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Power Birnbaum-Saunders Student t distribution

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Abstract. The fatigue life distribution proposed by Birnbaum and Saunders has been used quite effectively to model times to failure for materials subject to fatigue. In this article, we introduce an extension of the classical Birnbaum-Saunders distribution substituting the normal distribution by the power Student t distribution. The new distribution is more flexible than the classical Birnbaum-Saunders distribution in terms of asymmetry and kurtosis. We discuss maximum likelihood estimation of the model parameters and associated regression model. Two real data set are analysed and the results reveal that the proposed model better some other models proposed in the literature.
Keywords: Birnbaum-Saunders distribution, alpha-power distribution, power Student t distribution.
MSC2010: 62-07, 62F10, 62J02, 62N86.

Distribución Birnbaum-Saunders Potencia t de Student

Resumen. La distribución de probabilidad propuesta por Birnbaum y Saunders se ha usado con bastante eficacia para modelar tiempos de falla de materiales sujetos a la fátiga. En este artículo definimos una extensión de la distribución Birnbaum-Saunders clásica sustituyendo la distribución normal por la distribución potencia t de Student. La nueva distribución es más flexible que la distribución Birnbaum-Saunders clásica en términos de asimetría y curtosis. Presentamos los estimadores de máxima verosimilitud de los parámetros del modelo y su modelo de regresión asociado. El análisis de dos aplicaciones con datos reales revelan una superioridad del nuevo modelo con

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relación a otros modelos existentes en la literatura.
Palabras clave: Distribución Birnbaum-Saunders, distribución alfa potencia, distribución potencia t de Student.

1. Introduction

Motivated by problems of vibration in commercial airplanes that caused fatigue in the materials, Birnbaum and Saunders introduced in [3] a new probabilistic model for modelling the lifetime of certain structures under dynamic load. The probability density function (pdf) of a Birnbaum-Saunders random variable T depending on parameters λ and β is given by

$$f_T(t) = \frac{1}{\sqrt{2\pi}} exp\left[-\frac{1}{2\lambda^2}\left(\frac{t}{\beta}+\frac{\beta}{t}-2\right)\right]\frac{t^{-3/2}(t+\beta)}{2\lambda\sqrt{\beta}}, \ t>0, \tag{1}$$

where $\lambda > 0$ is the shape parameter that controls asymmetry and β is a scale parameter. The parameter β is also the median of the distribution. The pdf (1) is right skewed as λ increases and symmetric around β as λ gets close to zero. We shall use the notation $T \sim BS(\lambda, \beta)$. We have $kT \sim BS(\lambda, k\beta)$, for any k > 0, that is, the BS distribution is closed under scale transformations. It is also of interest to mention that if $T \sim BS(\lambda, \beta)$, then $T^{-1} \sim BS(\lambda, \beta^{-1})$. It implies that the BS distribution also belongs to the family of random variables closed under reciprocation; see [25].

The pdf defined in (1) can be obtained as the distribution of the random variable

$$T = \beta\left[\frac{\lambda}{2}U + \sqrt{\left(\frac{\lambda}{2}U\right)^2 + 1}\right]^2, \tag{2}$$

where U is a random variable with standard normal distribution, i.e. $U \sim N(0, 1)$. The expected value and variance of T are, respectively,

$$\mathbb{E}(T) = \beta\left(1+\frac{\lambda^2}{2}\right) \ \text{ and } \ \mathbb{V}(T) = (\lambda\beta)^2\left(1+\frac{5}{4}\lambda^2\right).$$

Notice that both mean and variance increase as λ increases. Furthermore, the skewness and kurtosis of T are, respectively,

$$skew(T) = \frac{16\lambda^2(11\lambda^2+6)}{(5\lambda^2+4)^3} \ \text{ and } \ kurt(T) = 3 + \frac{6\lambda^2(93\lambda^2+41)}{(5\lambda^2+4)^2}.$$

Sometimes, the kurtosis and skewness of the fatigue lifetimes data are not completely explained by the classical BS distribution. To address these two problems, Moreno-Arenas et al. [20] propose a generalization of the Birnbaum-Saunders distribution referred to as the proportional hazard Birnbaum-Saunders distribution, which includes a new parameter that provides more flexibility in terms of skewness and kurtosis.

To address the problem of kurtosis, Díaz-García and Leiva-Sánchez presents in [8] a generalized Birnbaum-Saunders (GBS) generated from an elliptically contoured distribution based on the search for faster-growing life distributions, with a greater or lesser kurtosis and/or with left tails that are more or less weighted than those of the Birnbaum-Saunders distribution based on the normal distribution. A GBS distribution is obtained as the distribution of the random variable T defined in equation (2) where $U \sim EC(0,1;g)$, i.e. U is a random variable following the g-distribution in the family of elliptical contour densities or simply elliptical distributions. As a special case of the GBS distribution, we find the Birnbaum-Saunders Student t (GBSt) distribution, this model is more robust than the classical BS distribution when the data contain a high kurtosis. Gómez et al. present in [11] an extension of the GBS distribution called the generalized slash Birnbaum-Saunders (GSBS) model with a view to make it even more flexible in terms of its kurtosis coefficient. Lemonte in [16] introduced a new extension for the Birnbaum-Saunders distribution called the Marshall-Olkin extended Birnbaum-Saunders distribution.

Asymmetric extensions of the classical Birnbaum-Saunders distribution were defined by Vilca-Labra and Leiva-Sánchez in [27] based on the family of skew-elliptical distributions; Castillo et al. considered in [5] the asymmetric epsilon-Birnbaum-Saunders distribution; Cordeiro and Lemonte [6] propose an exponentiated generalized Birnbaum-Saunders (EGBS) distribution where the EGBS density function can take various forms depending on its shape parameters. Also, Martínez-Flórez et al. proposed in [18] the alpha-power Birnbaum-Saunders distribution based in the fractional order statistics distribution of Durrans (see [9]) as defined in equation (3). The probability density function of an alpha-power random variable Z depending on α, F and f is given by

$$\varphi_f(z;\alpha) = \alpha f(z)\{F(z)\}^{\alpha-1},\ z \in \mathbb{R},\ \alpha \in \mathbb{R}^+, \tag{3}$$

where F is an absolutely continuous and differentiable distribution function with density function $dF = f$. We use the notation $Z \sim PF(\alpha)$. The parameter α is a shape parameter that controls the amount of asymmetry in the distribution. When $F = \Phi$ and $f = \phi$ are the density and distribution functions of the standard normal, respectively, then the $\varphi_\phi(z;\alpha)$ is called the power normal distribution, when $\alpha = 1$ is the normal distribution. Important properties of this distribution were studied in [12]. For the case $F = T_\nu$, the distribution function of the Student t with ν degrees of freedom, we have the power Student t family and denoted by $Z \sim PT_\nu(\alpha)$.

The importance of the PT_ν distribution lies in the fact that it models the asymmetry and high kurtosis present in the data, thus the power Student t distribution is more robust than the power normal distribution and normal distribution. As can be seen from Figure 1, parameter α controls kurtosis as well as asymmetry. Moreover, note that for $\alpha > 1$, the distribution kurtosis is greater than the kurtosis for the Student t distribution and, for $\alpha < 1$, the opposite occurs. On the other hand, the $PT_\nu(\alpha)$ is asymmetric for $\alpha > 1$ and symmetric otherwise.

