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On the uniqueness of sign-changing solutions to a semipositone problem in annuli

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Abstract. In this paper we establish the uniqueness of radial solutions for a semipositone Dirichlet problem in an annulus, having a prescribed large number of nodal regions. Shooting method and Prüfer transformation are the main tools used in this work.
Keywords: Semipositone, nonhomogeneous problem, uniqueness of sign changing solution, weighted Dirichlet problem, nonlinear elliptic problem.
MSC2010: 34B15, 35J25, 35J60, 35J61, 35J66.

Sobre la unicidad de soluciones que cambian de signo para un problema Semipositone en anillos

Resumen. En este artículo establecemos la unicidad de soluciones radiales para un problema de Dirichlet, de tipo Semipositone, en un anillo, con un número prescrito (grande) de regiones nodales. Las principales herramientas usadas en este trabajo son el método del disparo y la transformación de Prüfer.
Palabras clave: Semipositone, problema no homogéneo, unicidad de soluciones que cambian de signo, problemas de Dirichlet con peso, problemas elípticos no lineales.

1. Introduction and Statement of the Results

We consider the annulus $\Omega := \{x \in \mathbb{R}^N : 0 < a < \|x\| < b\}$, where $N \geq 3$. In this paper we study the problem

$$\begin{cases} \Delta u + f(\|x\|, u) = 0, \ x \in \Omega, \\ \qquad u(x) = 0 \quad \text{for} \quad x \in \partial\Omega, \\ u \text{ has exactly } k \text{ nodal regions in } \Omega, \end{cases} \tag{1.1}$$

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where $f(\|x\|,u)=K(\|x\|)\left(|u|^{p-1}u-C\,||x||^{-p(N-2)}\right)$ for some constant $C>0$. Here, Δ denotes the Laplacian operator and $K\in C^2([a,b])$ is a given positive weight which is nondecreasing.

Because $f(\|x\|,0)<0$ for every $x\in\Omega$, our problem is called semipositone. These problems are harder than positone even in the case of positive solutions (see [4], [26]). When we are looking for positive solutions the difficulty is due to the fact that in the semipositone case, solutions have to live in regions where the reaction term is negative as well as positive (see [7]).

A radial solution $u(r)$ of (1.1), where $r=\|x\|$, satisfies

$$\begin{cases} u''(r)+\dfrac{N-1}{r}u'(r)+K(r)\left(|u(r)|^{p-1}u(r)-Cr^{-p(N-2)}\right)=0, \quad a<r<b,\\ \hfill u(a)=u(b)=0,\\ u \text{ has exactly } k \text{ zeros in } (a,b). \end{cases} \tag{1.2}$$

To the best of our knowledge, uniqueness results about sign-changing solutions to (1.2) (more general, in the semipositone case) are not known. By considering exterior domains, a uniqueness result of nonnegative solution for a semipositone problem is achieved in [32] (also see references therein). More recently, in [35] the author obtained uniqueness of sign-changing radial solutions in some ball and annulus considering $K=1, C=0$ and the particular nonlinearity $f(u)=|u|^{p-1}u-u$. In the superlinear context it seems hard to get uniqueness of sign-changing radial solutions with a prescribed number of zeros. Some previous works as, for instance, [34], [35] have attained such uniqueness, but at the expense of giving up too much generality; some of these are homogeneous problems, very particular geometry of the domain, specific cases of nonlinearity or impossing low-dimensional domains. In this work we prove a uniqueness result for the problem (1.2) in a ring-shaped domain restricted to

$$2(b/a)^{N-1}-1<p<(N+2)/(N-2). \tag{1.3}$$

Note that these inequalities imply that $b<a(N/(N-2))^{1/(N-1)}$. This is our compromise; but we are able to get such uniqueness in a more general context, namely for a weighted semipositone problem. We use some ideas inspired by the work of H. Aduén, A. Castro and J. Cossio in [1]. We extend and improve a previous result exhibited in [1]. This improvement is reflected in several aspects: first, our nonlinearity $K(\|x\|)|u|^{p-1}u$ in place of $|u|^{p-1}u$; namely, the nonlinearity involves a weight. Second, our non-homogeneity also has a weight and mainly, the third reason, the result of uniqueness. Tanaka, in [34], considered the problem (1.2) in a ball with $C=0$ and also demonstrated uniqueness. Although our region is different, we compensate this difference considering an inhomogeneous problem, making it a more difficult problem of partial differential equation. In this sense we can say that we have improved a theorem obtained in [34].

In order to face (1.1) we rewrite it in the form

$$\begin{cases} \Delta u+K(\|x\|)|u|^{p-1}u=q(\|x\|),\ x\in\Omega,\\ \hfill u(x)=0 \quad \text{for} \quad x\in\partial\Omega,\\ u \text{ has exactly } k \text{ nodal regions in } \Omega, \end{cases} \tag{1.4}$$

with $q(\|x\|) := C \cdot K(\|x\|)\|x\|^{-p(N-2)}$ and $C > 0$ is a constant.

There are a lot of works related to the existence, nonexistence and multiplicity of radial solutions for differential equations with the structure appearing in (1.4), but without the nodal condition. Results about uniqueness of positive radial solutions are also known. Almost all those results involve homogeneous problems, i.e., $q \equiv 0$ (see, for instance, [6], [30], [31], [36]). In [6], the authors studied the equation $\Delta u + K(\|x\|)f(u) = 0$, with $K \in C^2$ and they showed that the problem has at most one positive solution, assuming f being sublinear, more precisely $f(s)/s > f'(s)$ for $s \neq 0$. In [30] and [31], considering the nonlinearity $f(u) = |u|^{p-1}u, N \geq 3$ and $p > 1$, the authors obtained uniqueness of positive radial solutions under one additional condition over $rK'(r)/K(r)$. In [36], the author studied the same problem with $K = 1$ and $f(u) = -u+u^p$ subject to the Dirichlet boundary condition on an annulus in $\mathbb{R}^N$. As a by-product, his approach also provides a much simpler, if not the simplest, new proof for the uniqueness of positive solutions to the same problem, in a finite ball or in the whole space $\mathbb{R}^N$. Without pretense of completeness, we refer for instance to the papers [2], [9], [12], [17], [24], [25], [28], [27], [38], [39] and [40], as well as the references therein, where results about uniqueness of positive radial solutions can be found. On the other hand, by considering $K = 1$ and $C = 0$, Kajikiya in [23] and the authors in [10] showed uniqueness to the differential equation in (1.1). In an annulus, also uniqueness was obtained by Ni and Nussbaum [31] with $K(t) = t^l, l \in \mathbb{R}$ and $C = 0$. Additionally, in [34] a uniqueness result for problem (1.1) was proved in the homogeneous context ($C = 0$). Also, in [22], [33], authors considered a sublinear nonlinearity. We must mention that existence results have been obtained by Y. Naito [29] in the homogeneous case. In relation to the case $q \neq 0$ considered here, infinitely many radially symmetric solutions were found in [8] while, more relevantly, [14], [15] showed results with a large prescribed number of zeros. Other works as [1], [3], [5], [11], [16], [18], [19] also consider nonhomogeneous problems. The references [3], [5] and [18] show existence of solutions for the non-homogeneous elliptic equation $\Delta u+|u|^{p-1}u+g(x) = 0$ in $\mathbb{R}^N$ and its weighted version with nonlinearity $K(x)|u|^{p-1}u$ in place of $|u|^{p-1}u$. The results in [3], [5] and [18] consider the range $N/(N-2) < p$ that covers the critical and supercritical variational ones $p > (N+2)/(N-2)$. In [3] and [5], it was considered bounded continuous force terms $g(x)$ while singular forces like $g(x) = \|x\|^{-\gamma}$ were treated in [18]. Papers [11], [13], [19] studied quasilinear equations but they do not showed a uniqueness result.

