6] define a ring to be called a *right* (resp. *left*) *principally quasi-Baer* (or simply *right* (resp. *left*) *p.q.-Baer*) ring if the right annihilator of each principal right (resp. left) ideal of B is generated by an idempotent.

Commutative and noncommutative rings Baer, quasi-Baer, p.p.-rings, and right p.q.-Baer have been investigated in the literature. For instance, polynomial extensions in the commutative case were studied in [2], [4], and Ore extensions $B[x; \sigma, \delta]$ of injective type, i.e., when σ is injective, of all this kind of rings can be found in several works (cf. [5], [6], [10], [13], and others). Some of these treatments consider the case $\delta = 0$ and σ an automorphism, or the case where σ is the identity. Nevertheless, it is important to say that the Baerness and quasi-Baerness of a ring B do not inherit the Ore extension of B . More exactly, there are examples which show that there exists a Baer ring B but the Ore extension $B[x; \sigma, \delta]$ is not right p.q.-Baer ([10], Example 8); similarly, there exist Ore extensions $B[x; \sigma, \delta]$ which are quasi-Baer, but B is not quasi-Baer ([10], Example 9 and [3], Example 11). In general, the Baerness of B and $B[x; \sigma, \delta]$ does not depend on each other. Since Ore extensions of injective type are particular examples of skew PBW extensions, the concepts of Baer, quasi-Baer, and p.p. and p.q. are interesting for the ring theoretical study of skew PBW extensions. With this in mind, the first author in [28] studied all these concepts using the notion of Σ -rigid rings and established necessary and sufficient conditions to guarantee their stability under skew PBW extensions, generalizing the results presented in [13].

Now, as we saw in expression (1), Σ -rigid are contained strictly in weak Σ -skew Armendariz rings, so our next task is to consider all above concepts over these more general rings. With this objective, it is useful the following notation: for a nonempty subset D of a ring B , we denote by $r_B(D) = \{b \in B \mid db = 0, \forall d \in D\}$, the right annihilator of D in B .

Theorem 5.1. *If A is a bijective skew PBW extension of a weak Σ -skew Armendariz ring R , then R is a Baer ring if and only if A is a Baer ring.*

Proof. Suppose that R is a Baer ring. Consider a nonempty subset C of A and let C^* be the set of all coefficients of elements of C . It is clear that C^* is a nonempty subset of R , and by assumption, $r_R(C^*) = eR$ for some idempotent $e \in R$. If $f = \sum_{k=0}^m a_k X_k$ is an element of A , Remark 2.8 (i), and the equalities $\sigma_i(e) = e$ and $\delta_i(e) = 0$ for $1 \leq i \leq n$ (Proposition 3.8), show that $X_k e = 0$ for every $1 \leq k \leq m$, and so $f e = 0$, that is, $e \in r_A(C)$, which implies $eA \subseteq r_A(C)$. Now, if $g = \sum_{j=0}^t b_j Y_j$ is a nonzero element of $r_A(C)$, then $f g = 0$ for any $f \in C$. Since R is Σ -skew Armendariz, $\sigma^{\alpha_k}(b_j) \in r_R(C^*) = eR$ for every $1 \leq k \leq m$ with $\alpha_k = \exp(X_k)$, whence $b_0, b_1, \dots, b_t \in eR$, since σ^{α_k} is an automorphism for every k and $\sigma_i(e) = e$ for $1 \leq i \leq n$. Therefore, g takes the form $g = \sum_{j=0}^t e d_j = e \sum_{j=1}^t d_j \in eA$ for some elements $d_j \in R$, which shows that $r_A(C) \subseteq eA$, and so $r_A(C) = eA$.

Conversely, if A is Baer and B is a nonempty subset of R , then $r_A(B) = eA$ for some idempotent $e \in R$ (Proposition 3.7). Using this fact, we obtain $r_R(B) = r_A(B) \cap R = eA \cap R = eR$, that is, R is Baer. \square

Corollary 5.2 ([28], Theorem 3.9). *Let R be a Σ -rigid ring. Then R is a Baer ring if and only if A is a Baer ring.*

Theorem 5.3. *If A is a bijective skew PBW extension of a weak Σ -skew Armendariz ring R , then R is a p.p.-ring if and only if A is a p.p.-ring.*

