THEORIA. Revista de Teoría, Historia y Fundamentos de la Ciencia
ISSN: 0495-4548
theoria@ehu.es
Universidad del País Vasco/Euskal Herriko Unibertsitatea
España

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THEORIA. Revista de Teoría, Historia y Fundamentos de la Ciencia, vol. 30, núm. 3, 2015, pp. 331-348
Universidad del País Vasco/Euskal Herriko Unibertsitatea
Donostia, España

Non-Representational Mathematical Realism*

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Received: 06/03/2015
Final Version: 30/07/2015

BIBLID 0495-4548(2015)30:3p.331-348
DOI: 10.1387/theoria.14105

ABSTRACT: This paper is an attempt to convince anti-realists that their correct intuitions against the metaphysical inflationism derived from some versions of mathematical realism do not force them to embrace non-standard, epistemic approaches to truth and existence. It is also an attempt to convince mathematical realists that they do not need to implement their perfectly sound and judicious intuitions with the anti-intuitive developments that render full-blown mathematical realism into a view which even Gödel considered objectionable (Gödel 1995, p. 150).

I will argue for the following two theses: (i) that realism, in its standard characterization, is our default position, a position in agreement with our pre-theoretical intuitions and with the results of our best semantic theories, and (ii) that most of the metaphysical qualms usually related to it depends on a poor understanding of truth and existence as higher-order concepts.

Keywords: abstract entities, anti-realism, existence, numbers, realism, representationalism, truth.

RESUMEN: Este artículo es un intento de convencer a los antirrealistas de que sus correctas intuiciones en contra del inflacionismo metafísico derivado de algunas versiones del realismo matemático no les obligan a abrazar aproximaciones epistémicas no estándar a la verdad y la existencia. Es también un intento de convencer a los realistas matemáticos de que no necesitan aplicar sus intuiciones, perfectamente correctas y juiciosas, a los antiintuitivos desarrollos que hacen del realismo matemático pleno una visión que el propio Gödel consideró objetable (Gödel 1995, p. 150)

Argumentaré a favor de las dos tesis siguientes: (i) que el realismo, en su caracterización estándar, es nuestra posición por defecto, una posición de acuerdo con nuestras intuiciones pre-teóricas y con los resultados de nuestras mejores teorías semánticas; y (ii) que la mayor parte de los escrúpulos metafísicos habitualmente relacionados con él dependen de una pobre comprensión de la verdad y la existencia en tanto que conceptos de orden superior.

Palabras clave: entidades abstractas, antirrealismo, existencia, números, realismo, representacionalismo, verdad.

* I am very grateful to Gabriel Uzquiano and to two anonymous referees for detailed and helpful comments. I presented a version of this paper in the XV Coloquio Compostelano. I am deeply indebted to Concha Martínez, Jose Miguel Sagüillo, Uxía Rivas, Jose Luis Falguera and Xavier de Donato for their many suggestions, remarks and encouraging criticisms. The research in this paper has been supported by the Spanish Ministerio de Economia y Competitividad (Project FFI2013-44836-P), Ayudas a la Movilidad, Project PRX14/00157, and by the Plan Propio (Sabáticos) of the University of Granada.

1. The Issue

This paper is an attempt to convince anti-realists that their correct intuitions against the metaphysical inflationism derived from some versions of mathematical realism do not force them to embrace non-standard, epistemic approaches to truth and existence. It is also an attempt to convince mathematical realists that they do not need to implement their perfectly sound and judicious intuitions with the anti-intuitive developments that render full-blown mathematical realism into a view which even Gödel considered objectionable[1]. "Few topics", Coffa (1991, p. 94) says, "have elicited more heat and less light among philosophers during the past two centuries than the subject of realism". In my effort to cast some light, I will contend that realist intuitions are our intuitions by default, and that they are correct as they stand. "The main point in favor of the realistic approach to mathematics", says Moschkovakis (1980, p. 605), "is the instinctive certainty of almost everybody who has ever tried to solve a problem that he is thinking about 'real objects', whether they are sets, numbers, or whatever" [*his scare quotes*]. Realism only mutates into an intractable imposture when combined with a picture theory of meaning, in any of its versions from the most naïve to the most sophisticated, and with an understanding of truth as picturing success. Trivially true is also the anti-realist insistence that abstract entities are not "real", in the sense of *actual* or *effective*. The difficulties that theorists encounter in this subject do not derive, I contend, from metaphysical depth but from semantic shallowness. Even so, or precisely because of it, realists and anti-realists cannot settle their differences by rational argumentation; their discrepancies lie in background assumptions that typically include strong feelings about truth, likelihood, and truthfulness that people assume to be characteristic of the paradigm they live in. My aim in what follows is to induce both parties to look at the debate from a fresh perspective, identifying the many topics that are typically involved in it, and also the reasons to support the many different theses entangled in this endless metaphysical discussion. My approach to language is of a pragmatist kind that takes the speaker's opinion seriously, and of a naturalist kind that explicitly acknowledges our nature as part of the natural world (for a development, see Frápolli 2014a). My kind of pragmatism narrows down to the thesis that the point of departure of any theoretical analysis should be what the agents do when involved in communicative exchanges.