To investigate the uniqueness of nodal radial solutions of the problem (1.4), we consider the following:

$$\begin{cases} v''(r) + \dfrac{N-1}{r}v'(r) + K(r)|v(r)|^{p-1}v(r) = q(r), \quad a < r < b, \\ v(a) = v(b) = 0, \ v'(a) =: \alpha > 0, \\ v \ \text{ has exactly } \ k \ \text{ zeros in } (a,b), \end{cases} \tag{1.5}$$

where $' \equiv \frac{d}{dr}$ and $v(r) := u(x)$ with $r = \|x\|$.

Since we apply the shooting method (cf. [20], [21], [37]), we study the initial value problem (1.5) with $v'(a) = \alpha > 0$ as the shooting parameter.

The following theorem is an existence result for (1.5) and it has been established by

Dambrosio [15, Theorem B].

Theorem 1.1 (Dambrosio). *If $1 < p < (N+2)/(N-2)$, then there exists $k^* \in \mathbb{N}$ such that for every integer $k \geq k^*$ the problem* (1.5) *has at least one solution.*

Let $v(\cdot,\alpha)$ be a function that satisfies (1.5). Hence $v, v' \in C^1([a,b] \times (0,\infty))$ and $v_\alpha(r,\alpha) = \frac{\partial v}{\partial \alpha}(r,\alpha)$ is a solution of the linearized problem

$$\begin{cases} w'' + \dfrac{N-1}{r} w' + pK(r)|v|^{p-1} w = 0, \quad a < r < b, \\ w'(a) = 1, \ w(a) = 0. \end{cases} \tag{1.6}$$

(See, for example, [37]).

Most of the lemmas or results presented here involving computations with the term of the right hand side of the equation (1.5) are valid for more general functions q, more precisely for continuous functions.

Our first result, aside from its own relevance, is crucial in our approach in order to get our second theorem. It establishes the oscillatory behavior of the solution w to the linearized equation (1.6). In other words, it exposes the interaction of the zeros of the solution v to the inhomogeneous differential equation (1.4) with any bounded external force $q > 0$, and the respective solution w to the linearized equation (1.6). Thus, we can prove the following theorem.

Theorem 1.2. *There exists $\tilde{\alpha_1} > 0$ such that if $|v'(a)| > \tilde{\alpha_1}$ and z_1, z_2 are consecutive zeroes of v, then the function w has a zero in (z_1, z_2).*

Remark 1.3. The previous result holds true for any bounded and positive function q in problem (1.5).

Let us define $V(r) := rK'(r)/K(r)$ for $r \in [a,b]$. Our main second result reads as follows:

Theorem 1.4. *Let $q(r) := C\, r^{-p(N-2)} K(r)$ with $K' \geq 0$. There exists $k^* \in \mathbb{N}$ such that if $k \geq k^*$ and if for all $r \in [a,b]$,*

$$[V(r) - p(N-2) - N + 4][V(r) - p(N-2) + N] - 2rV'(r) < 0, \tag{1.7}$$

then the solution of problem (1.2) *exists and it is unique.*

Remark 1.5. Let us consider $h \in \mathbb{R}$ and $K(r) := r^h$ for $r \in [a,b]$. If $p > \max\{N/(N-2), 2(b/a)^{N-1} - 1\}$ and

$$p(N-2) - N < h < p(N-2) + N - 4,$$

then the condition (1.7) is satisfied and therefore we get Theorem 1.4 with this weight K and $p \in (N/(N-2), (N+2)/(N-2))$. Hence, there are examples of functions K that give us uniqueness with p in the well-known gap $(N/(N-2), (N+2)/(N-2))$.

This paper is organized as follows. In Section 2 we prove some useful facts related to the energy of the solution. In Section 3 we present technical lemmas and Section 4 is devoted to show Theorem 1.2 which gives us a zero of the solution of the linearized equation, between two consecutive zeros of the solution to the problem (1.5). Section 5 concerns the study of a transformed problem, which is equivalent to (1.5), and finally, in Section 6 we prove Theorem 1.4.

2. Energy analysis

We will denote by $0 < m := \min K$ and $\|K\|_\infty = \|K\|$.

We define the energy function associated to a solution v:

$$E(r,\alpha) \equiv E(r) = \frac{1}{2K(r)}|v'(r)|^2 + \frac{1}{p+1}|v(r)|^{p+1}. \tag{2.1}$$

Lemma 2.1. *The energy function defined in* (2.1) *satisfies the following properties:*

1. $\lim_{\alpha\to+\infty} E(r,\alpha) = +\infty$ *uniformly for all* $r \in [a,b]$.

2. *There exist positive constants* α_0 *and* C_2 *such that for all* $\alpha \geq \alpha_0$ *and* $a \leq s < t \leq b$,
$$C_2 E(s,\alpha) \leq E(t,\alpha) \leq 2E(s,\alpha). \tag{2.2}$$

Proof. Differentiating (2.1) with respect to r and applying (1.5) it follows that

$$E'(r,\alpha) = \frac{q(r)v'(r,\alpha)}{K(r)} - \left(\frac{N-1}{rK(r)} + \frac{K'(r)}{2K^2(r)}\right)|v'(r,\alpha)|^2. \tag{2.3}$$

Taking into account (2.3) and from the regularity of K and q there exist positive constants $C := \|q\|/m$ and $D := (N-1)/(ma) + \|K\|/(2m^2)$, such that for all $r \in [a,b]$,

$$\begin{aligned} E'(r,\alpha) &\geq -C|v'(r,\alpha)| - D|v'(r,\alpha)|^2 \\ &\geq -\frac{C^2}{2} - \frac{1}{2}|v'(r,\alpha)|^2 - D|v'(r,\alpha)|^2 \\ &= -\bar{C} - \bar{D}|v'(r,\alpha)|^2. \end{aligned}$$