Proof. Suppose that R is a p.p.-ring and let $f = \sum_{k=0}^m a_k X_k$ a nonzero element of A . Then there exists an idempotent $e_k \in R$ such that $r_R(\{a_k\}) = e_k R$, for every $0 \leq k \leq m$. Let $e := e_0 e_1 \cdots e_m$. It is clear that $e^2 = e \in R$ and $eR = \bigcap_{k=0}^m r_R(\{a_k\})$, since R is Abelian (Proposition 3.9). By Proposition 3.8, we have $\sigma_i(e) = e$ and $\delta_i(e) = 0$ for $1 \leq i \leq n$; so, using a reasoning as above, $fe = 0$, which shows that $eA \subseteq r_A(\{f\})$. Let us see the another inclusion. If $g = \sum_{j=0}^t b_j X_j \in r_A(f)$, then $fg = 0$, and by assumption, $a_k \sigma^{\alpha_k}(b_j) = 0$ ($\alpha_k = \exp(X_k)$) whence $\sigma^{\alpha_k}(b_j) \in r_R(\{a_k\}) = e_k R$, for every $0 \leq k \leq m$, $0 \leq j \leq t$. In this way, $b_j \in e_k R$, since σ^{α_k} is an automorphism with $\sigma_i(e) = e$, for every $1 \leq i \leq n$. Hence $b_j \in eR = \bigcap_{k=0}^m r_R(\{a_k\})$, for all $0 \leq j \leq t$, which shows that $g \in eA$. Therefore $eA = r_A(\{f\})$, i.e., A is a p.p.-ring.

Now, if A is a p.p.-ring and r is an element of R , we know that there exists an idempotent $e \in R$ with $r_A(\{a\}) = eA$ (Proposition 3.7), which implies $r_R(\{a\}) = eR$, and hence R is a p.p.-ring. \square

Corollary 5.4 ([28], Theorem 3.12). *Let R be a Σ -rigid ring. Then R is a p.p.-ring if and only if A is a p.p.-ring.*

Corollary 5.5 ([28], Theorems 3.10 and 3.13). *Let A be a bijective skew PBW extension of a reduced and weak Σ -skew Armendariz ring R . Then R is a quasi-Baer (p.q.-Baer) ring if and only if A is a quasi-Baer (p.q.-Baer) ring.*

Remark 5.6. (i) ([11], Example 2.8). Let $B = \mathbb{k}[t]$ be the polynomial ring over a field \mathbb{k} and σ be the endomorphism given by $\sigma(f(t)) = f(0)$. Then B is quasi-Baer, but the ring $B[x; \sigma]$ is not a quasi-Baer ring. This example shows that the injectivity of the endomorphisms $\sigma_i \in \Sigma$ (Proposition 2.2) is not a superfluous condition in Corollary 5.5.

- (ii) Since prime rings are quasi-Baer, if A is a bijective skew PBW extension of a prime ring R , then A is prime ([27], Proposition 3.3 or [20], Corollary 4.2) and hence quasi-Baer.
- (iii) In [28], Theorems 3.9, 3.10, 3.12, and 3.13 the first author did not assume bijectivity on the injective endomorphisms σ_i of Σ . More exactly, it was only required the invertibility of the elements $c_{i,j}$ in Definition 2.1. In this way, the results presented in [28] are valid for general skew PBW extensions satisfying these conditions on the elements $c_{i,j}$. Now, as we saw in this paper, Σ -rigid rings are contained strictly in weak Σ -skew Armendariz rings, so Theorems 5.1, 5.3, and Corollary 5.5 generalizes [28], Theorems 3.9, 3.10, 3.12, and 3.13. It is important to say that these general results are formulated assuming the bijectivity of the injective endomorphisms $\sigma_i \in \Sigma$. One could consider that this requirement is very strong; however, this is not the case. Indeed, several remarkable results about skew PBW extensions are formulated considering this hypothesis: the Hilbert basis theorem, regularity, Serre's Theorem, Global, Krull, Goldie and Gelfand-Kirillov dimensions, Quillen's K -groups, characterization of prime ideals, etc. (see [20], [22], [26], [29], and others). Last, but not least important, it is the number of examples of bijective skew PBW extensions which have been found in the literature (see [22] for a detailed list).

Acknowledgements

The authors gratefully acknowledge the many helpful remarks of the referees. The first author would like to thank the Departamento de Matemáticas, Facultad de Ciencias, Universidad Nacional de Colombia for its financial support - Grant HERMES CODE 30366 -.

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