Abstract entities have been a source of puzzlement for philosophers since Plato. Issues such as the grounds of our knowledge of abstract entities, how discourse about them can be meaningful, and the kinds of properties they possess are some of the standard concerns that theorists have to face. This philosophical issue is one about which we have the impression not to have learnt anything in the last 2000 years. The empiricist approach to knowledge and language that analytic philosophy undertook at the beginning of the past century did not help reach a better understanding of what was at stake in the philosophy of abstract entities. The ground of meaning was placed in sense data or in the agent's causal links to physical objects ("acquaintance", in Russellian terms), and this left philosophers of mathematics

[1] Gödel said in a lecture of 1933: "our formalism works perfectly and is perfectly unobjectionable as long as we consider it as a mere game with symbols, but as soon as we come to attach a meaning to our symbols serious difficulties arise" (cited in Cassou-Nogues, p. 214, the reference is Gödel (1986-2003, III, p. 49): Gödel's Papers, box 7b, folder 26).

with a dilemma between two kinds of empiricism, reductionist or metaphorical. Reductionist empiricism holds the view that only physical entities exist and that non-empirical discourse and knowledge boil down to discourse and knowledge about the physical world. Russell and Quine held this approach. The non-reductionist alternative consists of granting abstract entities a realm of their own, but characterizing it as a pseudo-physical world in which entities possess pseudo-physical properties. Discourse about this realm is descriptive, mathematical intuition takes the place of acquaintance, and truth is faithful representation. Cantor and Gödel sometimes expressed themselves as metaphorical empiricists in this sense.

Modern proposals are technically more sophisticated (see Maddy 1990, for instance, and Frapolli 1992b for a review), but the basic framework does not seem to have moved: mathematical objects enjoy full-blown existence and, in their own way, they are as stable and independent as physical objects, or else mathematical objects are non-natural outcomes of the human mental activity, either fictions or constructions. Realism is the first position, according to which mathematicians do not invent mathematical objects or their properties. "Platonism" is often an alternative label for the same position, although sometimes this term suggests a specific kind of realism that stresses what Price (2011, p. 3) has recently called "the placement problem", i.e. where to locate the entities named by singular terms and the properties expressed by predicates.

As Benacerraf detected, the realism debate has two focuses: semantics and epistemology. The semantic focus is how to interpret the discourse about abstract entities; the epistemological focus is how knowledge about them is possible. When semantics is worked out in the standard representationalist lines, the semantic focus turns out to be a metaphysical focus. Representationalism makes Benacerraf's diagnosis unavoidable. "It is my contention", Benacerraf stated in (1973, p. 661), "that two quite distinct kinds of concerns have separately motivated accounts of the nature of mathematical truth: (1) the concern for having a homogeneous semantical theory in which semantics for mathematical sentences parallel the semantics for the rest of the language, and (2) the concern that the account of mathematical truth mesh with a reasonable epistemology. It will be my general thesis that almost all accounts of the concept of mathematical truth can be identified with serving one or another of these masters *at the expense of the other*."

In the playground delineated by Benacerraf's text, theorists have to choose between serving Semantics while abandoning Epistemology, or else serving Epistemology while abandoning Semantics. In the first case, theorists understand the semantics of abstract discourse along the lines in which we understand the semantics of empirical discourse. They should nevertheless explore a non-standard path in order to account for the epistemological issues. It is relatively straightforward to develop a story about how empirical terms can ultimately reach the physical objects and properties that are their meanings. Perception is usually the faculty that turns the trick in the classical empiricist proposal. Nevertheless, perception does not have an unequivocal counterpart in the abstract realm. Mathematical intuition is the common option; it is the notion used by Gödel, for instance, but there is no developed account of mathematical intuition that satisfies the epistemologist's standards.

In the second case, if a scientifically respectable epistemological approach is implemented, —one in which mathematical entities are accessible to the knower as ideas in the mind, as structures or signs that can be perceived by the senses, or as theoretical tools that help talk about physical objects in a general way—, standard, representationalist semantics would seem to be compromised. Theorists should then explain why sentences with apparently the same linguistic form, such as (1) and (2), require different semantic treatments.

(1) The number immediately following 4 in the series of natural numbers is 5.
(2) The child immediately next to Joan in the picture is Pablo.

In this scenario, there are background assumptions that very rarely emerge and that we have all the reasons to re-assess:

— From a metaphysical and epistemological perspective, the questionable assumption is that existence is some sort of placement or embodiment, an assumption that rarely is explicit. Realists assume that the objectivity granted to mathematical knowledge presupposes representation and correspondence. Anti-realists, on the other hand, reject the realist universe while struggling to get rid of the charge of relativism and subjectivity attached to their views. Both realists and anti-realists are misguided, and the debate is a perspicuous instance of a genuine philosophical problem placed in a distorted framework.
— From a semantic viewpoint, representationalism is the crux of the matter. Within the boundaries of representationalism the design of an acceptable proposal about the meaning of abstract discourse becomes, as the history of the topic shows, an impossible task. If representationalism is expelled from the picture, and replaced by a less naïve approach to meaning, a significant portion of the obstacles on the realist side vanishes. In addition, the often-tortuous anti-realist explanations prove unnecessary.

The metaphysical and epistemological dispute is objectivity vs. subjectivity; the semantic dispute is connected to representationalism and its consequences.

I will argue for the following two theses: (i) that realism, in the standard characterization that can be seen in the text of Maddy quoted below, is our default position, a position in agreement with our pre-theoretical intuitions and with the results of our best semantic theories, and (ii) that most of the metaphysical qualms usually related to it depends on a poor understanding of truth and existence as higher-order concepts.