Thus, if $k_1 := 2\,||K||_\infty \bar{D}$ then

$$e^{-k_1 r}\left(e^{k_1 r}E(r,\alpha)\right)' \geq (k_1/2K(r) - \bar{D})|v'(r,\alpha)|^2 - \bar{C} \geq -\bar{C}. \tag{2.4}$$

Hence $\left(e^{k_1 r}E(r,\alpha)\right)' \geq -\bar{C}e^{k_1 b} := -k_2$. Integrating we obtain positive constants k_3 and k_4 such that $E(r) \geq k_3|\alpha|^2 - k_4$. This proves the first part of the lemma. Again, a suitable integration of (2.4) on $[s,t]$ gives us positive constants c_1 and c_2 such that $E(t,\alpha) \geq c_1 E(s,\alpha) - c_2$. From the previous inequality and the conclusion of the first part of the lemma, there exists $\alpha_1 > 0$ such that for all $\alpha \geq \alpha_1$ holds $C_2 E(s,\alpha) \leq E(t,\alpha)$, where $C_2 := c_1/2$.

On the other hand, for all $r \in [a,b]$ we have

$$\begin{aligned} E'(r,\alpha) &\leq C|v'(r,\alpha)| - \frac{N-1}{b\|K\|}|v'(r,\alpha)|^2 \quad (\text{since } K' \geq 0) \\ &\leq \frac{C^2\varepsilon^2}{2} + \frac{1}{2\varepsilon^2}|v'(r,\alpha)|^2 - D_0|v'(r,\alpha)|^2 \\ &= \frac{C^2}{4D_0} =: c_3, \end{aligned}$$

where we took $\varepsilon^2 = 1/2D_0$. Integrating on $[s,t]$ we have $E(t,\alpha) \leq E(s,\alpha) + c_4$. From the first part of the lemma there exists $\alpha_2 > 0$ such that for all $\alpha \geq \alpha_2,\ E(s,\alpha) \geq c_4$. Defining $\alpha_0 := \max\{\alpha_1, \alpha_2\}$ we get $E(t,\alpha) \leq 2E(s,\alpha)$. This proves the second part of the lemma. ☑

3. Preliminary lemmas

From now on, when we mention a solution v with k zeros in (a,b), we denote $z_0 := a, z_{k+1} := b$ and z_i as the ith zero of $v(\cdot,\alpha)$ in (a,b) for $i = 1,2,\dots,k$. Due to the uniqueness of the initial value problem we note that $v(\cdot,\alpha)$ and $v'(\cdot,\alpha)$ cannot vanish simultaneously. Thus z_i is a simple zero and, moreover,

$$(-1)^i v'(z_i,\alpha) \equiv (-1)^i \frac{d}{dr} v(z_i,\alpha) > 0 \quad \text{for} \quad i = 0,1,2,\dots,k+1. \tag{3.1}$$

Another useful tool that we need is the Prüfer transformation for the solution $v(\cdot,\alpha)$ of the differential equation, with initial conditions in (1.5). We define the functions $\rho(r,\alpha)$ and $\theta(r,\alpha)$ by

$$\begin{aligned} v(r,\alpha) &= \rho(r,\alpha)\sin\theta(r,\alpha), \\ r^{N-1}v'(r,\alpha) &= \rho(r,\alpha)\cos\theta(r,\alpha). \end{aligned}$$

Thus we see that $\rho(r,\alpha)$ and $\theta(r,\alpha)$ can be written in the form

$$\rho(r,\alpha) = (v^2(r,\theta) + r^{2(N-1)}[v'(r,\theta)]^2)^{1/2} > 0$$

and

$$\theta(r,\alpha) = \arctan\left(\frac{v(r,\alpha)}{r^{N-1}v'(r,\alpha)}\right).$$

From $v, v' \in C^1([a,b]\times(0,\infty))$, it follows that $\rho,\theta \in C^1([a,b]\times(0,\infty))$. Straightforward calculations give

$$\frac{\partial\theta}{\partial r}(r,\alpha) \equiv \theta'(r) = \frac{r^{N-1}}{\rho^2}[(v'(r,\alpha))^2 - v(r,\alpha)\,q(r) + K(r)|v(r,\alpha)|^{p+1}],$$

for $r \in [a,b]$. We will see that $\theta(r,\alpha)$ is strictly increasing in $r \in [a,b]$ for each $\alpha > 0$ fixed and large enough. In fact, it is sufficient to show that $v(r,\alpha)\,q(r) < (v'(r,\alpha))^2 + K(r)|v(r,\alpha)|^{p+1}$. For simplicity of notation we omit the arguments (r,α). Let $s = (p+1)/p, s' = p+1$ and $\varepsilon > 0$ be such that $\varepsilon^{p+1} = 1/m(p-1)$, with $m = \min K$. From the first part of Lemma 2.1, there exists $\alpha^* > 0$ such that for $\alpha > \alpha^*$ we have $E(r,\alpha) > \varepsilon^s ||q||_\infty^s/(2sm)$ uniformly in r. By using Young's Inequality with this ε we find

$$\begin{aligned} v\,q = (v/\varepsilon)(q\varepsilon) &\leq \frac{|v|^{p+1}}{(p+1)\,\varepsilon^{p+1}} + \frac{\varepsilon^s\,||q||_\infty^s}{s} \\ &< \frac{m(p-1)}{p+1}|v|^{p+1} + 2mE(r,\alpha) \\ &\leq |v'|^2 + K(r)|v|^{p+1}, \end{aligned}$$

for every $\alpha > \alpha^*$ and uniformly in $r \in [a,b]$.

We note that $\rho(a,\alpha) = a^{N-1}\alpha$, and for simplicity we define $\theta(a,\alpha) = 0$; therefore it is simple to check that $v(r,\alpha)$ is a solution of (1.5) if and only if

$$\theta(b,\alpha) = k\pi. \tag{3.2}$$

Hence, the number of solutions of (1.5) is equal to the number of roots $\alpha > 0$ of (3.2).

The following lemma is proved in the same way as Lemma 2.3 in [1].

