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A NOTE ON DIRICHLET AND FEJÉR KERNELS

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Abstract

We exhibit a trigonometric identity which implies a link between the kernels of Dirichlet and Fejér.

Keywords: Fourier Series, Fejér and Dirichlet Kernels; Chebyshev polynomials.

Resumen

Exhibimos una identidad trigonométrica que implica una relación entre los núcleos de Dirichlet y de Fejér.

Mathematics Subject Classification: 42A20, 42A24, 43A50.

1 Introduction

In the original approach to Fourier series [1], it is convenient to consider the following partial sums for the interval $[-\pi, \pi]$:

$$f_n(y) = \frac{1}{2}a_0 + a_1 \cos y + \cdots + a_n \cos(ny) + b_1 \sin y + \cdots + b_n \sin(ny) \quad (1.1)$$

assuming for a_r, b_r the values:

$$a_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(mt) dt, \quad b_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin(mt) dt, \quad (1.2)$$

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and investigate what happens if n increases to infinity. From (1.1) and (1.2) we obtain:

$$f_n(y) = \int_{-\pi}^{\pi} f(t) K_n(t-y) dt, \quad (1.3)$$

with the Dirichlet kernel [2]-[4]:

$$K_n(\theta) = \frac{1}{2\pi} \frac{\sin(n + \frac{1}{2})\theta}{\sin \frac{\theta}{2}} \quad (1.4)$$

Then the convergence:

$$\lim_{n \rightarrow \infty} f_n(y) = f(y),$$

has to be restricted to a definite class of functions $f(y)$ verifying the known Dirichlet conditions [2]-[4].

Fejér [5] invented a new method of summing the Fourier series by which he greatly extended the validity of the series. Using the arithmetic means of the partial sums (1.1), instead of the $f_n(y)$ themselves, he could sum series which were divergent. The only condition the function still has to satisfy is the natural restriction that $f(y)$ shall be absolutely integrable. Then, in the Fejér approach we construct the sequence:

$$\begin{aligned} g_1(y) &= f_0(y), & g_2(y) &= \frac{1}{2}[f_0(y) + f_1(y)], & g_3(y) &= \frac{1}{3}[f_0(y) + f_1(y) + f_2(y)], \dots, \\ g_n(y) &= \frac{1}{n}[f_0(y) + f_1(y) + \dots + f_{n-1}(y)], \end{aligned} \quad (1.5)$$

accepting the expressions (1.1) and (1.2), therefore:

$$g_n(y) = \int_{-\pi}^{\pi} f(t) K_n(t-y) dt, \quad (1.6)$$

thus we see that Fejér's results come about by the fact that his method is related with the following Kernel [2]-[4],[5]:

$$K_n(\theta) = \frac{1}{2\pi n} \frac{\sin^2(n\frac{\theta}{2})}{\sin^2(\frac{\theta}{2})} \quad (1.7)$$

then a $f(y)$ absolutely integrable in $[-\pi, \pi]$ guarantess the convergence of $g_n(y)$ towards $f(y)$.

In the next section we exhibit a trigonometric identity which shows relationships between the kernels of Dirichlet and Fejér.

2 Links between Fejér and Dirichlet kernels

It is possible to demonstrate the following identity:

$$\frac{\sin(2n+1)\frac{\theta}{2}}{\sin \frac{\theta}{2}} = \frac{1}{\sin^2 \frac{\theta}{2}} \left[\sin^2(n+1)\frac{\theta}{2} - \sin^2(n\frac{\theta}{2}) \right], \quad (2.1)$$

which in terms of (1.4) and (1.7) means:

$$K_{n,D}(\theta) = (n+1)K_{n+1,F}(\theta) - nK_{n,F}(\theta), \quad (2.2)$$

that is, the Fejér kernel generates to Dirichlet kernel. For example:

$$\begin{aligned} K_{0,D} &= K_{1,F}, & \text{because} & & K_{0,F} &= 0, \\ K_{1,D} &= 2K_{2,F} - K_{1,F} & \therefore & & K_{2,F} &= \frac{1}{2}(K_{0,D} + K_{1,D}), \\ K_{2,D} &= 3K_{3,F} - 2K_{2,F} & \therefore & & K_{3,F} &= \frac{1}{3}(K_{0,D} + K_{1,D} + K_{2,D}), \quad \text{etc.} \end{aligned}$$

then the inverse relation to (2.2) adopts the form:

$$K_{n+1,F}(\theta) = \frac{1}{n+1} \sum_{r=0}^n K_{r,D}(\theta), \quad n = 0, 1, 2, \dots \quad (2.3)$$

thus the Dirichlet kernel permits to construct the Fejér kernel.

On the other hand, the fourth-kind Chebyshev polynomials are given by [6],[7]:

$$W_n(x) = W_n(\cos \theta) \equiv 2\pi K_{n,D}(\theta) = \frac{\sin(n + \frac{1}{2})\theta}{\sin \frac{\theta}{2}}, \quad (2.4)$$

with $\theta \in [0, \pi]$ and $x \in [-1, 1]$, therefore:

$$\begin{aligned} W_0 &= 1, & W_1 &= 2x + 1, & W_2 &= 4x^2 + 2x - 1, \\ W_3 &= 8x^3 + 4x^2 - 4x - 1, & W_4 &= 16x^4 + 8x^3 - 12x^2 - 4x + 1, & \text{etc.} \end{aligned} \quad (2.5)$$

The relationship (2.3) is important because with it we can introduce new polynomials, in fact:

$$\tilde{W}_n(x) = \tilde{W}_n(\cos \theta) \equiv 2\pi(n+1)K_{n+1,F}(\theta) = \frac{\sin^2(n+1)\frac{\theta}{2}}{\sin^2 \frac{\theta}{2}}, \quad (2.6)$$

$$= \sum_{r=0}^n W_r(x), \quad (2.7)$$

that we name “fifth-kind Chebyshev polynomials”, which are not explicitly in the literature. For example, (2.5) and (2.7) imply the expressions:

$$\begin{aligned} \tilde{W}_0 &= 1, & \tilde{W}_1 &= 2x + 2, & \tilde{W}_2 &= 4x^2 + 4x + 1, \\ \tilde{W}_3 &= 8x^3 + 8x^2, & \tilde{W}_4 &= 16x^4 + 16x^3 - 4x^2 - 4x + 1, & \text{etc.} \end{aligned} \quad (2.8)$$

and some of their properties are:

$$\begin{aligned}
 \tilde{W}_n(x) &= (n+1) \sum_{r=1}^{n+1} \frac{2^r}{n+r+1} \binom{n+r+1}{2r} (x-1)^{r-1}, \quad \tilde{W}_n(1) = (n+1)^2, \\
 \tilde{W}_{n+1}(x) &= 2 + 2x\tilde{W}_n(x) - \tilde{W}_{n-1}(x), \\
 \sum_{r=0}^{\infty} \tilde{W}_r(x) z^r &= \frac{1+z}{[(1-z)(1-2xz+z^2)]}, \quad |z| < 1, \\
 (x-1)\tilde{W}_n(x) &= 2^n \sum_{r=0}^{n+1} \binom{n+1}{r} {}_2F_1\left(r-n-1, -n-\frac{1}{2}; -2n-1; 2\right) x^r - 1.
 \end{aligned} \tag{2.9}$$

In a forthcoming work we will study some related topics such as Rodrigues formula, orthonormality, interpolation properties, etc., for this new set of polynomials $\tilde{W}_n(x)$.

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