2. *How realists see themselves*

How declared mathematical realists characterize their own positions will help identify their real concerns. Cantor and Frege in the 19th century and Gödel in the first half of the 20th century were probably the best known among realist mathematicians. None of them put forward a worked-out metaphysical view on numbers, but they described their attitude towards the objects of their professional interest in a way that has been understood as unequivocally realist. Cantor's defence of his powers and orderings as genuine numbers (Cantor 1883, Frápolli 1992a), Frege's exposition of the weaknesses of the psychologist and formalist approaches to the notion of number (in Frege 1884), and Gödel's discussion of Cantor's Continuum Hypothesis (Gödel 1947) are paradigms of what is customarily understood as "realism". Thus, understanding the concerns that led Cantor's, Frege's and Gödel's work is a good starting point to identify the main theses that characterize mathematical realism.

Cantor, Frege, and Gödel insisted that numbers are as objective as physical entities and that mathematical laws are as independent of the knower's will as physical laws. Cantor confessed to Mittag-Leffler in a letter of 1884: "I have been about the content of my work only a reporter and a civil servant" (Fraenkel 1932/1962, p. 480). The same insistence on objectivity is patent in Frege's *Grundlagen* (1884, § 26): "I distinguish what I call objective from what is handleable or spatial or actual. The axis of the earth is objective, so is the centre of mass of the solar system, but I should not call them actual in the way the earth itself is so. We often speak of the equator as an imaginary line; but it would be wrong to call it an imaginary line in the dyslogistic sense; it is not a creature of thought, the product of a psychological process, but is only recognized or apprehended by thought. If to be recognized were to be created, then we should be able to say nothing positive about the equator for any period earlier that the date of its alleged creation". This text shows that Frege set apart objectivity from other properties of empirical objects often related with existence, such as placement and actuality. Numbers exist, but they are not placed anywhere nor possess any causal effect. Gödel also found abstract and physical objects similar in some epistemological aspects. "It seems to me", Gödel says against Russell's constructivism, "that the assumption of such objects [mathematical objects] is quite as legitimate as the assumption of physical bodies and there is quite as much reason to believe in their existence. They are in the same sense necessary to obtain a satisfactory system of mathematics as physical bodies are necessary for a satisfactory theory of our sense perception" ("Russell's mathematical logic", 1944, p. 137). The import of Gödel's view falls better in place when one realizes that Gödel, like Cantor and Frege before him, is defending his position against psychologism and intuitionism. In addition, he explicitly defended mathematical intuition as a safe way to accede to mathematical entities: "I don't see any reason", he said in (1947, p. 484), "why we should have less confidence in this kind of perception, i.e. in mathematical intuition, than in sense perception. [. . .] It seems that, as in the case of physical experience, we form our ideas also of those objects on the basis of something else which is immediately given. Only this something else here is not, or not primarily, the sensations. That something besides the sensations actually is immediately given follows [. . .] from the fact that even our ideas referring to physical objects contain constituents qualitatively different from sensations or mere combinations of sensations, e.g. the idea of object itself, whereas, on the other hand, by our thinking we cannot create any qualitatively new elements, but only reproduce and combine those that are given." Unfortunately, Gödel said almost nothing about how he understood this epistemic capacity.

Cantor's transition from a nominalist understanding of transfinite powers and orderings (see Frápolli 1991, 1992a) towards a full acknowledgement of their nature as genuine numbers illustrates the scope of his metaphysical commitments. About the reality of integers he never held any doubt, and after disclosing the properties of powers and ordinal numbers (in *Grundlagen* (1883) and *Beiträge* (1985), (1987)) his confidence extended to the new numbers too. He declared to have been "forced" (Cantor 1966, p. 175) towards the new numbers and confessed that, at the beginning, he was unaware of the real significance of his auxiliary symbols. Cantor explained his conversion in epistemological terms, and insisted that he felt as much constrained by the nature of the new numbers as classical mathematicians by the nature of integers.

Mathematical existence is, in Cantor's work, a tricky issue. Freedom, he vindicated (Cantor 1966, p. 182), is the essence of mathematics, a vindication that led some scholars to interpret Cantor as defending that the consistency of a theory is enough to accept the objects postulated by it as existent (Dauben 1979, p. 129 and Grattan-Guinness 1982). But Cantor insisted on the freedom of mathematical work as much as on the resistance that mathematical objects present, two notions —freedom and resistance— that seem to pull in opposite directions. Hallett (M. Hallett 1984, p. 19) dispels the tension by appealing to the Plotinian Principle of Plenitude: every possible thing exists, a principle that blurs the differences between possibility and necessity and that Cantor explicitly endorsed (1966, p. 400). In spite of this apparent philosophical sophistication, it would be a mistake to assume that Cantor had an elaborated and detailed position about these difficult topics. He was not a metaphysician and the only aim of his incursions in metaphysics was to protect his numbers from the attack of theologians and mathematicians that rejected the infinite in act. By insisting in the mathematician's freedom, Cantor vindicated his right to develop his theory of transfinite numbers following his own criteria; by insisting on his role as a "Beamter", he ruled out that the new numbers were mere fictions without reality. Kronecker, who famously stated against Cantor that "the natural numbers were created by God; all the others are the invention of humans", was a fictionalist[2] *avant la letre*. Cantor, by contrast, could not accept that, after a struggle of years, his transfinite numbers were bogus entities like Odin or the Valkyries.