Lemma 3.1. *There exists $\hat{M} > 0$ such that if $|v'(a)| \geq \hat{M}$, then, between two consecutive zeroes of v there is exactly one critical point.*

Lemma 3.2. *Given $\delta > 0$ there exists $M_1(\delta) \equiv M_1 > 0$ such that if $|v'(a)| \geq M_1$, then $z_2 - z_1 \leq \delta$ for any two consecutive zeroes z_1, z_2 of v. Moreover, $M_1(\delta) \to +\infty$ as $\delta \to 0$.*

Proof. Let Γ be a fixed constant such that $\Gamma > (N-1)(N-3)\delta^2/(4a^2) + 64\pi^2$ and consider y satisfying, for all $r > 0$,

$$\begin{cases} y'' + \frac{N-1}{r} y' + \frac{\Gamma}{\delta^2} y = 0, \\ y(0) = 1, \quad y'(0) = 0. \end{cases}$$

By applying the Sturm's comparison theorem (cf. [20], [21], [37]) with the solution ϕ to the problem $y'' + \frac{64\pi^2}{\delta^2} y = 0$, $y(0) = 1$ and $y'(0) = 0$, we conclude

$$d - c < \frac{\delta}{4}, \tag{3.3}$$

where $c < d$ are consecutive zeroes of y. Let $K_2(\delta) \equiv K_2 > 0$ be such that for $|v| \geq K_2$ holds

$$\left(m|v|^{p-1} - \frac{\|q\|_\infty}{|v|} \right) \geq \frac{\Gamma}{\delta^2}. \tag{3.4}$$

Let $t \in (z_1, z_2)$ be such that $|v(s)| \leq K_2$ for all $s \in (z_1, t)$. Hence, recalling the definition of the energy and taking into account $2E(s) \geq C_2 E(a)$, we get

$$|v'(s)|^2 \geq K(s) \left(\frac{C_2 |v'(a)|^2}{2||K||_\infty} - \frac{2K_2^{p+1}}{p+1} \right). \tag{3.5}$$

Now, for some $\bar{s} \in (z_1, t)$ we have $|t - z_1| = |v(t)|/|v'(\bar{s})|$. Considering (3.5), and if

$$|v'(a)| \geq \left\{ \frac{4||K||}{C_2} \left(\frac{8K_2^2}{m\delta^2} + \frac{K_2^{p+1}}{p+1} \right) \right\}^{1/2} \equiv M_2, \tag{3.6}$$

we conclude $t - z_1 \leq \frac{\delta}{4}\sqrt{m/K(\bar{s})} \leq \delta/4$. Similarly, if (3.6) holds and $|v(s)| \leq K_2$ for all $s \in (t, z_2)$, then $z_2 - t \leq \delta/4$.

Thus, if $|v'(a)|$ satisfies (3.6) and $|v'(a)| \geq \hat{M}$ (from Lemma (3.1)) then v has a unique critical point in (z_1, z_2) and there exist $t_1 < t_2$ in the interval (z_1, z_2) such that $|v| \geq K_2$ in $[t_1, t_2]$ and $|v| \leq K_2$ on $[z_1, t_1] \cup [t_2, z_2]$. We claim that $t_2 - t_1 \leq \delta/2$. In fact, if $t_2 - t_1 > \delta/2$ then, by (3.3), y has at least two zeroes in $[t_1, t_2]$. Hence, by the definition of t_1, t_2 and the Sturm Comparison Theorem (keeping in mind (3.4)), v has a zero in (t_1, t_2), which is a contradiction. Hence, $t_2 - t_1 \leq \delta/2$ and $z_2 - z_1 \leq \delta$. Therefore, the first part of the lemma is proved.

From (2.2),

$$E(t, v'(a)) \geq C_2 |v'(a)|^2 / (2||K||) \geq \delta^{-1},$$

provided that

$$|v'(a)| \geq \sqrt{\frac{2||K||}{\delta C_2}}.$$

Thus, the second part is achieved by choosing

$$M_1(\delta) := \max\{\hat{M}, M_2, \sqrt{2||K||/(\delta C_2)}\ \}.$$

The lemma is proved. ☑

4. Proof of Theorem 1.2

Proof. Define $M^* := 2((N-1)/a)(b/a)^{N-1}$, $\delta_0 := (p-1)/(3M^*)$,

$$\Phi^2 := \frac{||K||}{C_2}\left[\frac{2}{p+1}\left(\frac{p+1}{p-1}\frac{\|q\|_\infty}{m}\right)^{\frac{p+1}{p}} + \frac{\|q\|_\infty^2(b-a)^2}{m}\left(\frac{p+1}{p-1}\frac{||K||}{m}+1\right)^2\right],$$

$$\Psi^2 := \frac{||K||}{C_2}\left[\frac{1}{m}\left(\frac{\|q\|_\infty}{M^*}\right)^2\left(\frac{p+1}{p-1}\frac{||K||}{m}+1\right)^2 + \frac{2}{p+1}\left(\frac{p+1}{p-1}\frac{\|q\|_\infty}{m}\right)^{(p+1)/p}\right]$$

and

$$\tilde{\alpha_1} := \max\{\alpha_0, \hat{M}, M_1(\delta_0), \Phi, \Psi\}. \tag{4.1}$$

Since z_1 and z_2 are consecutive zeroes of v we may assume, without loss of generality, that $v > 0$ in (z_1, z_2).

Suppose that $w > 0$ in (z_1, z_2) (similar arguments prove the case $w < 0$ in (z_1, z_2)). Let $\rho \in (z_1, z_2)$ be such that $v'(\rho) = 0$. Since $\tilde{\alpha_1} \geq \hat{M}$, by Lemma 3.1, $v(\rho) = \max\{v(r) : r \in (z_1, z_2)\}$. By (4.1) and Lemma 2.1,

$$E(\rho, v'(a)) = E(\rho) > \frac{1}{p+1}\left(\frac{(p+1)\|q\|_\infty}{m(p-1)}\right)^{(p+1)/p}.$$

From here follows $v^p(\rho) > [(p+1)/(p-1)][\|q\|_\infty/m]$. Thus, by the Intermediate Value Theorem there exist t_1, t_2 such that $z_1 < t_1 < \rho < t_2 < z_2$ and

$$v^p(t_1) = \frac{(p+1)\|q\|_\infty}{(p-1)m} = v^p(t_2). \tag{4.2}$$

Multiplying (1.5) by $r^{N-1}w$ and (1.6) by $r^{N-1}v$, and integrating by parts on $[s,t] \subset [t_1, t_2]$ we have

$$\begin{aligned} t^{N-1}(w'v - v'w)(t) &- s^{N-1}(w'v - v'w)(s) \\ &+ \int_s^t r^{N-1}[(p-1)K(r)|v|^p + q]w(r)dr = 0. \end{aligned} \tag{4.3}$$

Claim: $w'(\rho) < 0$.

Proof of the claim. Suppose that $w'(\rho) \geq 0$. Thus

$$-s^{N-1}(w'v - v'w)(s) + \int_s^\rho r^{N-1}[(p-1)K(r)|v|^p + q]w(r)dr \leq 0, \tag{4.4}$$

for any $s \in [t_1, \rho)$.