An extension of the BS distribution to regression models has been considered in [24] which became known as the log-linear BS regression model and Martínez-Flórez et al. in [19] generalized this model for the log-linear BS power model. In this paper, in addition to the existing BS models available in the statistical literature, we propose a power Birnbaum-Saunders Student t distribution and the log-linear power Birnbaum-Saunders Student t regression model. We expect that by replacing the normal distribution by such more general family, a more flexible BS is obtained. Asymmetry in the alpha-power

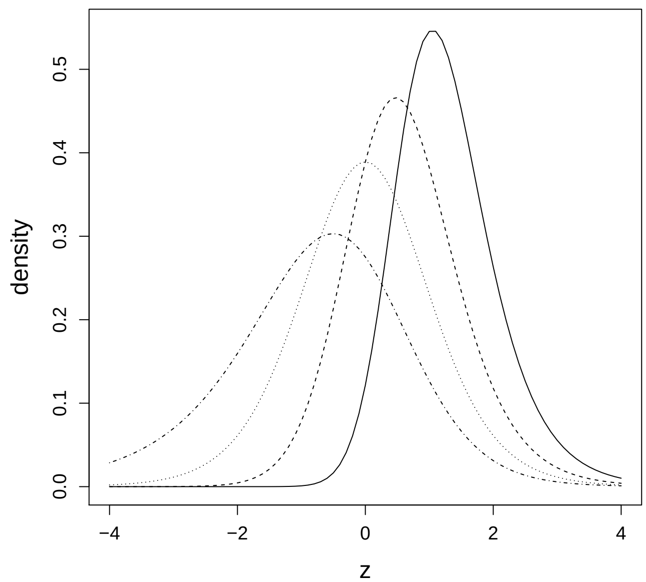


Figure 1. Density $PT_\nu(\alpha)$ with $\nu = 10$ and $\alpha = 5$, (solid line), $\alpha = 2$ (dashed line), $\alpha = 1$ (dotted line) and $\alpha = 0.5$ (dotted dashed line).

family is controlled by a parameter, which will also control asymmetry in the extended BS family. The flexibility of the proposed family is demonstrated by plotting the density for different parameters combinations. Model flexibility make it adequate for fitting data with asymmetry as well as outlying observations. Such aspects of the proposed model are seen in the application studied for which the proposed model outperforms previous competitors in terms of fitting.

The paper is outlined as follows. In Section 2, we introduce the power Birnbaum-Saunders Student t distribution and some general properties are presented. In Section 3 maximum likelihood estimation is discussed. In Section 4, we introduce the log-linear power Birnbaum-Saunders Student t regression model. The potentiality of the new models is illustrated by means of applications in three real data sets in Section 5. Finally, Section 6 ends the paper with some concluding remarks.

2. Power Birnbaum-Saunders Student t distribution

Suppose that we have a system with α independent components and that the system fails if all individual components fail. Suppose that $T_1, \ldots, T_\alpha$ represent the survival times of the individual components, which for $\lambda > 0$ and $\beta > 0$ can be represented by using the random variables

$$a_{T_j} = \frac{1}{\lambda}\left(\sqrt{\frac{T_j}{\beta}} - \sqrt{\frac{\beta}{T_j}}\right), \tag{4}$$

which are independent and identically distributed (iid) with Student t distribution with ν degrees of freedom. Therefore, if T denotes the system failure time, and since the T_j are iid with Student t distribution, then its distribution function is given by:

$$F_T(t) = P[T \leq t] = P[T_1 \leq t, T_2 \leq t, \ldots, T_\alpha \leq t] = \prod_{i=1}^{\alpha} T_\nu(a_{t_i}) = \{T_\nu(a_t)\}^\alpha .$$

Therefore, the probability density function of T is given by the expression

$$\varphi_T(t; \lambda, \beta, \alpha) = \alpha t_\nu(a_t)\{T_\nu(a_t)\}^{\alpha-1} \frac{t^{-3/2}[t+\beta]}{2\lambda\beta^{1/2}}, \tag{5}$$

where

$$a_t = a_t(\lambda, \beta) = \frac{1}{\lambda}\left(\sqrt{\frac{t}{\beta}} - \sqrt{\frac{\beta}{t}}\right),$$

which is an expression similar to the probability density function of the random variable $PT_\nu(\alpha)$. Hence, following [9] approach, we propose to generalize the BS distribution considering an α-fractionary order statistics for $\alpha \in \mathbb{R}^+$. The new BS model is an alternative to the classical BS distribution adequate to situations where the fatigue process presents high degree of asymmetry and kurtosis that are outside the ranges allowed by the classical BS distribution. Note that in the particular case when $\alpha = 1$, the pdf (5) coincides with the Generalized BS Student t distribution proposed in [8]. Moreover, for $\alpha = 2$, we obtain the asymmetric GBSt model with asymmetry parameter equal to 1, studied in [27]. The inclusion of the shape parameter α makes more flexible the asymmetry of the distribution. Moreover, the inclusion of the degrees of freedom ν of the Student t distribution makes it possible to have more flexibility with the kurtosis.

Definition 2.1. It is said that a random variable T follows a power Birnbaum-Saunders Student t distribution, if T can be written as

$$T = \beta\left[\frac{\lambda}{2}Z + \sqrt{\left(\frac{\lambda}{2}Z\right)^2 + 1}\right]^2, \tag{6}$$

where $Z \sim PT_\nu(\alpha)$, λ and α are parameters that control distributional shape and $\beta > 0$ is a scale parameter. We denote $T \sim PBST_\nu(\lambda, \beta, \alpha)$.

Properties:

- **Property 1.** Let $T \sim PBST_\nu(\lambda, \beta, \alpha)$. Then, the probability density function for T is given by

$$\begin{aligned}\varphi(t; \boldsymbol{\theta}) &= \alpha t_\nu(a_t)\{T_\nu(a_t)\}^{\alpha-1} \frac{t^{-3/2}[t+\beta]}{2\lambda\beta^{1/2}} \\ &= \frac{\alpha\Gamma(\frac{v+1}{2})}{(v\pi)^{1/2}\Gamma(\frac{v}{2})}\left[1 + \frac{1}{v\lambda^2}\left(\frac{t}{\beta} + \frac{\beta}{t} - 2\right)\right]^{-\frac{v+1}{2}} \{T_\nu(a_t)\}^{\alpha-1} \frac{t^{-3/2}[t+\beta]}{2\lambda\beta^{1/2}},\end{aligned}$$

  where $\boldsymbol{\theta} = (\lambda, \beta, \alpha)$.

- **Property 2.** Let $T \sim PBST_\nu(\lambda, \beta, \alpha)$. Then, the cumulative distribution function of T is given by

$$\mathbb{F}(t; \lambda, \beta, \alpha) = \{T_\nu(a_t)\}^\alpha.$$

Hence, the inversion approach can be used to generate random numbers with distribution $T \sim PBST_\nu(\lambda, \beta, \alpha)$ from the standard $PT_\nu(\alpha)$.

- **Property 3.** Let $T \sim PBST_\nu(\lambda, \beta, \alpha)$. If $\nu \to \infty$, then

$$PBST_\nu(\lambda, \beta, \alpha) \to PNBS(\lambda, \beta, \alpha),$$

where PNBS means the power normal Birnbaum-Saunders distribution defined by [18].