In the painful process of unfolding the nature of his new numbers, Cantor had to adjust the scope of some largely assumed mathematical intuitions. "The old and oft-repeated proposition 'Totum est majus sua parte' [the whole is larger than the part]", said Cantor in (1882, p.92), "may be applied without proof only in the case of entities that are based upon whole and part; then and only then is it an undeniable consequence of the concepts 'totum' and 'pars'. Unfortunately, however, this 'axiom' is used innumerably often without any basis and in neglect of the necessary distinction between 'reality' and 'quantity', on the one hand, and 'number' and 'set', on the other, precisely in the sense in which it is generally false". The Euclidian principle mentioned in this text was not the only one that stood between mathematical tradition and the Cantorian revolution. The thesis that a one-one matching between two multiplicities implies that they have the same cardinal number, responsible of Galileo's Paradox, also had dramatic consequences for the mathematics and the philosophy of infinity, as Bolzano's case illustrates. The Czech mathematician was able to acknowledge different sizes of infinity (1851, §19), he even defined arithmetic operations between different sizes or multiplicities ("Grösse"), but he did not possess the theoretical framework that would allow him to draw the conclusion that his *Grosse* were *Zahlen* (1851, §19 for instance).

There is a substantial lesson that the history of Cantorian numbers teaches and that Cantor explicitly stated, i.e. that elements in a theoretical realm should not be blamed for lacking properties that define elements in a completely different one. "All so-called proofs against the possibility of actually infinite numbers are faulty", he told to Eneström in 1885, "as can be demonstrated in every particular case, and as can be concluded on general grounds as well. It is their πρῶτον ψευδος[3] that from the outset they expect or even impose all the properties of finite numbers upon the numbers in question, while on the other hand

[2] Fictionalism will be explained at the end of this section.

[3] Literally, "the first lie".

the infinite numbers, if they are to be considered in any form at all, must (in their contrast to the finite numbers) constitute an entirely new kind of number, whose nature is entirely dependent upon the nature of things and is an object of research, but not of our arbitrariness or prejudices." (Letter to Eneström, 1885; cited in Dauben, p. 125). Trivial as this conclusion might seem, we fail to draw it time after time. A similar πρῶτον ψευδος lies behind a significant part of the debates between realists and anti-realists, where theorists often assume that abstract objects should dwell somewhere. As they cannot be found in the physical world, realists postulate non-physical worlds to place them and anti-realists end up by denying their existence. Chairs and persons are objective entities that are placed in space and time; numbers and sets are objective entities that nevertheless are nowhere. The objectivity that Cantor, Frege, and Gödel vindicate is not placement but the anti-psychologist intuition that can be stated as [ER]:

> *Epistemological Resistance* [ER]: Abstract sciences deal with objects and concepts whose properties do not depend on the particular minds that entertain them or on the will of particular thinkers.

In this respect, there is nothing special in the case of numbers, and in fact they behave like many other abstract entities. From the proposition that Victoria is a girl, follows the proposition that she is a human being. This is a semantic fact that imposes on us with the force of logical (semantic) necessity. The proposition that number two is an even prime number imposes on us with the force of mathematical necessity. Humans do not decide the meaning of concepts; we might decide, to a certain extent, that "girl" means GIRL or that "two" means TWO. Clearly, these two words could have meant something else. Nevertheless, once they belong to a language and they have a precise role inside our conceptual system they mean what they mean, and their use bears consequences that the speaker cannot help but to assume. All this is compatible with the assumption that concepts evolve and change, if the theories that explain their behaviour suffer variations. Nevertheless, independently of the language that we use, it is not up to speakers to decide whether the concept GIRL is subsumed under the concept HUMAN BEING, as it is not up to speakers to decide whether the number two is a even number. Abstract concepts and objects constitute a further domain beside the world and our representation of it. Physical objects are located in the physical world; our representations of them are ideas in our minds, but abstract entities are neither inside nor outside, neither in the world nor inside our heads. The objectivity of concepts is Frege's lessons against Kant's philosophy of mathematics. Linking objectivity to placement completely misses Frege's point.

Some developments of full-blown realism are admittedly quite unpalatable. Even Gödel acknowledged that "as soon as we come to attach a meaning to our symbols serious difficulties arise" (see footnote 2). Full-blown anti-realism scores no better. Fictionalism[4] is an illustration of a kind of anti-realism that has broken out energetically in the philosophical debate. It intervenes in the debate as an attempt to explain why realism seems so

[4] Balaguer offers the following definition: "Fictionalism is the view that (a) our mathematical sentences and theories do purport to be about abstract mathematical objects, as Platonism suggests, but (b) there are no such things as abstract objects, and so (c) our mathematical theories are not true."("Fictionalism", *Stanford Encyclopedia of Philosophy*).

appealing and, at the same time, in which sense anti-realists are right in their suspicions. Mathematics is, according to this view, a fictive narrative about fictional entities, a narrative in which truth does not play any role. Theorists and users of mathematical discourse, fictionalists hold, are seriously misled about the nature of their speech acts. In the Benacerraf scenario, fictionalism applies to mathematical discourse the semantics of empirical discourse and deactivates the epistemological horn. Fictionalism is a coherent view. Unfortunately, inner coherence cannot be the only criterion to assess fictionalism as a theory of abstract discourse. Semantics should be more than a non-contradictory approach, since language is a natural ability socially developed, a substantial part of human evolution (see Mendivil-Giro 2014), and not a disembodied theory about combinatorial properties of signs. The intuitions of speakers should be the point of departure of any theoretical proposal aiming to explain the use of language. According to the way in which it is defined, fictionalism is a kind of error theory, i.e. a theory that considers speakers systematically wrong about their linguistic intuitions. Error theories have a difficult accommodation in pragmatist or naturalist views of language, because they annihilate the possibility of semantics by dismissing the philosophical ground on which it grows. A basic tenant of contemporary pragmatism is the Availability Principle: "What is said must be intuitively accessible to the conversational participants (unless something goes wrong and do not count as 'normal interpreter'" (Recanati 2004, p. 20). "'What is said'", he declares, *"is* consciously available to the participants in the speech situation" (op. cit., p. 13). And he continues: "The availability of what is said follows from Grice's idea that saying itself is a variety of non-natural meaning" (loc. cit.). Thus, fictionalism clashes against some well-established contemporary theoretical positions, and also against our pre-theoretical intuitions. As a way out of the debate between realism and anti-realism, it does not represent a genuine improvement.