On the other hand, for $s \in [t_1, \rho]$ we have (from (1.6))

$$s^{N-1} w'(s) = \rho^{N-1} w'(\rho) + p \int_s^\rho r^{N-1} K(r) |v(r)|^{p-1} w(r)\, dr. \tag{4.5}$$

Since the right hand side of (4.5) is positive, it follows that $w'(s) > 0$ for $s \in [t_1, \rho]$, which implies that w is increasing on that interval. Thus $w(s) \geq w(t_1)$ for all $s \in [t_1, \rho]$. Multiplying the ODE in (1.5) by r^{N-1} and integrating we get

$$t_1^{N-1} v'(t_1) = \int_{t_1}^\rho r^{N-1} (K(r) v^p(r) - q(r))\, dr. \tag{4.6}$$

By using (4.2) and $v(t_1) \leq v(r)$ the right hand side is less than or equal to $\int_{t_1}^\rho r^{N-1} [K(r) v^p(r) + m \frac{p-1}{p+1} v^p(r)]\, dr$, and hence

$$t_1^{N-1} v'(t_1) \leq \frac{2p}{p+1} \int_{t_1}^\rho r^{N-1} K(r) v^p(r)\, dr. \tag{4.7}$$

Now,

$$\begin{aligned}
&\int_{t_1}^\rho r^{N-1} [(p-1) K(r) |v|^p + q(r)] w(r) dr \\
&\geq w(t_1) \int_{t_1}^\rho r^{N-1} [(p-1) K(r) |v|^p + q(r)]\, dr \\
&= w(t_1) \left(\int_{t_1}^\rho r^{N-1} (p-1) K(r) |v|^p\, dr + \int_{t_1}^\rho r^{N-1} q(r)\, dr \right) \\
&\geq \frac{p-1}{2} t_1^{N-1} w(t_1) v'(t_1) \quad (\text{ by (4.6) and (4.7)}).
\end{aligned} \tag{4.8}$$

We also observe that we have used the fact

$$(p-1) K(r) |v|^p + q(r) \geq (p-1) m |v|^p(t_1) + q(r) \geq (p+1) ||q|| - ||q|| \geq 0.$$

Combining (4.4) and (4.8) we obtain

$$-t_1^{N-1} (w'v - v'w)(t_1) + \frac{p-1}{2} t_1^{N-1} w(t_1) v'(t_1) \leq 0. \tag{4.9}$$

Since $r \mapsto r^{N-1} w'(r)$ is decreasing in (z_1, z_2) (see (1.6)) then for $t \in (z_1, t_1]$, $t^{N-1} w'(t) \geq t_1^{N-1} w'(t_1)$, and thus, by (4.9),

$$w'(t) \geq (\frac{p-1}{2}) w(t_1) v'(t_1) / v(t_1).$$

Therefore

$$w(t_1) \geq w(t_1) - w(z_1) \geq \frac{(p-1) w(t_1) v'(t_1)}{2 v(t_1)} (t_1 - z_1). \tag{4.10}$$

By Taylor's formula

$$\begin{aligned}
0 = v(z_1) &= v(t_1) + v'(t_1)(t_1 - z_1) + \frac{v''(\zeta)}{2} (t_1 - z_1)^2 \\
&= v(t_1) + v'(t_1)(t_1 - z_1) \\
&\quad - \frac{1}{2} \left\{ \frac{N-1}{\zeta} v'(\zeta) + K(\zeta) |v^p(\zeta)| - q(\zeta) \right\} (t_1 - z_1)^2,
\end{aligned} \tag{4.11}$$

for some $\zeta \in (z_1, t_1)$. Also, by (1.5),

$$\begin{aligned} \zeta^{N-1} v'(\zeta) &= t_1^{N-1} v'(t_1) + \int_\zeta^{t_1} r^{N-1}[K(r)v^p(r) - q(r)]dr \\ &\leq t_1^{N-1}\left[v'(t_1) + (\frac{p+1}{p-1}\frac{||K||}{m} + 1)\|q\|_\infty (t_1 - z_1)\right]. \end{aligned} \tag{4.12}$$

The above and (4.1) give

$$v'(\zeta) \leq (b/a)^{N-1} 2v'(t_1). \tag{4.13}$$

In order to see this, according to the definition of $\tilde{\alpha_1}$ and by the second part of Lemma 2.1, $C_2\Phi^2 \leq C_2|v'(a)|^2 = 2C_2K(a)E(a) \leq 2C_2||K||E(a) \leq 2||K||E(t_1)$, which implies

$$\frac{\|q\|_\infty^2 (b-a)^2}{m}\left(\frac{p+1}{p-1}\frac{||K||}{m} + 1\right)^2 + \frac{2}{p+1}\left(\frac{p+1}{p-1}\frac{\|q\|_\infty}{m}\right)^{(p+1)/p} \leq 2E(t_1),$$

and due to (4.2),

$$\begin{aligned} \frac{\|q\|_\infty^2 (b-a)^2}{m}\left(\frac{p+1}{p-1}\frac{||K||}{m} + 1\right)^2 &\leq 2E(t_1) - \frac{2}{p+1}\left(\frac{p+1}{p-1}\frac{\|q\|_\infty}{m}\right)^{\frac{p+1}{p}} \\ &= |v'(t_1)|^2/K(t_1). \end{aligned}$$

From this and (4.12) we have (4.13).

On the other hand,

$$C_2\Psi^2 \leq C_2|v'(a)|^2 = 2C_2K(a)E(a) \leq 2C_2||K||E(a) \leq 2||K||E(t_1).$$

Reasoning as before we prove that

$$||q||\left(\frac{p+1}{p-1}\frac{\|K\|}{m} + 1\right) \leq M^* v'(t_1),$$

and combining it with (4.13) and (4.11) we get

$$\begin{aligned} v(t_1) &\leq \frac{1}{2}\left(\frac{N-1}{a}(b/a)^{N-1}2v'(t_1) + ||q||\Big(\frac{p+1}{p-1}\frac{\|K\|}{m} + 1\Big)\right)(t_1 - z_1)^2 \\ &\leq M^*\, v'(t_1)(t_1 - z_1)^2. \end{aligned} \tag{4.14}$$

From (4.10) it is clear that $t_1 - z_1 \leq 2v(t_1)/((p-1)v'(t_1))$. Taking into account (4.14),

$$v(t_1) \leq \frac{4M^* v'(t_1) v^2(t_1)}{(p-1)^2|v'(t_1)|^2},$$

or equivalently, $v'(t_1) \leq 4M^* v(t_1)/(p-1)^2$. Then, by using (4.14),

$$(z_2 - z_1)^2 \geq (t_1 - z_1)^2 \geq \frac{v(t_1)}{M^* v'(t_1)} \geq \frac{(p-1)^2}{4(M^*)^2},$$

which implies

$$z_2 - z_1 \geq \frac{p-1}{2\,M^*} > \delta_0,$$

This contradicts Lemma 3.2 and thus the claim is proved. Therefore

$$w'(\rho) < 0.$$

Using the previous inequality, taking $t = t_2$ and $s = \rho$ in (4.3), we have

$$\begin{aligned} t_2^{N-1}(w'v - v'w)(t_2) &= \rho^{N-1}w'(\rho)v(\rho) \\ &\quad - \int_\rho^{t_2} r^{N-1}[(p-1)K(r)v^p(r) + q(r)]w(r)dr \\ &< - \int_\rho^{t_2} r^{N-1}[(p-1)K(r)v^p(r) + q(r)]w(r)dr. \end{aligned} \tag{4.15}$$