- **Property 4.** Let $T \sim PBST_\nu(\lambda, \beta, \alpha)$. If $\nu = 1$ then, we obtain the power Cauchy Birnbaum-Saunders distribution with probability density function given by

$$\varphi(t; \lambda, \beta, \alpha) \quad = \alpha \frac{\left[\frac{1}{2} + \frac{1}{\pi}\arctan(a_t)\right]^{\alpha-1}}{\left[1 + \frac{1}{v\lambda^2}\left(\frac{t}{\beta} + \frac{\beta}{t} - 2\right)\right]} \frac{t^{-3/2}[t+\beta]}{2\lambda\beta^{1/2}}.$$

- **Property 5.** The p-th percentile of the $PBST_\nu(\lambda, \beta, \alpha)$, $t_p = T_\nu^{-1}(p; \lambda, \beta, \alpha)$, is given by

$$t_p = \beta\left[\frac{\lambda}{2} z_{p,\nu} + \sqrt{\left(\frac{\lambda}{2} z_{p,\nu}\right)^2 + 1}\right]^2,$$

where $z_{p,\nu}$ is the p-th percentile of the distribution $PT_\nu(\alpha)$, given by $z_{p,\nu} = T_\nu^{-1}(p^{1/\alpha})$.

- **Property 6.** Let $T \sim PBST_\nu(\lambda, \beta, \alpha)$. Then, $kT \sim PBST_\nu(\lambda, k\beta, \alpha)$ for $k > 0$.

- **Property 7.** The survivor function, cumulative risk function, risk and inverted risk functions for model PBST_ν are given, respectively, by

$$S(t) = 1 - \{T_\nu(a_t(\lambda, \beta))\}^\alpha, \;\; H(t) = -\log[S(t)],$$

$$r(t) = r_{GBSt}(t) \frac{\alpha\{T_\nu(a_t)\}^{\alpha-1} - \{T_\nu(a_t)\}^\alpha}{1 - \{T_\nu(a_t)\}^\alpha} \quad \text{and} \quad R(t) = \alpha R_{GBSt}(t),$$

where $r_{GBSt}(t)$ and $R_{GBSt}(t)$ are the indices for the risk and inverted risk for the GBSt model, that is, the inverse risk rate is proportional to the risk rate for the GBSt distribution. Hence, the intervals where $R(t)$ is decreasing or increasing, are the same intervals where $R_{GBSt}(t)$ is decreasing or increasing.

The properties 1–7 follow from the definition of the distribution PBST in Equation 6 upon using suitable transformations. The details of the calculations can be obtained from the authors upon request.

Figure 2 depicts PBST_ν with 5 degrees of freedom and with α equal to 0.8, 1, 2 and 3. From these figures, note that as λ changes also changes the asymmetry and kurtosis for the distribution. Increasing λ makes the distribution more platykurtic. Differences can be noticed with the GBSt with $\alpha = 1$, depicted with dotted line.

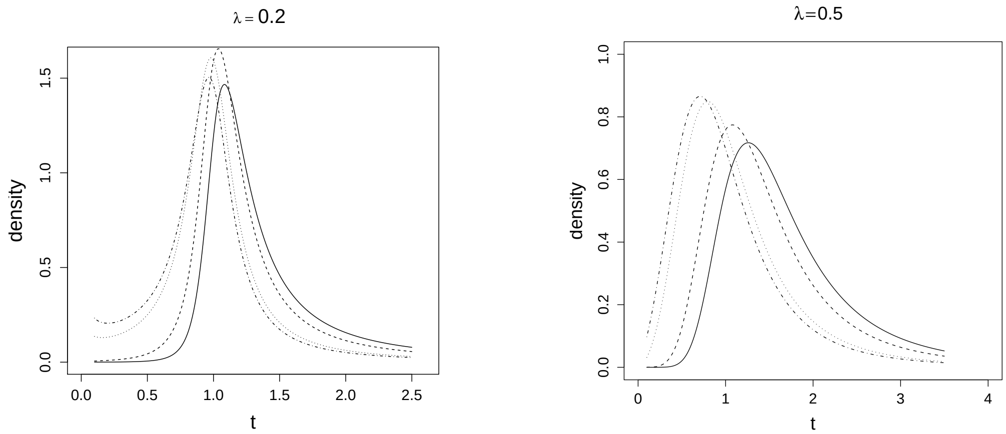


Figure 2. Plots for density function $\mathrm{PBST}_{\nu=5}(t;\lambda,\beta,\alpha)$, for $\beta = 1$ and with $\alpha = 0.8$ (dashed and dotted line), $\alpha = 1$ (dotted line), $\alpha = 2$ (dashed line) and $\alpha = 3$ (solid line).

Moments of the PBST$_\nu(\lambda,\beta,\alpha)$ distribution

Moments of the random variable $Z \sim PT_\nu(\alpha)$ have no closed form, but they can be generally represented as

$$E(Z^n) = \alpha \int_0^1 [T_\nu^{-1}(z)]^n z^{\alpha-1} dz. \tag{7}$$

Theorem 2.2. *Let $T \sim PBST_\nu(\lambda,\beta,\alpha)$ and $Z \sim PT_\nu(\alpha)$. Then, $\mathbb{E}(T^n)$ exists if and only if*

$$\mathbb{E}\left[\left(\frac{\lambda Z}{2}\right)^{k+l}\left(\left(\frac{\lambda Z}{2}\right)+1\right)^{\frac{k-l}{2}}\right] \tag{8}$$

exists for $k = 1, 2, \ldots, n$ with $l = 0, 1, \ldots, k$.

Theorem 2.3. *Let $T \sim PBST_\nu(\lambda,\beta,\alpha)$ and $Z \sim PT_\nu(\alpha)$. If $\mathbb{E}[Z^r]$ exists for $r = 1, 2, \ldots$, then*

$$\begin{aligned}\mu_r &= \mathbb{E}(T^r) = \beta^r \sum_{[0\le k\le r/2]} \binom{r}{2k}\left(\frac{1}{2}\right)^{2k} \sum_{j=0}^{2k}\binom{2k}{j}\mathbb{E}[(\lambda Z)^{4k-j}(\lambda^2 Z^2+4)^{j/2}] \\ &+ \beta^r \sum_{[0\le k< r/2]} \binom{r}{2k+1}\left(\frac{1}{2}\right)^{2k+1} \sum_{j=0}^{2k+1}\binom{2k+1}{j}\mathbb{E}[(\lambda Z)^{4k+2-j}(\lambda^2 Z^2+4)^{j/2}],\end{aligned} \tag{9}$$

where $[\cdot]$ corresponds to the sum of the integer part of the subscripts.

Remark 2.4. The central moments, $\mu'_r = \mathbb{E}(T - \mathbb{E}(T))^r$, for $r = 2, 3, 4$ can be obtained using $\mu'_2 = \mu_2 - \mu_1^2$, $\mu'_3 = \mu_3 - 3\mu_2\mu_1 + 2\mu_1^3$ and $\mu'_4 = \mu_4 - 4\mu_3\mu_1 + 6\mu_2\mu_1^2 - 3\mu_1^4$. Hence, variance, variation coefficient, asymmetry and kurtosis can be obtained by using: $\sigma_T^2 = \mathbb{V}(T) = \mu'_2$, $cv = \frac{\sqrt{\sigma_T^2}}{\mu_1}$, $\sqrt{\beta_1} = \frac{\mu'_3}{[\mu'_2]^{3/2}}$ and $\beta_2 = \frac{\mu'_4}{[\mu'_2]^2}$.