Nominalism can be seen as a mild version of fictionalism that, nevertheless, boils down to an identical assumption, that scientific statements are useful fictions (Goodman 1964). Field (1983), a different kind of nominalism, discloses a sophisticated argument against Quine-Putnam indispensability argument. And again the upshot comes down to the same, that there is no need to consider mathematics as a system of truths (Field, 1983, p. viii). Quine (in for instance 1960, p. 242) and Putnam (1979) reluctantly accept mathematical objects because science cannot do without them, Field devotes his many philosophical skills to show that they are not right. As I hope to make clearer at the end of this essay, neither Putnam's and Quine's painful acceptation nor Field's smart rejection are justified.

3. *Six theses associated with realism*

Realism about abstract entities involves some of the assumptions expressed in the following theses [T1]-[T6].

[T1], [T2] and [T3] are explicit semantic theses:

[T1] *Bivalence*: Mathematical sentences are true or false.

In 1982, Dummett introduces bivalence as a characterization of realism, which he interprets as a local position. It follows from Dummett's characterization that one can be a

realist about a particular kind of discourse and anti-realist about some other kind. Kronecker's statement against Cantorian numbers is an example of realism about integers and anti-realism about irrational and transfinite numbers. "Realism", Dummett says, "involves acceptance, for statements of the given class, of a principle of bivalence, the principle that every statement is determinately either true or false" (Dummett 1982, p. 55). The explicit linguistic perspective represented in [T1] is present in other characterizations of realism that nevertheless centre their analysis on the reference of mathematic terms; Dummett's originality lies in his focus on complete sentences. Gödel's approach to the Continuum problem is an example of realism in Dummett's terms. Bivalence is not only a semantic position but includes an epistemic aspect, too, for it usually implies the rejection of constructivism of mathematical entities. The case of Gödel is clear: if the truth-value of the Continuum Hypothesis cannot be determined, this means that the axioms of set theory do not offer a complete characterization of the realm (Gödel 1947, p. 519). The Hypothesis must be either true or false, even though mathematicians do not possess the tools to settle the issue.
Theses, [T2] and [T3] are general assumptions that serve as background in the debate of realism vs. anti-realism.

[T2] *Descriptivism*: Declarative sentences describe how the world is,
[T3] *Correspondence*: Truth is accurate description.

Both theses are connected, [T2] sets the semantic background and [T3] defines truth within it. But in spite of their apparent obviousness, there are solid reasons to reject [T2] and [T3]. See for instance Austin's formulation of the descriptive fallacy (Austin 1962, p. 3), Strawson's criticism of Austin's proposal about truth, and Price's rejection of the placement problem. I have argued against them in Frápolli (2013) and Frápolli (2014a).

Explicit metaphysical theses are [T4] and [T5]:

[T4] *Outerness*: Mathematical entities are outside the human mind.

The scope of [T4] becomes clearer if the view it excludes is explicitly stated. In general, by means of [T4], theorists show their rejection of psychologism and constructivism. Frege, for instance, says that "number is no more an object of psychology or a product of mental processes than, let us say, the North Sea is. The objectivity of the North Sea is not affected by the fact that it is a matter of our arbitrary choice which part of all water on the earth's surface we mark off and elect to call the 'North Sea'" (Frege 1884, §26). And he insists: "If number were an idea, then arithmetic would be psychology. But arithmetic is no more psychology than, say, astronomy is. Astronomy is concerned, not with ideas of the planets, but with the planets themselves, and by the same token the objects of arithmetic are not ideas either" (1884, § 27). Thus, despite the way in which [T4] is usually stated, [T4] does not open a debate about placement. Being spatially located is a property of physical entities, and *in* and *out* are relations of physical objects. My purse can be either inside or outside my bag. Nevertheless, when we say that numbers are out of the human mind we surely wish to say something different. "In" and "out" apply metaphorically to abstract entities to stress their similarities with those objects for which "in" and "out" make literal sense, i.e. physical objects. As always happens, the metaphor is incomplete. Numbers are like chairs and dogs in that we do not invent their properties or their relations, but they are different in all aspects that characterize numbers as abstract entities. An alternative way of expressing [T4] is [T5],

[T5] *Independence*: Mathematical objects are independent of the human mind.

Sober (1982, p. 369), for instance, defined realism as "a declaration of independence", meaning the independence of mental images, of epistemic abilities, and of linguistic conventions. [T5] is a standard metaphysical claim, that numbers are substances that exist on their own. [T5] rules out (i) that numbers are properties of objects, i.e. that like size and weight they need a substance to bear them, and (ii) that numbers are psychological entities, i.e. that like qualia and ideas they need an individual mind to bear them. In the Fregean picture, individual numbers are objects in the same sense in which the North Sea or the distance between the Earth and the Moon are. Numerical properties are properties of concepts, and they belong to the same logical category to which existence belongs.

