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Two versions of minimal intuitionism with the CAP. A note

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ABSTRACT. Two versions of minimal intuitionism are defined restricting Contraction. Both are defined by means of a falsity constant F . The first one follows the historical trend, the second is the result of imposing special constraints on F . Relational ternary semantics are provided.

Keywords: Intuitionistic logic, Contraction Axiom, Converse Ackermann Property, Constructive Falsity.

1. Motivation

As it is known, minimal intuitionism can be viewed as a definitional extension of positive intuitionistic logic. The idea is to add to the positive language a propositional falsity constant F along with the definition $\neg A \equiv_{df} A \rightarrow F$. Then, given the intuitionistic positive theorems,

- a. $[A \rightarrow (B \rightarrow C)] \rightarrow [B \rightarrow (A \rightarrow C)]$
- b. $A \rightarrow [(A \rightarrow B) \rightarrow B]$
- c. $(A \rightarrow B) \rightarrow [(B \rightarrow C) \rightarrow (A \rightarrow C)]$
- d. $[A \rightarrow (A \rightarrow B)] \rightarrow (A \rightarrow B)$
- e. $[A \rightarrow (B \rightarrow C)] \rightarrow [(A \rightarrow B) \rightarrow (B \rightarrow C)]$

we have, for example (and to limit ourselves to conditional-negation theorems) the following minimal intuitionistic theses:

- a'. $(A \rightarrow \neg B) \rightarrow (B \rightarrow \neg A)$
- b'. $A \rightarrow \neg\neg A$
- c'. $(A \rightarrow B) \rightarrow (\neg B \rightarrow \neg A)$
- d'. $(A \rightarrow \neg A) \rightarrow \neg A$
- e'. $(A \rightarrow \neg B) \rightarrow [(A \rightarrow B) \rightarrow \neg A]$



Minimal intuitionism departs (from this syntactical point of view) from full intuitionistic logic in the absence of, e. g.,

$$f. A \rightarrow (\neg A \rightarrow B)$$

or

$$g. (A \wedge \neg A) \rightarrow B$$

though, we note,

$$h. A \rightarrow (\neg A \rightarrow \neg B)$$

or

$$i. (A \wedge \neg A) \rightarrow \neg B$$

are present.

On the other hand, let us focus on the CAP. A positive logic with a falsity constant F has the Converse Ackermann Property (CAP) if all the formulas of the form $(A \rightarrow B) \rightarrow C$ are unprovable whenever C contains neither \rightarrow nor F . The CAP can intuitively be interpreted as the non-derivability of necessitive propositions from non-necessitive ones (A is *necessitive* if A is of the form NB , that is, if A is equivalent to $(B \rightarrow B) \rightarrow B$ (Anderson and Belnap 1975, §4.3)).

The question about which systems do possess the CAP is first proposed in Anderson and Belnap (1975, §8.12). In Méndez (1987) it is answered for implicative and positive logics. Syntactically speaking, the solution consists in restricting *contraction* ((d) above) and *assertion* ((b) above) to the case in which B is an implicative formula (A is implicative iff A is of the form $B \rightarrow C$). Thus, logics with the CAP are contractionless logics. Actually, they are the natural bridge between strict contractionless logics and logics with contraction.

But what about negation in these logics? That is, which kind(s) of negation(s) is (are) compatible with the CAP? We briefly note the results we are aware of. In Méndez (1988) a sort of semiclassical negation, in Kamide (2002), Kamide (2003) a so-called “strong negation” is added to the positive logics of Méndez (1987). Now, the aim of this paper is to define minimal negation in the positive intuitionistic logic with the CAP of Méndez (1987).

As the title makes clear, we consider two possibilities. The first one follows the historical trend commented above: we definitionally extend positive intuitionistic logic with the CAP, I^0_+ , with F . This gives us the logic I^0_{m1} . Now let us try to motivate the second one. As (a)-(e) above are not provable in I^0_+ , (a’)-(e’) are not provable either. Nevertheless, (a’)-(e’) are, of course, not only minimally acceptable but also desirable indeed. Moreover, they are CAP-compatible. In consequence, we shall show how to impose additional constraints on F to make these formulas valid, which give us the second possibility. This gives us the logic I^0_{m2} .