Multiplying the ODE in (1.6) by r^{N-1} and integrating in $[\rho, s]$ with $s \in [\rho, t_2]$ we prove that w is decreasing in $[\rho, t_2]$. Using the same procedure with the ODE in (1.5), we arrive to

$$\begin{aligned} -t_2^{N-1}v'(t_2) &= \int_\rho^{t_2} r^{N-1}[K(r)v^p(r) - q(r)]\,dr \\ &\leq \int_\rho^{t_2} r^{N-1}(K(r)v^p(r) + ||q||)\,dr \\ &= \int_\rho^{t_2} r^{N-1}[K(r)v^p(r) + \frac{m(p-1)}{p+1}v^p(t_2)]\,dr \\ &\leq \frac{2p}{p+1}\int_\rho^{t_2} r^{N-1}K(r)v^p(r)\,dr. \end{aligned} \tag{4.16}$$

In a similar fashion as in (4.8), and using (4.16), we conclude

$$\int_\rho^{t_2} r^{N-1}[(p-1)K(r)v^p(r) + q(r)]w(r)\,dr \geq -\frac{p-1}{2}t_2^{N-1}v'(t_2)w(t_2).$$

This implies that (4.15) becomes

$$t_2^{N-1}(w'v - v'w)(t_2) \leq \frac{p-1}{2}t_2^{N-1}v'(t_2)w(t_2). \tag{4.17}$$

Now, we concentrate on the corresponding subinterval to the right of ρ. Since the map $r \mapsto r^{N-1}w'(r)$ is decreasing in (z_1, z_2), taking $t \in [t_2, z_2)$ we get $t_2^{N-1}w'(t_2) \geq t^{N-1}w'(t)$, and by (4.17),

$$t_2^{N-1}w'(t_2) \leq \frac{1}{v(t_2)}\left[\frac{p-1}{2}t_2^{N-1}w(t_2)v'(t_2) + t_2^{N-1}w(t_2)v'(t_2)\right],$$

and thus,

$$w'(t) \leq -\frac{p+1}{2}\left(\frac{t_2}{t}\right)^{N-1}\frac{w(t_2)|v'(t_2)|}{v(t_2)} \leq -\frac{p+1}{2}\left(\frac{a}{b}\right)^{N-1}\frac{w(t_2)|v'(t_2)|}{v(t_2)}.$$

Therefore

$$-w(t_2) \leq w(z_2) - w(t_2) \leq -\frac{p+1}{2}\left(\frac{a}{b}\right)^{N-1}\frac{w(t_2)|v'(t_2)|}{v(t_2)}(z_2 - t_2). \tag{4.18}$$

As before, by Taylor's formula,

$$\begin{aligned} 0 = v(z_2) &= v(t_2) + v'(t_2)(z_2 - t_2) + \frac{v''(\tau)}{2}(z_2 - t_2)^2 \\ &= v(t_2) + v'(t_2)(z_2 - t_2) \\ &\quad - \frac{1}{2}\left\{\frac{N-1}{\tau}v'(\tau) + K(\tau)|v^p(\tau)| - q(\tau)\right\}(z_2 - t_2)^2, \end{aligned} \tag{4.19}$$

for some $\tau \in (t_2, z_2)$. On the other hand,

$$\tau^{N-1}v'(\tau) = t_2^{N-1}v'(t_2) + \int_{t_2}^{\tau} r^{N-1}[q(r) - K(r)v^p(r)]\,dr$$

and $\int_{t_2}^{\tau} r^{N-1}[K(r)v^p(r) - q(r)]\,dr \leq \tau^{N-1}(||K||\frac{p+1}{(p-1)m} + 1)||q||(\tau - t_2)$ imply

$$\tau^{N-1}v'(\tau) \geq t_2^{N-1}v'(t_2) - \tau^{N-1}\left(\frac{||K||}{m}\frac{p+1}{p-1} + 1\right)||q||(\tau - t_2).$$

Moreover, $|v'(\tau)| \leq (b/a)^{N-1}|v'(t_2)| + \left(\frac{||K||}{m}\frac{p+1}{p-1} + 1\right)||q||(\tau - t_2)$. From the definition of Φ it is clear that

$$C_2\Phi^2 \leq C_2|v'(a)|^2 = 2C_2K(a)E(a) \leq 2C_2||K||E(a) \leq 2||K||E(t_2),$$

and thus

$$\begin{aligned} \frac{\|q\|_\infty^2(b-a)^2}{m}\left(\frac{p+1}{p-1}\frac{||K||}{m} + 1\right)^2 &\leq 2E(t_2) - \frac{2}{p+1}\left(\frac{p+1}{p-1}\frac{\|q\|_\infty}{m}\right)^{\frac{p+1}{p}} \\ &= 2E(t_2) - \frac{2}{p+1}v^{p+1}(t_2) \\ &= |v'(t_2)|^2/K(t_2) \leq |v'(t_2)|^2/m. \end{aligned}$$

Consequently, $\left(\frac{||K||}{m}\frac{p+1}{p-1} + 1\right)||q||(\tau - t_2) \leq |v'(t_2)|$, and therefore

$$|v'(\tau)| \leq [(b/a)^{N-1} + 1]|v'(t_2)|. \tag{4.20}$$

Also, as we mentioned, it is simple to check that

$$||q||\left(\frac{p+1}{p-1}\frac{\|K\|}{m} + 1\right) \leq M^*|v'(t_2)|.$$

Combining (4.20), (4.19), the previous inequality, $v(\tau) \leq v(t_2)$ and replacing the second equality of (4.2), we have

$$\begin{aligned} v(t_2) &\leq \frac{1}{2}\left\{\frac{N-1}{a}[(b/a)^{N-1} + 1]|v'(t_2)| + ||q||\left(\frac{||K||(p+1)}{m(p-1)} + 1\right)\right\}(z_2 - t_2)^2 \\ &\quad - v'(t_2)(z_2 - t_2) \\ &\leq \Theta|v'(t_2)|(z_2 - t_2)^2 - v'(t_2)(z_2 - t_2). \end{aligned}$$

Here $\Theta := \frac{1}{2}\left\{\frac{N-1}{a}[(b/a)^{N-1}+1]+M^*\right\} = \frac{N-1}{2a}[3(b/a)^{N-1}+1]$. Now, an elementary computation shows that

$$\left(z_2 - t_2 + \frac{1}{2\Theta}\right)^2 \geq \frac{1}{4\Theta^2} - \frac{v(t_2)}{\Theta\, v'(t_2)}.$$

Next, using (4.18), we deduce

$$\left(z_2 - t_2 + \frac{1}{2\Theta}\right)^2 \geq \frac{1}{4\Theta^2} + \frac{1}{\Theta}\frac{p+1}{2}\left(\frac{a}{b}\right)^{N-1}(z_2 - t_2).$$

This implies that $z_2 - t_2 \geq (1/\Theta)\left(\frac{p+1}{2}\left(\frac{a}{b}\right)^{N-1} - 1\right) > 0$, provided that $p+1 > 2(b/a)^{N-1}$ (see (1.3)), which contradicts Lemma 3.2.

