ANTI-REALISM AND INFINITARY PROOFS

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Abstract

In the discussion about Yablo's Paradox, a debated topic is the status of infinitary proofs. It is usually considered that, although a realist could (with some effort) accept them, an anti-realist could not do it at all. In this paper I will argue that there are plausible reasons for an anti-realist to accept infinitary proofs and rules of inference.

KEY WORDS: Anti-realism; Realism; Infinitary logic.

Resumen

En la discusión sobre la Paradoja de Yablo, un tópico debatido es el estatus de las pruebas infinitarias. Se suele considerar que, aunque un realista podría (con cierto esfuerzo) aceptarlas, un anti-realista no podría hacerlo en absoluto. En este artículo, argumento que hay razones plausibles para que un anti-realista acepte pruebas y reglas de inferencia infinitarias.

PALABRAS CLAVE: Anti-realismo; Realismo; Lógica Infinitaria.

1. Introduction

One of the deepest topics of discussion around Yablo's Paradox is the debate about the possible length of arithmetical proofs. The very concept of proof is usually associated with the idea of an always finite set of sentences, where there are permitted transformations between them. Nevertheless, the finiteness of proofs is not conceptually unavoidable: infinitary systems, for example, make use of infinitary proofs. In fact, the non-circular proof of Yablo's paradox that Cook designs (which was also announced by Priest, Sorensen and others) consists in an infinite number of steps, and uses sentences of infinite length and infinitary rules. It's easy to see that a proof like that could raise some objections.

Graham Priest, for example, says that nobody can actually perform an infinitary rule, and that when we talk 'as if' we performed one, it's actually finite information what is justifying us. Sorensen (1998) and Cook (forthcoming), on the other side, defend a somewhat realist position, which says that even if no human being could perform an inference like that,

that doesn't mean that the proof doesn't exist. The proof would be, then, one of the many things that exist but we can't know (or maybe some superhuman entity can). Bringsjord and Van Heuveln (2003) make an original point: they claim that human beings are not Turing machines but supermachines, that is, machines that can perform infinite tasks in a finite amount of time; if that is true, they say, there are no reasons to reject infinitary proofs.

In this article I will focus on this discussion, but I will take a different account. Following Sorensen, who observed that any realist can be justified to reject infinitary proofs (claim with which I don't necessarily agree[1]), I will ask which position can an anti-realist take with respect to this issue. Prima facie, it's unconceivable that an anti-realist would accept infinitary rules or sentences. But, I think that if we focus on the internal discussions of anti-realism (which authors like Tennant, Wright and Dummett developed), the rejection of this kind of rules is much less obvious than it appears.

2. Manifestation

One of the central elements of anti-realism (or at least, of a Dummettian anti-realism) is the manifestation requirement[2]. It says that a subject knows the meaning of a sentence only if she can recognize a proof of it, and evaluate whether it is correct or not (that is, when she can manifest her knowledge of the verification-conditions of the sentence). Nevertheless, this requirement seems too restrictive, and was discussed inside the anti-realist tradition. In particular, there are some statements whose proofs are much longer than anything a human being (or the whole number of human beings) could read in their entire life (Ketland 2005). In those cases, it's insensitive to ask people (or even experts!) to recognize a proof when they see it; because, plausibly, they won't be able to see the whole proof in order to recognize it in its parts and check it step by step. So, an expert competence could be insufficient for comprehension of a statement.

[1] In particular, there use to be a strong rejection between realists of infinitary proofs. That is mainly because even if they consider that this kind of inferences can preserve truth (Tarski's 1936 case is paradigmatic), it's supposed to be something 'human' or 'feasible in principle' in the notion of proof. This article is, in part, an answer to the realists who hold that position.

[2] I am consciously ignoring a vital point of this account: the rejection of bivalence. But I follow Tennant (1997) in drawing a difference between anti-realism as the rejection of recognition-transcendent truths and anti-realism as the rejection also of bivalence.

The traditional way of proceeding in these cases is idealization: it's supposed that the speaker, if she had a great amount of time and a memory that exceeds by far human capacities, could recognize if the proof is correct or not[3].