[T5] has also an epistemic interpretation, that mathematical laws are not affected by our knowledge of them. "Speaking quite generally", Parsons says, "philosophers often talk as if we all know what it is to be a realist, or a realist about a particular domain of discourse: realism holds that the objects the discourse talks about exist, and are as they are, independently of our thought about them and knowledge of them, and similarly truths in the domain hold independently of our knowledge" (Parsons 1995, p. 46).

The independence of linguistic conventions is more difficult to assess, for "convention" sometimes means something that we can change at will, and sometimes something which does not belong to the natural world. We cannot change at will the meaning of our terms or the content of our concepts, even though there seems to be no physical necessity in the particular way in which language has effectively evolved.

Realism often includes thesis [T6],

[T6] *Aboutness*: Mathematics is about numbers.

[T6] rules out the interpretation of mathematics as a game with empty signs, i.e. some versions of formalism and fictionalism. "Realism", Maddy says as a preliminary characterization, "is the view that mathematics is the science of numbers, sets, functions, etc., just as physical science is the study of ordinary physical objects, astronomical bodies, subatomic particles, and so on. That is, mathematics is about these things, and the way these things are is what makes mathematical statements true or false. This seems a simple and straightforward view. Why should anyone think otherwise?" (Maddy, 1990, p. 2). Maddy's view is richer than what is derived from this quote (for a review of Maddy's realism and of Maddy's naturalism, see Frápolli 1992b and Frápolli 2001), but it is still an outstanding expression of the intuitions that support most kinds of realism. Maddy's evolution from realism to open naturalism in mathematics reflects the tortuous path that theorists with solid scientific commitments have to take in order to assume the objectivity of mathematical thought without renouncing their scientist credo. Usually, nevertheless, the path leads nowhere, because neither the background assumptions nor the theoretical tools assumed in the journey enable us to understand the semantics of abstract discourse, which is the crux of the matter.

Representationalism has the effect of blurring the frontiers between semantics and metaphysics.

4. My kind of realism

Default realism is a spontaneous attitude. It is not a further theory to be put on the shelf with the rest of the metaphysical approaches held by philosophers over the centuries, but the kind of background assumption theorists and laypersons live with. Pragmatism accepts the intuitions of agents as the point of departure of any semantic analysis, as the Availability Principle expresses (see p. 10 above). It is true that philosophy is not common sense, but if it has to enlighten our comprehension of the world, it should be informed common sense. Like sciences, it should explain the agents' systematic intuitions and respect the results of our best scientific theories. "Let us not pretend to doubt in philosophy what we do not doubt in our hearts", advised Peirce in (1868, p. 140). Following his advice, I see no reasons to object to the simple truisms expressed in [T4], [T5] and [T6]. Default mathematical realism is then the conjoined acceptance of [T4], [T5], and [T6].

> *Default Mathematical Realism [DMR]*: Mathematical discourse is about mathematical entities; mathematical entities are objective, governed by their own laws, and independent of our knowledge of them.

[DMR] accounts for the "systematic intuitions" part, the part of "our best scientific theories" is provided by the Aristotelian approach to truth, assumed by most of the truth theorists in the past century, and by the Fregean approach to existence, widely considered as the starting point of contemporary logic (see for example Thiel 2009, p. 197).

> *Aristotelian Truth [AT]*: To say of what it is that it is is the truth (*Metaphysiscs* 1011b 27)
> *Fregean Existence [FE]*: Affirmation of existence is denial of the number nought (*Grundlagen*, § 53)

The conjunction of Default Mathematical Realism [DMR], Aristotelian Truth [AT], and Fregean Existence [FE] is Non-representational Mathematical Realism [NRMR]. I support [NRMR], and fully accept the intuitions that inspired the work of Cantor, Frege, and Gödel. In addition, I consider the truism expressed in [AT] difficult to resist, and praise [FE] as the best explanation of the semantics of the existential quantifier available so far.

Some readers might feel that I have not given an argument for [NRMR]. And they would be right. But I hope to have shown that in order to maintain the sensible intuitions that fuel realism, and at the same time the consequences of the Aristotelian take on truth and the Fregean characterization of existence, [NRMR] is enough. If you still want to add to this picture the standard metaphysical developments that define full-blown realism, there must be other reasons. And my guess is that the added reasons are related with the theory of meaning that seems to be most natural, the representationalist view. Nevertheless, the representationalist view, even if it might be harmless to explain the semantics of the empirical discourse, applied to discourse about abstract realms imposes a misguiding analogy. This analogy is the πρῶτον ψευδος mentioned by Cantor.

From [NRMR] it does not follow that numbers possess the first-order properties that characterize physical objects. Numbers do not have weight or colour, and even if realists insist that they are "eternal" and "self-standing" objects, they do not stand anywhere and they do not have temporal properties. The metaphorical approach proper of standard realism should be developed in a non-metaphorical way; otherwise, it is either empty or nonsensical.

4.1. Aristotelian Truth

The Aristotelian characterization of truth, "To say of what is that it is not, or of what is not that it is, is false, while to say of what is that it is, and of what is not that it is not, is true" (*Metaphysics* 1011b25), makes a grammatical claim that can be rendered in many different forms. One of them is Tarski's T-Convention, an unimaginative instance is the standard (1)

(1) "Snow is white" is a true sentence if and only if snow is white (Tarski 1933/1983, p. 155)

Another, less boring version is Ramsey's definition, (2),

(2) "What he believed is true" $\equiv_{df}$ "If p is what he believed, p" (Ramsey 1929, p. 10).