This second contradiction implies that w cannot be positive on (z_1, z_2). Replacing w by $-w$ in the above arguments we see that w cannot be negative at all points in (z_1, z_2). Hence w must have a zero in (z_1, z_2), which proves our Theorem 1.2. ☑

5. Transforming the problem

We recall that $q : [a, b] \to \mathbb{R}$ is a differentiable function. By using the transformation $\phi : [a, b] \to [a, a^{3-N}b^{N-2}]$ given by $\phi(t) = a^{3-N}t^{N-2}$, we transform (1.5) into a new annulus and a new problem. In fact, we define

$$U(t, \alpha) = a^{-1}(N-2)tv(a^{1-\beta}t^{\beta}, \alpha), \qquad W(t) = a^{-1}(N-2)tw(a^{1-\beta}t^{\beta}),$$

with $\beta = 1/(N-2)$, v the solution of (1.5) and w the solution of (1.6). Notice that $U(a) = 0 = U(b_1)$ and $U'(a) = \alpha$, where $b_1 := a^{3-N}b^{N-2}$. Then $U = U(t, \alpha)$ and $W = W(t)$ satisfy

$$U''(t) + M(t)|U|^{p-1}U(t) = Q(t),\ a < t \leq b_1, \tag{5.1a}$$
$$U(a) = 0,\ U'(a) = \alpha, \tag{5.1b}$$
$$W''(t) + pM(t)|U|^{p-1}W(t) = 0,\ a < t \leq b_1, \tag{5.1c}$$
$$W(a) = 0,\ W'(a) = 1, \tag{5.1d}$$

where $M(t) = \beta^{p+1}a^{p-2\beta+1}t^{2\beta-p-1}K(a^{1-\beta}t^{\beta})$ and $Q(t) = \beta a^{1-2\beta}t^{2\beta-1}q(a^{1-\beta}t^{\beta})$. We define $Z_i = \phi(z_i) = a^{3-N}z_i^{N-2}$, $i = 0, 1, 2, \ldots, k+1$. Then we see that

$$U(Z_i, \alpha) = 0, \quad \text{for} \quad i = 0, 1, 2, \ldots, k+1,$$
$$(-1)^{i-1}U(t, \alpha) > 0 \ \text{ for } t \in (Z_{i-1}, Z_i),\ i = 1, 2, \ldots, k+1.$$

Also, there exist $S_i \in (Z_{i-1}, Z_i)$ such that $U'(S_i, \alpha) = 0$, for $i = 1, 2, \ldots, k+1$ and α large. Actually, S_i is unique as we show next. It is easy to check that $v'(a^{1-\beta}t^{\beta}) = a^{\beta}t^{-\beta}[U'(t) - t^{-1}U(t)]$. Therefore,

$$E(a^{1-\beta}t^{\beta}, \alpha, v) \equiv E(a^{1-\beta}t^{\beta}) = \frac{a^{2\beta}t^{-2\beta}}{2K(a^{1-\beta}t^{\beta})}[U'(t) - t^{-1}U(t)]^2 + \frac{a^{p+1}t^{-p-1}}{(p+1)(N-2)^{p+1}}|U(t)|^{p+1}.$$

Particularly, if $\tau \in (a, b_1)$ is a critical point of U, then

$$E(a^{1-\beta}\tau^{\beta}) = \frac{a^{2\beta}\tau^{-2\beta-2}}{2K(a^{1-\beta}\tau^{\beta})}U^2(\tau) + \frac{a^{p+1}\tau^{-p-1}}{(p+1)(N-2)^{p+1}}|U(\tau)|^{p+1}. \tag{5.2}$$

Applying the Young's Inequality with $\varepsilon = \tau^2, s = (p+1)/2$ and its conjugate $r = (p+1)/(p-1)$ in the first term in (5.2), we find positive constants λ_1 and λ_2 independent on α and τ such that

$$E(a^{1-\beta}\tau^{\beta}) \leq \lambda_1 + \lambda_2 |U(\tau)|^{p+1}.$$

On the other hand, by Lemma 2.1 we can say that $|U(\tau)|^{p+1}$ is large for $\alpha >> 1$. Finally, taking into account (5.1a), it follows that $U(\tau)U''(\tau) < 0$ for every critical point τ. Thus, U has only one critical point in (Z_{i-1}, Z_i), which concludes the claim.

In addition, it is clear that $U'(t,\alpha) > 0$ for $t \in (a, S_1)$ and

$$(-1)^i U'(t,\alpha) > 0 \quad \text{for } t \in (S_i, S_{i+1}), \quad i = 1, 2, \dots, k, \tag{5.3}$$

$$(-1)^k U'(t,\alpha) > 0 \quad \text{for } t \in (S_{k+1}, Z_{k+1}]. \tag{5.4}$$

Lemma 5.1. *Let W be the solution of* (5.1c), (5.1d). *Then, for each $i \in \{1, 2, \dots, k+1\}, W$ has at least one zero in (Z_{i-1}, Z_i).*

Proof. We fix $i \in \{1, 2, \dots, k+1\}$. As a consequence of Theorem 1.2, there exists $r_i \in (z_{i-1}, z_i)$ such that $w(r_i) = 0$. If we define $R_i = a^{3-N} r_i^{N-2}$, we see that $R_i \in (Z_{i-1}, Z_i)$ and $W(R_i) = 0$. ☑

Lemma 5.2. *The Inequality* (1.7) *holds if and only if* $([M(t)]^{-1/2})'' < 0$ *for* $a < t < b$.