Nevertheless, the idealization seems to go too far away from the actual competence: what is going to be evaluated is if the speaker has potentially a level of comprehension and capacities that may not reach in a lifetime. Much worse, because of the difficulty for evaluating this kind of idealized potential capacities, that makes difficult to fulfill of an important manifestationist desideratum: that we should be able to verify whether a speaker understands a statement.

Because of those reasons, I think that it's better to think of manifestation in a different way. A better response to this problem, which doesn't appeal to idealization, was proposed by Tennant (1997) by the name of 'factorization'. He says that an individual understands a mathematical statement only if he can recognize the relevant aspects of the proof of it, and is able to verify, locally, the correctness of any of its parts, so that he understands if he is dealing with a valid proof or not.

With this solution, it seems less clear why we should reject infinite statements whose proofs are infinitary. For, (e.g.) regarding Cook's proof of Yablo's Paradox, it seems that a competent logician, even if she can't evaluate that proof step by step, is able to differentiate the relevant aspects of it, and recognize it as a valid proof. That is, I think, much closer to our intuitions. If we are faithful to the traditional intuitionistic account, we should say that we actually don't understand infinite sentences like S1 ('S2 is false and S3 is false and…'), because in order to recognize a proof of it, we should pay attention to infinite applications of the universal instantiation rule or the conjunction introduction rule. But it seems reasonable to hold that we do understand a sentence like that, partly because in a manifestationist sense, we can recognize a proof of it, not checking every step in particular, but understanding in a general way the relevant aspects and checking if they are right or not.

One might object that when we meet an application of an infinitary rule, no factorization can be done[4]. In fact, no human being can see an

[3] Idealization often includes some increase in the mathematical capacities of the subject, specially in the practice and the familiarity with the theorems and axioms that are used in the proof. Only with that idealization we can explain that a lot of mathematicians actually understand what Fermat's theorem says, but can't evaluate if Wiles' proof is correct or not.

[4] This objection was raised by Ignacio Ojea.

infinite numbers of premises, in order to evaluate if the rule was correctly applied; there seems to be no finite part to separate and evaluate step by step. That seems different from what happens with normal, finitary rules, where a normal subject can go back and see the finite (and maybe few) steps that served as premises. But actually, factorization does not mean exclusively separating the parts of the proof and checking the correctness of the rules applied. It means also to see the relevant structure of the proof and evaluate it. Surely, it's impossible to check all the infinite steps needed for an application of the omega-rule; but yet it's possible to check if there is something like a uniform method which can be applied for each number at its time (and check, for any of the infinite instantiations of that method, if the subproof is valid), as normally do infinitary logicians. That is empirically the same way in which a regular mathematician evaluates the validity of an unfeasible finite proof; so, if in that case we could argue that the subject recognizes it as valid, the same could be said in the infinitary case.

Obviously, it remains a substantial difference between infinitary proofs and unfeasible finite proofs: the former is infinite, and the latter is finite. So, the rejection of infinitism from an anti-realist standpoint should be, not in the manifestation principles, but in the infinitism itself.

3. Finitism

Anti-realism is deeply committed to finitism. Since its birth, with Brouwer, the claim that we can't make up infinite mental constructions was related to the claim that time and reality are finite (Brouwer 1928). With Dummett, the concept of an infinite proof would go, conceptually, against the verificationism that characterizes his position. Every proof should be decidable in a finite time; that is, for every proof, in a finite amount of time it should be possible to determine whether it is correct.[5]

Nevertheless, I think that anti-realism can, not easily but reasonably, dismiss this old commitment. The main reason is that the finiteness of proofs, or in anti-realist terms, the 'in principle decidability' of them, obeys to an idealization that is, as C. Wright says, bizarre: proofs that no one could ever perform, because incredibly great memory, concentration, time and inferential capacity are needed, are decidable 'in principle', and it's supposed that they are legitimate because the needed

[5] Hilbertian formalism has a less visible commitment to finitism; in some papers, like (Hilbert 1931), Hilbert suggest the inclusion of rules whose finiteness is matter of discussion (see Ignjatovic 1992).

extension of cognitive capacities is only finite (in a not so clear way[6]). The most natural anti-realist reaction to that idealization is the one of the strict finitist: they consider that there's a limit for human feasibility, established by biological and physical laws, and that no proof exceeding that limit could be legitimate.[7]

Strict finitism has serious difficulties to ground arithmetics; in spite of that, and as Wright (1982) observes against Dummett (1975), it can be reasonably defended and characterized. Which is to say that it cannot be ruled out as incoherent. The strategy of a traditional finitist to overcome the strict finitist objections has to be grounded, then, in the same idealization as the one of the last section: even if it's true that no person (or machine) could perform proofs of more than n steps (let's suppose, 10^{100}), she can do it 'in principle', in a (incredibly large) finite time, with infinite memory, etc. That draws a line between what is feasible in a finite time with 'finite capacities' and what is not.