These incontrovertible characterizations have systematically been interpreted as supporting a metaphysically inflated correspondentist view. Nevertheless, Ramsey explicitly rejected that his definition offered a version of the Theory of Truth as Correspondence (*loc. cit*, p. 11), and Tarski explained that his approach was independent of any philosophical implementation.

The Aristotelian "what is", sentences in T-Convention, and instances of *p* in Ramsey's approach could well be sentences (3), (4) or (5), their names or the propositions expressed by them, depending on the favoured views.

(3) Love is in the air,
(4) God is almighty,
(5) Every even number is the sum of two primes.

If love is in the air, to say that love is in the air is true; "God is almighty" is a true sentence if and only if God is almighty; and "What he believes is true" means that if he believes that every even number is the sum of two primes, then every even number is the sum of two primes. Nothing in the classical approaches suggests that the sentences involved should depict states-of-affairs. Assertion is the kind of speech act in which truth-discourse makes sense, but assertion does not imply description, as Austin 1962, Strawson 1950, or Price 2011 make clear. The translation of these ordinary intuitions about the behaviour of truth in standard logical languages is the combination of the rules of Introduction and Elimination for the truth operator. Field (forthcoming, p. 3) offers the following formulations:

[True Elimination] t = <A>, True (t) |— A
[True Introduction] t = <A>, A |— True (t),

where "t" is a singular term denoting sentence A, and "<,>" is a function from sentences to names of them. The only constraint for A and t is that they be well-formed expressions of the language under discussion. The formal rules and the informal or semi-formal characterizations mentioned so far exhaust the semantic behaviour of the standard notion of truth. All of them make the same grammatical point, devoid of any factual content. For this reason, they have attracted so much support and their rejection throws away with them the standard, ordinary notion. Asserting (6),

(6) It is true that the Moon is the Earth's only satellite,

and rejecting (7),

(7) The Moon is the Earth's only satellite,

is a contradiction. Asserting (8),

(8) London has more than ten million inhabitants,

and rejecting (9),

(9) It is true that London has more than ten million inhabitants,

is a contradiction. The rules that govern assertion together with the meaning of truth imply that by asserting (6) an agent is committed to (7), and that by asserting (8) an agent is committed to (9). The technical details that explain the ordinary behaviour of truth require a sophisticated theory of meaning, which also accounts for the appeal of versions of the correspondence and redundancy theories of truth, and the immunity of [AT] against the standard criticisms derived from the semantic paradoxes. But this discussion lies outside the scope of this paper (for detailed discussion, see Frapolli 2013).

The semantic confusion derived from implementing the rules that govern truth with an empiricist account of the grounds of meaning and knowledge reveals itself in all its strength in the realm of the philosophy of science. Van Fraassen's proposal, constructive empiricism, is an outstanding example. There is no contradiction in claiming with van Fraassen that a theory can be empirically adequate without being true. "In contrast, constructive empiricism", van Fraassen explains in (1980, p. 6), "which also opts for a literal understanding of scientific language, is the following view: *Science aims to give us theories which are empirically adequate; and acceptance of a theory involves as belief only that it is empirically adequate* (but also has a pragmatic dimension, to be elucidated)". Nevertheless, to say that scientists could accept the theories they work with and at the same time remain neutral about their truth is hardly defensible. Of course, theorists can put forward theories tentatively to test their potential, but if they work with a theory, if they seriously use it in their explanations and predictions, then they cannot reject its truth since this acceptance is what truth means. "What this means is that acceptance of science, and appreciation of its worth", says van Fraassen (1994, p. 133), "does not require us to believe that it is true. On the contrary, the important point about scientific activity is not that it provides theories which every generation in turn can take as true, but rather that it accustoms us to giving up our beliefs, changing and altering them, valueing them without being in bondage to them". To assert the truth of a content is to avow an explicit commitment to its consequences. This is compatible with the healthy acknowledgement of our fallibility and the provisional nature of our dearest theories. With the exception of his theses on truth, I am comfortable with the view of science that van Fraassen put forward, and it is discouraging to see that the resistance that his (allegedly anti-realist) approach continuously encounters derives from a rejection of his use of truth. The focus on truth is invariably a symptom of an inflated approach that adorns the ordinary notion with metaphysical (realism) or epistemic (anti-realism) ornaments.

Aristotle, Ramsey, and Tarski captured the common notion of truth, formally codified in the Introduction and Elimination rules of the standard formal theory. To accept a theory as true is to endorse it, not to believe that the theory replicates some section of the physical world. By accepting a theory as true the subject becomes committed to the assertion (actual or potential) of its consequences. The assertion of any proposition *p* entitles the agent to assert that *p* is true and commits the agent to the content of that assertion; the assertion of the truth of a proposition entitles the agent to assert the proposition at stake and commits him to the content of the proposition asserted. Enriching these simple mechanisms with metaphysical and semantic narratives about language mirroring the world is an option, though independent of the meaning of truth (see Frapolli 2013, chapter 2).

4.2. Fregean Existence

Existence is the other key ingredient in the realism debate. The Fregean account of quantifiers as higher-order concepts is a nuclear piece of the way in which logicians and philosophers understand existence and universality. At the same time, philosophers seem to feel an irresistible attraction towards the identification of existence with some kind of localization: if something exists, it should be somewhere. [FE] and existence as localization are nevertheless incompatible views. The celebrated Fregean account makes the notion a higher-order concept, the account of existence as localization makes existence a property of objects. One or the other, but one cannot at the same time praise [FE] as the correct analysis and get entangled in heated discussions about the existence of abstract objects.