The logics we present here have not been, to our knowledge, defined in the literature. Nevertheless, it is not hard to derive them from some published work, as we show below.

2. The logics I_{m1}^0 , I_{m2}^0

Positive intuitionistic logic with the CAP, I_+^0 , can be axiomatized as follows (see Méndez (1987)).

Axioms:

$$A1. A \rightarrow A$$

$$A2. A \rightarrow (B \rightarrow A)$$

$$A3. (A \rightarrow B) \rightarrow [(B \rightarrow C) \rightarrow (A \rightarrow C)]$$

$$A4. A \rightarrow [[A \rightarrow (B \rightarrow C)] \rightarrow (B \rightarrow C)]$$

$$A5. [A \rightarrow [A \rightarrow (B \rightarrow C)]] \rightarrow [A \rightarrow (B \rightarrow C)]$$

$$A6. (A \wedge B) \rightarrow A \quad / \quad (A \wedge B) \rightarrow B$$

$$A7. [(A \rightarrow B) \wedge (A \rightarrow C)] \rightarrow [A \rightarrow (B \wedge C)]$$

$$A8. A \rightarrow (A \vee B) \quad / \quad B \rightarrow (A \vee B)$$

$$A9. [(A \rightarrow C) \wedge (B \rightarrow C)] \rightarrow [(A \vee B) \rightarrow C]$$

$$A10. [A \wedge (B \vee C)] \rightarrow [(A \wedge B) \vee C]$$

Rules:

Modus ponens (MP): if $A \rightarrow B$ and A , then B .

Adjunction (Ad): if A and B , then $A \wedge B$.

Now, I_{m1}^0 is a definitional extension of I_+^0 with the propositional falsity constant F along with the definition $\neg A \equiv_{\text{df}} A \rightarrow F$. For instance, the following theorems belong to I_{m1}^0 :

$$T1. \neg F$$

$$T2. (A \rightarrow B) \rightarrow (\neg B \rightarrow \neg A)$$

$$T3. \neg B \rightarrow [(A \rightarrow B) \rightarrow \neg A]$$

$$T4. F \rightarrow \neg A$$

$$T5. \neg A \rightarrow (A \rightarrow \neg B)$$

$$T6. A \rightarrow (\neg A \rightarrow \neg B)$$

$$T7. (\neg A \wedge \neg B) \leftrightarrow \neg(A \vee B)$$

$$T8. (\neg A \vee \neg B) \rightarrow \neg(A \wedge B)$$

$$T9. \neg(A \wedge B) \rightarrow (A \rightarrow \neg B)$$

$$T10. (\neg A \vee \neg B) \rightarrow (A \rightarrow \neg B)$$

$$T11. (A \vee \neg B) \rightarrow (\neg A \rightarrow \neg B)$$

$$T12. (A \wedge \neg A) \rightarrow \neg B$$

$$T13. [(A \vee \neg B) \wedge \neg A] \rightarrow \neg B$$

On the other hand, I_{m2}^0 is the result of adding to I_{m1}^0 the axioms:

$$A11. A \rightarrow [(A \rightarrow F) \rightarrow F]$$

i.e., $A \rightarrow \neg\neg A$, and

$$A12. [A \rightarrow (A \rightarrow F)] \rightarrow (A \rightarrow F)$$

i.e., $(A \rightarrow \neg A) \rightarrow \neg A$.

In addition to T1-T13 of I_{m1}^0 , we have, for example, the following theorems:

$$T14. (A \rightarrow \neg B) \rightarrow (B \rightarrow \neg A)$$

$$T15. (A \rightarrow B) \rightarrow [(A \rightarrow \neg B) \rightarrow \neg A]$$

$$T16. (A \rightarrow \neg B) \rightarrow [(A \rightarrow B) \rightarrow \neg A]$$

$$\text{T17. } (A \wedge B) \rightarrow \neg(A \rightarrow \neg B)$$

$$\text{T18. } \neg(A \wedge \neg A)$$

$$\text{T19. } \neg\neg\neg A \rightarrow \neg A$$

$$\text{T20. } \neg\neg(A \vee \neg A)$$

3. Converse Ackermann Property

It is proved in Salto, Méndez and Robles (2001) that LC_m^0 has the CAP. I_{m2}^0 is (syntactically) included in LC_m^0 . Therefore, I_{m1}^0 and I_{m2}^0 have the CAP.