Proof. Let $t = a^{3-N} r^{N-2}$. Then $[M(t)]^{-1/2} = C r^{\rho} [K(r)]^{-1/2}$, where $C = \beta^{-p/2-1/2} a^{-\rho}$ and $\rho = \frac{1}{2}[p(N-2) + N - 4]$. Hence, we obtain

$$\frac{d}{dt}[M(t)]^{-1/2} = C a^{N-3} \beta [\rho r^{\rho-N+2} K^{-1/2} - \frac{1}{2} r^{\rho-N+3} K^{-3/2} K'].$$

Moreover,

$$\frac{d^2}{dt^2}[M(t)]^{-1/2} = C a^{2(N-3)} \frac{\beta^2 r^{\rho-2N+4}}{K^{1/2}} \times \left[\rho(\rho - N + 2) - \frac{1}{2}(2\rho - N + 3)\frac{rK'}{K} + \frac{3}{4}\left(\frac{rK'}{K}\right)^2 - \frac{1}{2}\frac{r^2 K''}{K}\right].$$

Since $r^2 K''/K = rV' - V + V^2$, we have

$$\begin{aligned}
\frac{4C^{-1}\beta^{-2}a^{2(3-N)}K^{1/2}}{r^{\rho-2N+4}} & \frac{d^2}{dt^2}[M(t)]^{-1/2} \\
&= 4\rho(\rho - N + 2) - 2(2\rho - N + 2)V + V^2 - 2rV' \\
&= (V - 2\rho)(V - 2\rho - 4 + 2N) - 2rV' \\
&= [V - p(N-2) - N + 4][V - p(N-2) + N] - 2rV'.
\end{aligned}$$

From this, the lemma follows. ☑

6. Uniqueness result

In this section we prove our uniqueness result for the problem

$$\begin{cases} v''(r) + \dfrac{N-1}{r} v'(r) + K(r)|v(r)|^{p-1} v(r) = C\, r^{-p(N-2)} K(r), \quad a < r < b, \\ \qquad\qquad v(a) = v(b) = 0,\ v'(a) =: \alpha > 0, \\ v \text{ has exactly } k \text{ zeros in } (a,b), \end{cases} \tag{6.1}$$

where $C > 0$ is constant.

First, we establish some facts where the particular form of $q(r) := C\, r^{-p(N-2)} K(r)$ is crucial. Due to the definition of q, we obtain:

Lemma 6.1. *Let U be a solution of* (5.1a), (5.1b)*, and W be a solution of* (5.1c), (5.1d)*. Then, for $a \le t \le b_1$,*

$$\frac{d}{dt}\Big([M(t)]^{-1/2} [W' U' - W U''] - ([M(t)]^{-1/2})' W U' \Big) = -([M(t)]^{-1/2})'' W U'. \tag{6.2}$$

Proof. By (5.1a), we note that $U''' = Q' - M'(t)|U|^{p-1}U - pM(t)|U|^{p-1}U'$ for every $a \le t \le b_1$. From here, and replacing U'' and W'' from (5.1a) and (5.1c) respectively, the assertion follows from direct computations. ☑

Lemma 6.2. *Assume that* (1.7) *holds. Let W be a solution of* (5.1c), (5.1d)*. Then the following hold:*

(i) $W(t) > 0$ for $t \in (a, S_1]$.

(ii) W has at most one zero in $(S_i, S_{i+1}]$ for each $i \in \{1, 2, \ldots, k\}$.

(iii) W has at most one zero in $(S_{k+1}, Z_{k+1}]$.

Proof. (i) Suppose that there exists $t_2 \in (a, S_1]$ such that $W(t_2) = 0$ and $W(t) > 0$ for $t \in (a, t_2)$. Then we have $W'(t_2) < 0$. Since $t_2 \in (a, S_1]$, then $U'(t_2) \ge 0$, and thus $W'(t_2)\, U'(t_2) \le 0$. Integrating (6.2) over $(a, t_2]$ and using Lemma 5.2, we get $W'(t_2)\, U'(t_2) > 0$, which is a contradiction. The proof of (i) is complete.

(ii) Assume that there exist t_1 and t_2 such that $S_i < t_1 < t_2 \le S_{i+1}, W(t_1) = W(t_2) = 0$ and $W(t) \ne 0$ for $t \in (t_1, t_2)$. We may suppose that $W(t) > 0$ for $t \in (t_1, t_2)$. Then we have $W'(t_1) > 0$ and $W'(t_2) < 0$. Let U be a solution of (5.1a), (5.1b). Integrating (6.2) over $[t_1, t_2]$, then multiplying by $(-1)^i$ and using Lemma 5.2 and (5.3), we obtain

$$0 > (M(t_2))^{-1/2} W'(t_2)(-1)^i U'(t_2) - (M(t_1))^{-1/2} W'(t_1)(-1)^i U'(t_1) > 0,$$

which is a contradiction. The case where $W(t) < 0$ for $t \in (t_1, t_2)$ is treated in a similar way. The proof of (ii) is complete.

(iii) The proof is similar to the previous one and taking into account (5.4). ☑

Lemma 6.3. *If $v(r, \alpha)$ is a solution of* (6.1) *with k zeros in (a, b), $w = v_\alpha$ is a solution of* (1.6) *and* (1.7) *holds, then $(-1)^i w(z_i) > 0$ for $i = 1, 2, \ldots, k+1$.*

Proof. By Lemmas 5.1 and 6.2, there exists a number $C_1 \in (S_1, Z_1)$ such that $W(t) > 0$ for $t \in (a, C_1), W(C_1) = 0$ and $W(t) < 0$ for $t \in (C_1, S_2]$. In particular, we have $W(Z_1) < 0$. Also, from Lemmas 5.1 and 6.2 we see that there exists a number $C_2 \in (S_2, Z_2)$ such that $W(t) < 0$ for $t \in (S_2, C_2), W(C_2) = 0$ and $W(t) > 0$ for $t \in (C_2, S_3]$. Since $C_2 < Z_2 < S_3$, we have $W(Z_2) > 0$. Repeating the process, we conclude that $(-1)^i W(Z_i) > 0$ for each $i = 1, 2, \ldots, k+1$. This implies that $(-1)^i w(z_i) > 0$ for each $i = 1, 2, \ldots, k+1$. The lemma is proved. ☑

Using the previous lemma in the same way presented in [34, Lemma 2.2], the following important ingredient in the proof of the main theorem is shown.

Lemma 6.4. *Let $k \in \mathbb{N}$ and let $v(r, \alpha_0)$ be a solution of* (1.5) *for some $\alpha_0 > 0$. If* (1.7) *holds, then $\theta_\alpha(b, \alpha_0) > 0$.*

Proof of Theorem 1.4. Recalling Theorem 1.1, we see that (1.5) has at least one solution. Now we show that the solution of (1.5) is unique. Assume, to the contrary, that there exist numbers $0 < \alpha_1 < \alpha_2$ such that $v(\cdot, \alpha_1)$ and $v(\cdot, \alpha_2)$ are solutions to (1.5). Then

$$\theta(b, \alpha_1) = \theta(b, \alpha_2) = k\pi.$$

Lemma 6.4 implies that $\theta_\alpha(b, \alpha_1) > 0$ and $\theta_\alpha(b, \alpha_2) > 0$. By the regularity of $\theta_\alpha(b, \cdot)$ we have that $\theta(b, \alpha_0) = k\pi$ and $\theta_\alpha(b, \alpha_0) \leq 0$ for some $\alpha_0 \in (\alpha_1, \alpha_2)$. This contradicts Lemma 6.4 and, consequently, (1.5) has only one solution. The proof of Theorem 1.4 is complete.

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