Let us recall the main traits of [FE]. In 1884, Frege distinguished between properties of objects and characteristics of concepts, a distinction that opened the door to the analysis of quantifiers as higher-order concepts. The differences in inferential behaviour of expressions that grammatically belong to the same category made Frege reject grammar as a guide to logical properties. In one of the expressions analysed in the *Foundations of Arithmetic*, "four thoroughbred horses" (§ 52), the two grammatical adjectives "four" and "thoroughbred" modify the noun "horses" in contrasting ways, supporting inferences of diverse kinds. The adjective "thoroughbred" expresses a property of particular horses whereas "four" expresses a property of the concept itself that does not apply to the objects in its extension. From (10) and (11),

(10) Four thoroughbred horses draw the King's carriage,
(11) Abendlied is one of the horses that draws the King's carriage,

(12) but not (13) follows,

(12) Abendlied is thoroughbred,
(13) Abendlied is four.

In fact, (13) displays a category mistake even though it is a well-formed English sentence. That existence and numbers belong to the logical category of quantifiers, which are always higher-order notions, is the core of the much acclaimed [FE]. "In this respect", Frege says in 1884, § 53, "existence is analogous to number. Affirmation of existence is in fact nothing but denial of the number nought. Because existence is a property of concepts the ontological argument for the existence of God breaks down". Existence and number indicate the size of a concept's extension; existence is just instantiation, the "denial of the number nought".

The status of numbers in the Fregean universe might seem puzzling. When assimilated to existence, numbers seem to be higher-order concepts. When what is at stake is their status as "self-subsistent objects" (as Frege says in § 58 of his 1884 book, *Grundlagen*), they seem to be objects. This ontological duplicity parallels the linguistic duplicity of "four" as a name and as an adjective. But the confusion appears only when the focus of analysis is too narrow: "Never to ask for the meaning of a word in isolation, but only in the context of a proposition" (1884, p. xxii). Out of a context, the question of whether "four" is a name or an adjective does not make sense. In (10), "four" stands for a property; in (14),

(14) Four is an even number,

on the other hand, "four" works as the proper name of a number. If existence is what Frege says it is, i.e. a property of concepts and not of objects, then the standard formulation of mathematical realism — "Do numbers exist?"— becomes either trivial, if the enquiry is about the concept of number, or nonsense, if the enquiry is about particular numbers.

As it happens with truth, the semantic behaviour of existence is seen in the rules that govern its introduction and elimination. The existential generalization of (10) and (14) are (15) and (16),

(15) There are four horses that draw the King's carriage,
(16) There is at least an even number.

Examples (10) and (16) express true propositions. The standard assumption that their truth presupposes the descriptive character of the speech acts in which they occur is a constraint that representationalism adds to the Fregean account of existence.

In addition to being higher-order, Fregean existence is a homogenous notion that makes a constant semantic contribution in all contexts. Consider examples (17) and (18),

(17) There are numbers,
(18) There are black swans,

In both cases what is being said is that the concepts "being a number" and "being a black swan" are not empty, i.e. that some objects fall under them. In the first case, these objects are abstract objects such as four, the sum of three and five, or the square root of nine. In the second case, these objects are some animals with particular characteristics. The concepts share the property of being instantiated, but their instances do not need to share any feature. Swans are spatio-temporal entities and numbers are not, and numbers and swans exist because instances of the two concepts can be provided.

Mathematical realists are sometimes misled by the analogy with scientific realism. Scientific realism focuses on natural sciences, and natural sciences attempt to explain physical reality. For a natural scientist, it surely is an extraordinary achievement to identify the spatio-temporal coordinates of an object predicted by the theory, be it the planet Pluto, the

Higgs bosom or a black hole. When this happens, scientists have proof of existence, but the existence of which they have proof relates to the concepts "being a planet in such-and-such-position", "being a Higgs bosom", or "being a black hole". These concepts are first-order concepts that represent properties of physical entities, and possessing a localization in space and time is an essential feature of the entities that occupy the physical universe. The equivalent proof for a mathematician cannot be any kind of placement, not because mathematical existence is *sui generis*, but because numbers are abstract entities. As Ramsey said of his analysis of truth: "All this is really so obvious that one is ashamed to insist on it, but our insistence is rendered necessary by the extraordinary way in which philosophers produce definitions [...] in no way compatible with our platitudes" (1929/1991, p. 13).

Non- representational Mathematical Realism, the position I have proposed in this paper, respects our ordinary intuitions, the motivations that led Cantor, Frege, and Gödel to put forward their views, and also the semantic core of the contemporary paradigm of higher-order concepts such as truth and existence. The increasingly sophisticated realist and anti-realist developments surely point to deep philosophical issues worth considering, but in spite of the insistence of most theorists in linking the metaphysical debate with the meanings of truth and existence, the metaphysical debate is independent of the semantic characterization of the higher-order notions involved. The issue of the meanings of truth and existence should be settled before engaging in metaphysical discussions about the correct way of understanding the role of abstract entities. [AT] and [FE] are not only intuitively correct but also theoretically appropriate. The intuitions that explain the force of realism are better understood as a defence of the objectivity of science than as a defence of a specific type of existence proper of a specific type of objects. The meagre advances made in the development of a minimal framework in which realists and anti-realists can understand each other and the lack of impact that the philosophical debate has had in the effective development of scientific theories suggest that maybe the whole story should be re-thought and the debate re-cast in different terms. I hope that [NRMR] make a contribution, however small, to the cutting of the Gordian not in this complex and fascinating issue.

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