But, if we forget all the physical and cognitive boundaries for the 'decidable in principle' requirement, and we claim that every finitely conceivable proof is acceptable, there cease to be strong reasons to stay there and don't go further. For on the one side, if the world was physically different and, e.g. the time were infinitely divisible, an ideal human being could perform supertasks (i.e. infinite tasks in a finite time)[8]. On the other hand, even if the very notion of performing infinite tasks in a finite time was an inconsistent concept, there aren't strong reasons to be always committed to finiteness. If we are talking from an ideal point of view, because we have stopped to worry about what a human being can actually do, we could broaden our notion of 'decidable in principle', including not only what can be shown to be false or true in a finite amount of time, but also what can be verified or falsified by intuitively acceptable rules and axioms in an infinite time. After all, the anti-realist who tries to answer to the strict finitist has to relax substantively the commitment to actual human capacities; so the attachment to what is finitely

[6] In Wright's words (1982, p. 223) : 'It is not obviously sensible to speak of a finite extension of at least some of the particular capacities determining the scope of our overall practical powers. (…) The only purchase, it seems, which we have on the idea of a finite increase in such powers is via the finiteness of the tasks that would then be within our compass; now, though, on pain of circularity, an independent explanation is required of the finiteness of tasks -independent , that is, of any appeal to the finiteness of the capacities of creatures who could actually perform'.

[7] Positions as those were plausibly defended by Wittgenstein (1964) and van Dantzig (1956), although it's obviously not the predominant anti-realist standpoint

[8] In fact, Moore (1990) claims that this is the main challenge for intuitionism.

decidable is not as well motivated as it was, and it's reasonable to look for less restrictive principles. This is obviously different from the traditional anti-realism, and could be seen as an argument against it[9]. But I'm not saying that traditional anti-realists should embrace infinitary proofs; they are free to narrow their concept of decidability in the typical way, and I think that there are good motivations to do that (the whole history of anti-realism is in their side!). My point is that those motivations are not conclusive and can be perfectly challenged; so, apart from the typical finitist anti-realism, there is also room for an infinitist anti-realism that is motivated by the same philosophical considerations (semantical manifestation principle, rejection of recognition-transcendent truths).

Something good about this anti-realist defense of infinitism is that we don't have to ask any more if we can 'actually' apply the omega-rule. If we relax the requirements of manifestation and decidability in principle, it results that our use of rules like those in some occasions (for example, where we know that they preserve truth) doesn't alter any fundamental principle of anti-realism. Because, as in other mathematical proofs, in this one we can see its relevant aspects, analyze every one of its finite parts (although not the entire proof), and determine if they are correct or not. This also can be useful to answer to the JC. Beall's claim that we 'can't see and infinite sequence of sentences' (*cf.* 2001): we can say that, actually, we can see it in the same way that we can see schematically finite proofs that we can't write with all the paper in the world.

A last objection that any anti-realist could raise is that, even if we concede that we can evaluate infinite proofs and that they can exist, we can't actually construct infinite objects. To this objection we can answer the same as before. It's true that we can't construct a proof of omega steps as we construct a proof of three or four steps (that is, writing it in a sheet of paper), but any version of finitism that wants to avoid strict finitism has to admit proofs that can't be physically constructed. And so, if we can actually recognize proofs in a purely general and schematic way (by factorization), we can also construct them in the same way: a schema of proof (or the proof for the first steps, followed by 'and so on…') may perfectly be enough. This idea of construction seems to be closer to the practice of infinitary mathematicians. In this way, some proofs will be in strict sense larger than anything we could see o construct physically, but they will be as well constructed, recognizable and evaluable as every other.

[9] This was suggested by Lucas Rosenblatt.

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