4. Semantics for I_+^0

Given a pair $\langle K, R \rangle$ where K is a non-empty set and R a ternary relation on K , let us define the binary relation \leq , the quaternary relation R^2 and the five element relation R^3 by, for every $a, b, c, d \in K$,

$$\text{d1. } a \leq b \text{ iff } (\exists x \in K) Rxab$$

$$\text{d2. } R^2abcd \text{ iff } (\exists x \in K)(Rabx \text{ and } Rxcd)$$

$$\text{d3. } R^3abcde \text{ iff } (\exists x \exists y \in K)(Rabx \text{ and } Rxcy \text{ and } Ryde)$$

An I_+^0 model is a triple $\langle K, R, \leq \rangle$ where K is a non-empty set, R is a ternary relation on K satisfying the following conditions for every $a, b, c, d \in K$:

$$\text{P1. } a \leq a$$

$$\text{P2. } a \leq b \text{ and } Rbcd \Rightarrow Racd$$

$$\text{P3. } R^2abcd \Rightarrow (\exists x \in K)(Racx \text{ and } Rxbd)$$

$$\text{P4. } R^2abcd \Rightarrow R^2bacd$$

$$\text{P5. } R^2abcd \Rightarrow R^3abbd$$

$$\text{P6. } Rabc \Rightarrow a \leq c$$

Finally, \leq is a valuation relation from K to the sentences of I_+^0 satisfying the following conditions for all formulas p, A, B and a point a in K :

- i. $a \models p$ and $a \leq b \Rightarrow b \models p$
- ii. $a \models A \vee B$ iff $a \models A$ or $a \models B$
- iii. $a \models A \wedge B$ iff $a \models A$ and $a \models B$
- iv. $a \models A \rightarrow B$ iff for all $b, c, e \in K$, $(Rabc$ and $b \models A) \Rightarrow c \models B$

A is valid in I_+^0 iff $a \models A$ for all $a \in K$ in all models.

Remark: the postulates

$$P7. R^2abcd \Rightarrow (\exists x \in K)(Rbcx \text{ and } Raxd)$$

$$P8. Rabc \Rightarrow b \leq c$$

are derivable.

It is not difficult to prove along the lines of [3] that a formula A is valid iff A is a I_+^0 theorem.

5. Semantics for I_{m1}^0

A I_{m1}^0 model is a quadruple $\langle K, S, R, \models \rangle$ where S is a non-empty subset of K and $\langle K, R, \models \rangle$ is a I_+^0 model such that the relation \models satisfies in addition the clauses

- v. $a \leq b$ and $a \models F \Rightarrow b \models F$
- vi. $a \models F$ iff $a \notin S$

A formula A is I_{m1}^0 *valid* iff $a \models A$ for all $a \in K$ in all models. Semantic consistency (semantic soundness of I_{m1}^0 relative to the semantics of I_{m1}^0 models) is immediately derived from that of I_+^0 . (Note that being S non-empty, F is not valid).

6. Semantics for I_{m2}^0

A I_{m2}^0 model is similar to a I_{m1}^0 model but with the addition of the postulates

$$P9. (Rabc \text{ and } c \in S) \Rightarrow (\exists x \in S) Rbax$$

$$P10. (Rabc \text{ and } c \in S) \Rightarrow (\exists x \in K)(\exists y \in S)(Rabx \text{ and } Rxyb).$$

A formula A is I_{m2}^0 valid iff $a \vdash A$ for all $a \in K$ in all models. Semantic consistency is left to the reader (the validity of A11 is proved with P9; the validity of A12 with P10).

7. Completeness of I_{m1}^0 and I_{m2}^0

As noted in §3, I_{m1}^0 and I_{m2}^0 are sublogics of LC_m^0 . In Salto, Méndez and Robles (2001) it is proved the completeness of LC_m^0 with respect to an extension of the semantics here provided for I_{m2}^0 . Now, it is not difficult to prove completeness theorems for I_{m1}^0 and I_{m2}^0 using appropriate restrictions of the theorems and lemmas there employed (just let aside any references in Salto, Méndez and Robles (2001) to P7-P10 in the case of I_{m1}^0 and to P10 in the case of I_{m2}^0).

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