3. Inference for the PBST$_\nu(\lambda,\beta,\alpha)$ distribution

We now address the process of estimating the parameters of the PBST$_\nu(\lambda,\beta,\alpha)$ model. At this point it should be emphasized that being ν an unknown value in the model, this should be estimated in this process. However, several authors have pointed out the difficulty in estimating ν due to problems of unbounded and local maximum in the likelihood function, see for example, [2], [10], [13], [22], [26], among others. In practical situations, we can also establish different values of ν in an interval $[n_1,n_2]$ with $n_1,n_2\in\mathbb{N}$. This process, although a bit extensive in some cases is easy to carry out. The suggestion given by [2] and [13] is to initially establish this value and find the maximum likelihood estimators of the model as if ν was known. In this sense, suppose that ν is known and let $\mathbf{T}=(t_1,\ldots,t_n)^\top$ is a random sample of size n from the PBST$_\nu(\lambda,\beta,\alpha)$ distribution. The log-likelihood function for $\boldsymbol{\theta}=(\lambda,\beta,\alpha)^\top$ given $\mathbf{T}$ can be written as

$$\begin{aligned}\ell(\boldsymbol{\theta};\mathbf{T})= \;& n\left[\log(\alpha)-\log(2\lambda)-\tfrac{1}{2}\log(\beta)-\tfrac{1}{2}\log(2\pi)\right]+\textstyle\sum_{i=1}^n\log(t_i+\beta)\\ &-\tfrac{3}{2}\textstyle\sum_{i=1}^n\log(t_i)+n\left[\log\left(\Gamma\left(\tfrac{\nu+1}{2}\right)\right)-\tfrac{1}{2}\log(\nu\pi)-\log\left(\Gamma\left(\tfrac{\nu}{2}\right)\right)\right]\\ &-\tfrac{\nu+1}{2}\textstyle\sum_{i=1}^n\log\left[1+\tfrac{a_{t_i}}{\nu}\right]+(\alpha-1)\textstyle\sum_{i=1}^n\log(T_\nu(a_{t_i})).\end{aligned}\tag{10}$$

Formally, the partial derivative with respect to $\boldsymbol{\theta}$ of the $\ell(\boldsymbol{\theta};\mathbf{T})$ is called the score. The score function is denoted by $U(\boldsymbol{\theta})=(U(\lambda),U(\beta),U(\alpha))^\top$, so that the score equations follow by equating scores to zero, leading to the following equations:

$$U(\alpha)=\frac{n}{\alpha}+\sum_{i=1}^n\log\left(T_\nu(a_{t_i})\right)=0,$$

$$U(\lambda)=-\frac{n}{\lambda}+\frac{1}{\lambda}\sum_{i=1}^n p_i a_{t_i}^2-\frac{\alpha-1}{\lambda}\sum_{i=1}^n a_{t_i}w_i=0$$

and

$$U(\beta)=\quad -\frac{n}{2\beta}+\sum_{i=1}^n\frac{1}{\beta+t_i}+\frac{1}{2\lambda^2}\sum_{i=1}^n p_i\left[\frac{t_i}{\beta^2}-\frac{1}{t_i}\right]-\frac{\alpha-1}{2\lambda\beta}\sum_{i=1}^n\left[\sqrt{\frac{t_i}{\beta}}+\sqrt{\frac{\beta}{t_i}}\right]w_i=0,$$

where $p_i=\frac{\nu+1}{\nu+a_{t_i}^2}$ and $w_i=\frac{t_\nu(a_{t_i})}{T_\nu(a_{t_i})}$. Numerical procedures are required for solving the above equations.

If $\ell(\boldsymbol{\theta};\mathbf{T})$ is twice differentiable with respect to $\boldsymbol{\theta}$, and under certain regularity conditions, then the Fisher information may be written as

$$\mathcal{I}(\boldsymbol{\theta})=-\mathbb{E}\left[\frac{\partial^2}{\partial\theta^2}\ell(\mathbf{T};\boldsymbol{\theta})\bigg|\,\boldsymbol{\theta}\right].$$

It can be shown that for $\alpha=1$ the rows (or columns) of the matrix $\mathcal{I}(\boldsymbol{\theta})$ are linearly independent. This guaranties existence of the inverse of the Fisher information matrix when $\alpha=1$ so that ordinary large sample properties of the likelihood ratio and Wald type statistics are satisfied. Thus, the hypothesis $H_0:\ \alpha=1$, versus $H_1:\ \alpha\neq1$ that compares the generalized Birnabum-Saunder Student t model against the PBST_ν model can be performed using the likelihood ratio statistics.

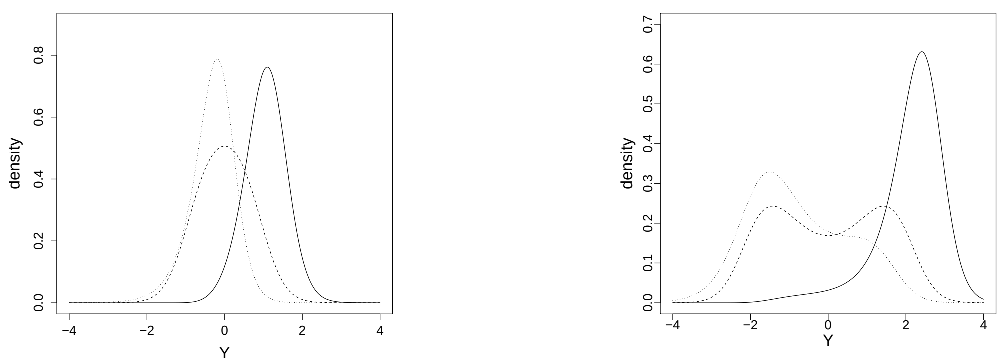
The rejection of the hypothesis $H_0: \ \alpha = 1$, corroborates the presence of asymmetry in the data and therefore a power Birnabum-Saunder Student t distribution can better fit the data compared with the generalized Birnabum-Saunder Student t distribution.

4. Log-linear PBST_ν regression model

Similarly as the sinh-normal distribution was defined by [24], now we define the power-sinh-Student t (PSHT) distribution as the random variable with density function given by

$$f_Y(y) = \alpha \frac{\frac{2}{\lambda}\cosh\left(\frac{y-\gamma}{\eta}\right)}{\eta} t_\nu\left(\frac{2}{\lambda}\sinh\left(\frac{y-\gamma}{\eta}\right)\right)\left\{T_\nu\left(\frac{2}{\lambda}\sinh\left(\frac{y-\gamma}{\eta}\right)\right)\right\}^{\alpha-1}. \quad (11)$$

We denote by $Y \sim PSHT_\nu(\lambda, \gamma, \eta, \alpha)$. Figure 3 illustrates the behaviour of the density (11), we call attention to the bimodal behaviour of the density for parameter values of $\lambda > 2$, in (b) above. On the other hand, for values of $\lambda < 2$ in (a) the distribution is unimodal and in some cases symmetric.



(a) $\alpha = 0.5$ and $\lambda = 0.75$ (dotted line) $\alpha = 1.0$ and $\lambda = 1.5$ (dashed line) $\alpha = 5$ and $\lambda = 2$ (solid line)

(b) $\alpha = 0.5$ and $\lambda = 3.0$ (dotted line) $\alpha = 1.0$ and $\lambda = 4.5$ (dashed line) $\alpha = 5$ and $\lambda = 7.5$ (solid line)

Figure 3. $f_Y(y;\gamma,\ \lambda,\ \eta,\ \alpha)$, for values $\nu = 5$, $\gamma = 0$ and $\eta = 1$.

To develop the log-linear PBST_ν regression model to follow, it is important to note that $Y = \log(T) \sim PSHT_\nu(\lambda, \log(\beta), 2, \alpha)$ when $T \sim PTBS_\nu(\lambda, \beta, \alpha)$.

Consider $T_i \sim PTBS_\nu(\lambda_i, \beta_i, \alpha_i)$ and $\beta_i = \exp(x_i^\top \theta)$ for $i = 1, 2, \ldots, n$, where $\theta^\top = (\theta_1, \theta_2, \ldots, \theta_p)$ is a vector of unknown parameters to be estimated and $x_i = (x_{i1},\ x_{i2}, \ldots,\ x_{ip})^\top$ is a vector of explanatory variables independent of T_i. Therefore, since $cT_i \sim PTBS_\nu(\lambda_i, c\beta_i, \alpha_i)$, we may write $T_i = \exp(x_i^\top\theta)\delta_i$, where $\delta_i \sim PTBS_\nu(\lambda, 1, \alpha)$.

Suppose now that $Y_i = \log(T_i)$, so that we can define the following regression model:

$$y_i = x_i^\top \theta + \log(\delta_i) = x_i^\top \theta + \epsilon_i,\ \ i = 1, \ldots, n, \tag{12}$$

where y_i is the log-survival (or log-censoring time) for the i-th individual and $\epsilon_i \sim PSHT_\nu(\lambda, 0, 2, \alpha),\ \ i = 1, \ldots, n$.

We assume that the explanatory variables are independent of the shape parameters and also assume independence between the observed survival and censoring time. We can conclude that $Y_i \sim PSHT_\nu(\lambda, x_i^\top \theta, 2, \alpha)$. It can be shown that $\mathbb{E}(Y_i) \neq x_i^\top \theta$, so that the intercept has to be corrected in order that Y_i becomes unbiased for its expectation. Straightforward algebraic manipulations yield $\mathbb{E}(\epsilon_i) = \mu_\epsilon$ and $\mathbb{V}(\epsilon_i) = 4w_2(\lambda, \alpha)$ for $i = 1, \ldots, n$. Moreover, since $Y_1, \ldots, Y_n$ are independent random variables, then $cov(\epsilon_i, \epsilon_j) = 0$. Therefore, making $\theta_0^* = \theta_0 + 2w_1(\lambda, \alpha)$, we have that $\mathbb{E}(y_i) = x_i^\top \theta^*$, so that a linear estimator for $\theta^* = (\theta_0^*, \theta_1^\top)^\top$ can be obtained using the ordinary least squares approach, with solution given by

$$\hat{\theta}^* = (X^\top X)^{-1} X^\top Y$$

and covariance matrix

$$Cov(\hat{\theta}^*) = 4w_2(\lambda, \alpha)(X^\top X)^{-1}.$$

We then have that an unbiased estimator of $w_2(\lambda, \alpha)$ is given by

$$\hat{w}_2(\lambda, \alpha) = \frac{1}{4\Phi_2(\alpha)(n-p)} \sum_{i=1}^{n} (y_i - x_i^\top \hat{\theta}^*)^2,$$

where $\Phi_2(\alpha)$ is the variance of a standard power Student t random variable.

5. Application to real data

In what follows, we shall present three applications of the proposed models in this paper to real data for illustrative purposes. We use two real data sets to compare the fit of the PBST distribution with PNBS, BS and GBSt distributions and an illustration of the PSHT regression model to compare with the SHN and SHT regression models.

5.1. Fatigue life of 6061-T6 aluminum coupons

We shall consider some actual data analysed previously by Birnbaum and Saunders [3], related to the fatigue life of 6061-T6 aluminum coupons cut parallel to the direction of rolling and oscillated at 18 cycles per second to the maximum pressure of 21,000 psi, with a sample size of 101 units. Descriptive statistics results are summarized in Table 1. There is indication of a slight asymmetry and that the kurtosis exceeds that of normality, which might be an indication of a good fitting for the PBST$_\nu$ model.

We adjust the Birnbaum-Saunders distribution (BS), the power normal Birnbaum-Saunders distribution (PNBS), generalized Birnbaum-Saunders Student t distribution (GBSt) and power Birnbaum-Saunders Student t distribution (PBST). The maximum likelihood estimators were computed by maximizing likelihood using the function *optim*

n	$\bar{t}$	s^2	$\sqrt{b_1}$	b_2
101	1400.91	1529.10	0.142	2.81

Table 1. Descriptive statistics for data set.

in R program [23]. To begin the maximization process we use as initial points the estimates of λ and β from BS distribution obtained by the method of modified moments (see [21]), and for α we use the method of elementary percentiles with known λ and β parameters (see [4]). The degrees of freedom were taken as 6 for GBSt and 18 for PBST models after having carried out multiple intents, from 1-30 degrees of freedom. Results are presented in Table 2 with AIC (Akaike information criterion defined in [1]) values.

Parameters	BS	PNBS	GBSt_6	PBST_{18}
λ	0.310(0.021)	0.099(0.000)	0.2577(0.021)	0.177(0.037)
β	1336.563(40.757)	2135.990(18.591)	1368.291(40.015)	1704.202(118.524)
α	–	0.052(0.005)	–	0.340(0.133)
AIC	1506.664	1501.092	1501.855	1497.781

Table 2. ML estimates for BS, PNBS, GBSt_ν and PBST_ν models.

We use two criteria with the purpose of verifying which model better fits the data. To compare the performance of the nested models, we use a criterion of [1], namely

$$AIC = -2\hat{\ell}(\cdot) + 2k,$$

where $\hat{\ell}(\cdot)$ is the maximized likelihood function and k is the number of parameters in the model. According to this criterion the model with the smallest AIC value is the best model for fitting data. Then, the AIC indicates a better fit for the PBST_{18} over GBSt_6 model, given that these are nested models. Hence, it seems to pay off using the PBST_{18} model over GBSt_6 model in spite of the fact that it involves an extra parameter.

We also compare the PNBS model with the PBST_ν model. Then according to property 3 we can conclude using the AIC criterion that the model PBST_ν fits better than the PNBS model. In addition, it can be corroborated that this model presents better adjustment than the BS and GBSt models, which justifies the use of a skew model to fit the data set.

For non-nested models, we use a generalized LR statistic test defined by Vuong [28]. This test was derived to compare competing models that are strictly non-nested. Being F_θ and G_ζ two non-nested models, $f(y_i|x_i, \theta)$ and $g(y_i|x_i, \zeta)$ two densities corresponding to these non-nested models, the likelihood ratio statistics to compare both models is given by

$$LR\left(\widehat{\theta}, \widehat{\zeta}\right) = \left\{ \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \log \frac{f\left(y_i|x_i, \widehat{\theta}\right)}{g\left(y_i|x_i, \widehat{\zeta}\right)} \right\},$$

which does not follow a chi-square distribution. To overcome this problem, Vuong proposed an alternative approach based on the Kullback-Liebler information criterion. Based on the distance between each model and the true process generating the data, namely the model $h^0(y|x)$, he arrived at the statistics

$$T_{LR,NN} = \frac{1}{\sqrt{n}} \frac{LR\left(\widehat{\theta}, \widehat{\zeta}\right)}{\widehat{w}}, \tag{13}$$

where

$$\widehat{w}^2 = \frac{1}{n}\sum_{i=1}^{n}\left(\log \frac{f\left(y_i|x_i, \widehat{\theta}\right)}{g\left(y_i|x_i, \widehat{\zeta}\right)}\right)^2 - \left(\frac{1}{n}\sum_{i=1}^{n}\log \frac{f\left(y_i|x_i, \widehat{\theta}\right)}{g\left(y_i|x_i, \widehat{\zeta}\right)}\right)^2.$$

For strictly non-nested models, the statistic (13) converges in distribution to a standard normal distribution under the null hypothesis of equivalence of the models. Thus, the null hypothesis is not rejected if $|T_{LR,NN}| \leq z_{p/2}$. On the other hand, we reject at significance level p the null hypothesis of equivalence of the models in favor of model F_θ being better (or worse) than model G_ζ if $T_{LR,NN} > z_p$ (or $T_{LR,NN} < -z_p$).

We now use (13) for comparing the PBST_{18} versus BS models fitted to the data, since they are two non-nested models. Let $f\left(y_i|x_i, \widehat{\theta}\right)$ the density corresponding to PBST_{18} distribution and $g\left(y_i|x_i, \widehat{\zeta}\right)$ the BS distribution. The generalized LR test statistic value is $T_{LR,NN} = 2,277$ (p-value $<$ 0,011) then the PBST_{18} model is significantly better than the BS models.

To compare the PBST_{18} versus GBSt_6 models fitted to the data, now let $g\left(y_i|x_i, \widehat{\zeta}\right)$ the density corresponding to GBSt_6 distribution. The generalized LR test statistic value is $T_{LR,NN} = 14.590$ (p-value ≈ 0), then again the PBST_{18} model is significantly better than the GBSt_6 model. Thus, the PBST_{18} model is the better model compared with BS and GBSt models.

More information is provided by a visual comparison of the histogram of the data with the adjusted density functions. It can be seen in Figure 4 that the BS and GBSt models fail to adjust the asymmetry in the data. Instead, the PBST_{18} model captures the asymmetry and fits the entire kurtosis of the data. Clearly, the new distribution provides a closer fit to the histogram.

Figure 5-(a) and 5-(b) depicts the *qq-plot* calculated with the estimates of the parameters in BS and PNBS models, respectively, note that the adjustment is very poor. Figure 5-(c) and 5-(d) depicts the *qq-plot* calculated with the estimates of the parameters in GBSt_6 and PBST_{18} models, respectively. Note that *qq-plot* for PBST_{18} model provides better fit for the data set than the BS, GBSt or GBSt_6 models.

5.2. Diameter at breast height of trees

The following application is associated with the distribution of the diameter at breast height (DBH) of trees. Leiva et al. in [14] consider that the trees die due to several factors caused by stress according to a phenomenon similar to material fatigue. The authors consider that the force (rate) of mortality of trees quickly increases at a first stage and then reaches a maximum. In that moment, this rate slowly decreases until

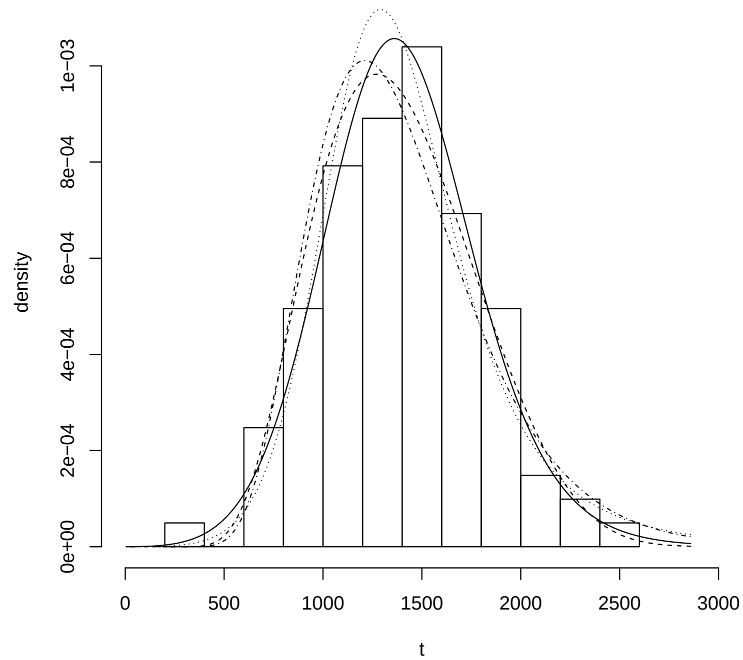


Figure 4. Histogram and adjusted models: PBST_{18} (solid line), PNBS (dashed line), BS (doted line) and GBSt_6 (doted and dashed line).

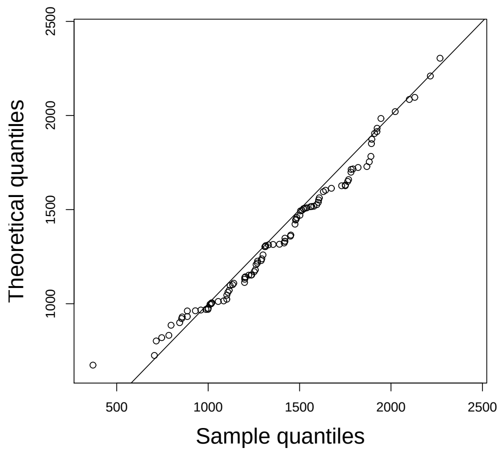
stabilizing at a constant value in the long term establishing a second stage of such a rate. BS models have their genesis from a problem of material fatigue and present a failure or hazard rate (equivalent to the force of mortality) that has the same behaviour as that of the DBH of trees. This linkage has been possible because the hazard rate of this distribution has two clearly marked phases that coincide with the force of mortality of trees. This mortality is related to the diameter at breast height of trees. For its part, the basal area allows the volume of a tree to be determinate setting thus the production of a forest. In [14] the data are presented and a statistical methodology is applied based on the Birnbaum-Saunders and Birnbaum-Saunders Student t distributions.

Median	Mean	SD	CV	CS	CK	Minimum	Maximum	n
13.70	14.54	3.61	24.84%	2.88	13.97	10.50	39.30	160

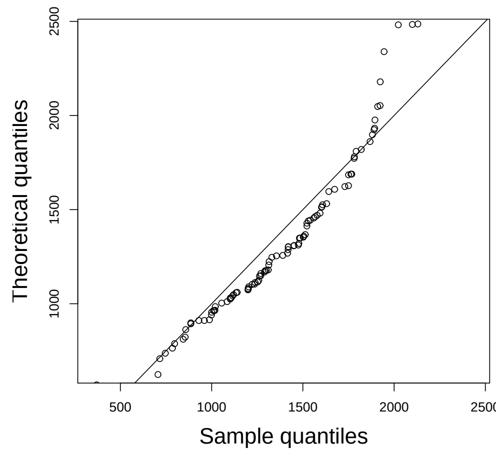
Table 3. Descriptive statistics for the DBH data.

After making a descriptive analysis (see Table 3) we adjust the BS, GBSt_ν, PNBS and PBST_ν distributions, the results are presented in Table 4.

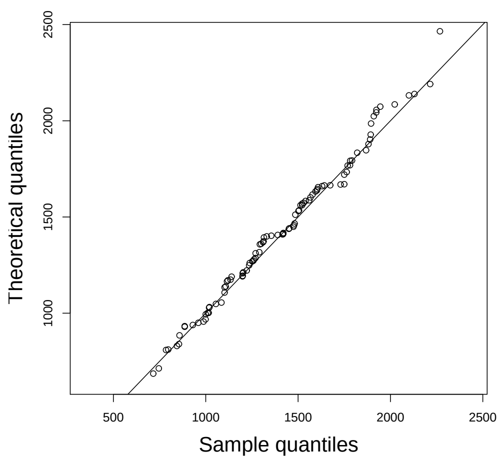
In Figure 6 shows the histogram of the data with the adjusted density functions. The



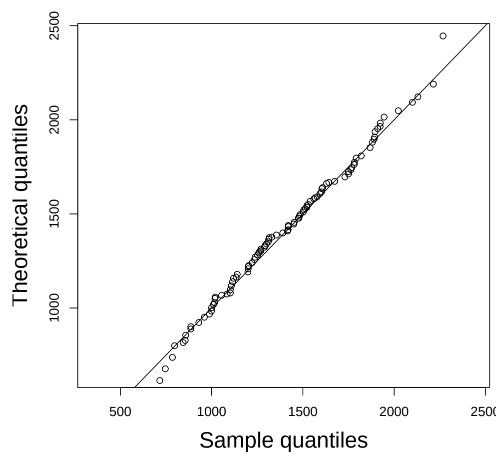
(a) BS



(b) PNBS



(c) GBSt_6



(d) PBST_{18}

Figure 5. *qq-plots* of the adjusted models for fatigue life of 6061-T6 aluminum coupons data.

Parameters	BS	PNBS	GBSt_4	PBST_4
λ	0.208(0.011)	0.363(0.033)	0.150(0.011)	0.202(0.015)
β	14.229(0.233)	7.795(0.787)	13.817(0.195)	9.618(0.428)
α	–	13.139(4.436)	–	8.238(1.981)
AIC	803.552	783.760	782.876	769.088

Table 4. ML estimates for BS, GBSt_4, PNBS and PBST_4 models.

BS, PNBS and GBSt models fail to adjust the asymmetry in the data and instead, the PBST_4 model captures the asymmetry and fits the entire kurtosis of the data. Clearly, the new distribution provides a closer fit to the histogram.

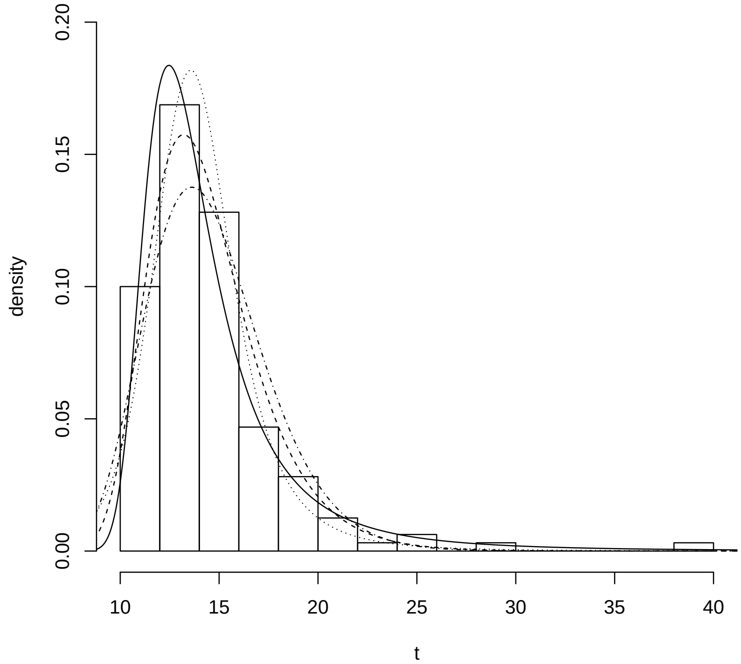


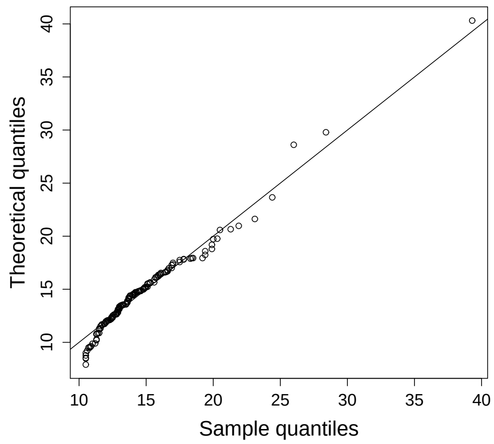
Figure 6. Histogram and adjusted models: BS (dotted and dashed line), PNBS (dashed line), GBSt_4 (dotted line) and PBST_4 (solid line).

Figure 7 depicts the qq-plot for BS, PNBS, GBSt_4 and PBST_4 models show clearly that the PBST_4 model fits well the data.

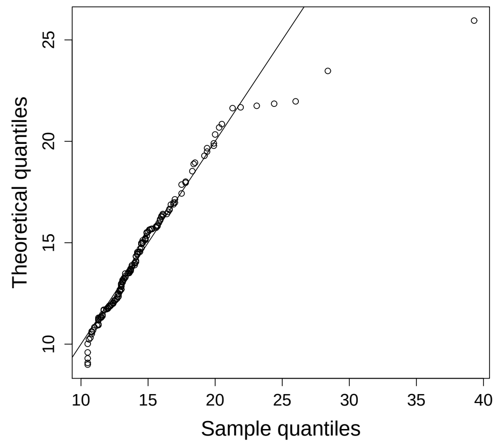
We justify the use of the PBST_ν model, this justification is carried out by means of the hypothesis test

$$H_0: \ \alpha = 1 \text{ versus } H_1: \ \alpha \neq 1$$

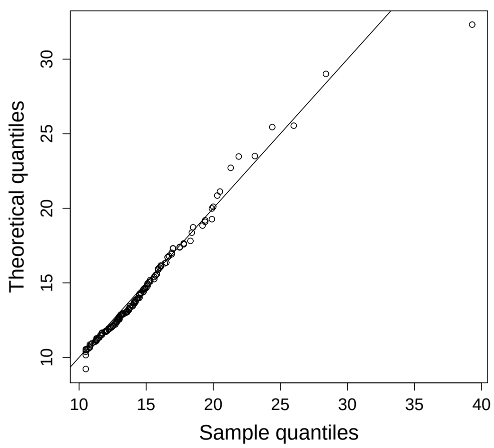
which compares the GBSt_ν distribution against the PBST_ν distribution. Using the like-



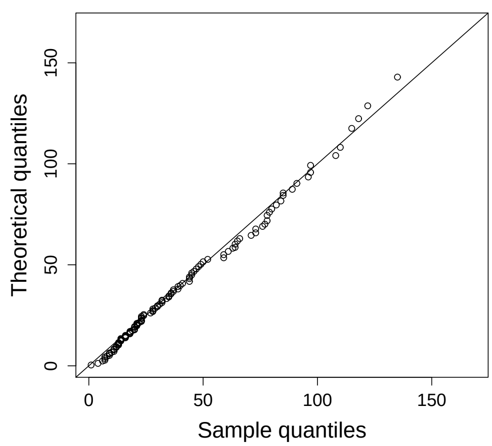
(a) BS



(b) PNBS



(c) GBSt_4



(d) PBST_4

Figure 7. qq-plot of the adjusted models for diameter at breast height of trees data.

lihood ratio statistics, based on

$$\Lambda = \frac{L_{\mathrm{GBSt}_4}(\hat{\lambda}, \hat{\beta})}{L_{\mathrm{PBST}_4}(\hat{\lambda}, \hat{\beta}, \hat{\alpha})},$$

which upon replacing by the corresponding MLEs leads to $-2\log(\Lambda) = 15.788$, being then greater than the corresponding chi-square 5% critical value which is given by 3.84, so that null hypothesis is rejected and we conclude that the PBST_4 model fits the data better than the GBSt_4 model. These results agree with AIC values calculated for these two models.

We use (13) again for comparing the BS, PNBS, $GBSt_4$ and $PBST_4$ non-nested models fitted to the data. For the $PBST_4$ versus BS model, $T_{LR,NN} = 6.766$ (p-value $< 6,62 \times 10^{-12}$), for the $PBST_4$ versus $GBSt_4$ model, $T_{LR,NN} = 5.473$ (p-value $< 2.21 \times 10^{-8}$) and for the $PBST_4$ versus PNBS model, $T_{LR,NN} = 2.254$ (p-value $<$ 0,012). Therefore, the $PBST_4$ model is significantly better than the BS, $GBSt_4$ and PNBS models according to the generalized LR statistic. Furthermore, the AIC criterion indicates that $PBST_4$ model is significantly better than the $GBSt_4$ model. Consequently, the $PBST_4$ model is the better model.

5.3. Times to failure in rolling contact fatigue

Lemonte considers in [15] the data set consisting of times to failure (T) in rolling contact fatigue of ten hardened steel specimens tested at each of four values of four contact stress points (X). The data were obtained using a 4-ball rolling contact test rig at the Princeton Laboratories of Mobil Research and Development Co. The data set was given initially in [17] and reported in [7]. These data set was also analyzed in [15], who like [7] considered the regression model

$$y_i = \beta_0 + \beta_1 \log(x_i) + \epsilon_i, \ i = 1, \ldots, 40,$$

where $y_i = \log(T_i)$ and $\epsilon_i \sim SSN(\alpha, -c(\alpha, \lambda), 2, \lambda), \ i = 1, \ldots, 40$, with

$$c(\alpha, \lambda) = 4 \int_{-\infty}^{\infty} \{\sinh(\alpha z/2)\}^{-1} \phi(z)\Phi(\lambda z) dz.$$

For this model we have that the MLE and asymptotic standard errors (SE) of the model parameters are $\hat{\beta}_0 = 0.1657(0.1759)$, $\hat{\beta}_1 = -13.8710(1.5887)$, $\hat{\alpha} = 2.0119(0.3487)$ and $\hat{\lambda} = 1.6423(0.5679)$ with $AIC = 125.36$. For more details, properties and uses of the SSN model, see [15].

For the regression model described above we adjust the sinh-normal (SHN) distribution, sinh-Student t (SHT_ν) distribution and alpha-power sinh-Student t ($PSHT_\nu$) distribution for $\epsilon_i, \ i = 1, 2, \ldots, n$. The SHT_ν model can be obtained as a special case of the $PSHT_\nu$ model when $\alpha = 1$. The estimates for the parameters of these models are presented in Table 5.

Parameters	SHN	SHT_{12}	$PSHT_{12}$
β_0	0.0978(0.1707)	0.2067(0.1661)	-2.3444(0.7924)
β_1	-14.1164(1.5714)	-13.2759(1.4704)	-13.6774(1.3864)
λ	1.2791(0.1438)	1.1193(0.1428)	2.9624(1.0410)
α			5.5156(1.6616)
AIC	129.2352	125.9024	121.3212

Table 5. MLE estimators for models, SHN, SHT_ν and $PSHT_\nu$.

Making the correction in the intercept of the $PSHT_\nu$ model we find that $\hat{\beta}_0^* = 0.1979$, so we estimate the vector $\hat{\theta}^* = (0.1979, -13.6774)^\top$. It is noteworthy that there are no important differences in the adjustment of regression models with SSN and PSHT_{12} distributed errors, but we can conclude that the regression model with PSHT_{12} error distribution provides a better fit than the regression model with SSN error distribution, because the PSHT_{12} model yields the smallest values of the AIC statistics and should be preferred.

We consider now the problem of testing the null hypothesis of no difference between the SHT_{12} model and the PSHT_{12} model, that is,

$$H_0: \ \alpha = 1 \ \text{ versus } \ H_1: \ \alpha \neq 1,$$

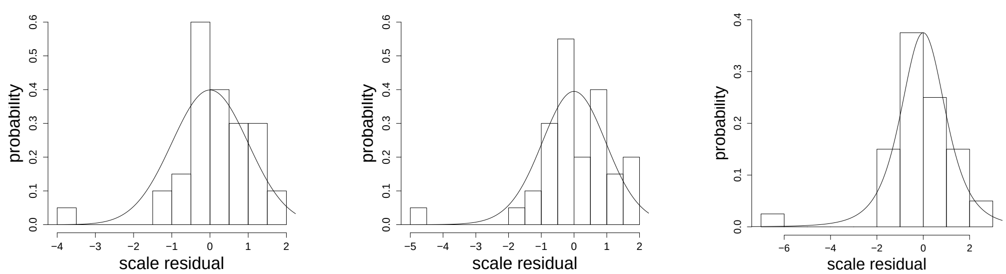
using the likelihood ratio statistics

$$\Lambda = \frac{L_{SHT_{12}}(\hat{\theta})}{L_{PSHT_{12}}(\hat{\theta})}.$$

Numerical evaluations indicate that

$$-2\log(\Lambda) = -2(-59.9512 + 56.6606) = 6.5812,$$

which is greater than the 5% critical value 3.84. Hence, the null hypothesis is rejected and we conclude that the PSHT_{12} model (which involves an extra parameter, making it more flexible in terms of asymmetry and kurtosis) fits the data better (in fact, much better) than the SHT_{12} model.



(a) SHN (b)SHT_{12} (c) PSHT_{12}

Figure 8. Probability histograms for the scaled residuals Z from (a) SHN and (b) SHT_ν models.

To confirm the good fit of the distributions used for the error term, we plotted the transformed standardized residual scale $Z_i = (2/\lambda)\sinh(Y_i - x^\top\beta)/2$ for the distribution of the estimated errors. Under this scale, the distribution of Z_i is normal for the SHN model, while for the SHT_ν model, Z_i is the Student t distribution with $\nu = 12$ degrees of freedom and for the PSHT_{12} model, Z_i is the PT_{12} distribution. Figure 8 shows the scaled residuals Z for the set of models with the indicated theoretical distributions. One can see the good fit of the model with errors PSHT_{12}. Thus, this model is presented as a viable alternative to study censored data when the distribution of the response variable is asymmetric.

6. Main conclusions

In this paper we develop a new family of distributions which can be used in a variety of practical situations which present flexible amounts of asymmetry and kurtosis. This new family corresponds to a generalization of the Birnbaum-Saunders Student t family of distributions. The density function for this new family is derived, and moments are studied, particularly, the mean, variance and asymmetry and kurtosis coefficients. Parameter estimation is considered by using the maximum likelihood approach and model comparison is implemented by using the generalized likelihood ratio statistics. Real data applications reveal good performances of the proposed model.

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