Modified Symplectic Structures in Cotangent Bundles of Lie Groups.

F.J. Vanhecke, C. Sigaud, and A.R. da Silva
Instituto de Física, Instituto de Matemática, UFRJ, Rio de Janeiro, Brazil
(Received on 10 April, 2008)

In earlier work [1], we studied an extension of the canonical symplectic structure in the cotangent bundle of an affine space $Q = \mathbf{R}^N$, by additional terms implying the Poisson non-commutativity of both configuration and momentum variables. In this article, we claim that such an extension can be done consistently when Q is a Lie group G.

Keywords: Symplectic Mechanics; Noncommutative Configuration Space.

1. INTRODUCTION

As applied to physics, noncommutative geometry is understood mainly in two ways. The first one is the spectral triple approach of A.Connes [2] with the Dirac operator playing a central role in unifying, through the universal action principle, gravitation with the standard model of fundamental interactions. The second one is the quantum field theory on noncommutative spaces [3] with the Moyal product as main ingredient. Besides these, a proposition by several authors [4, 5] was made to generalise quantum mechanics in such a way that the operators corresponding to space coordinates no longer commute: $[\widehat{x}^k,\widehat{x}^\ell] \neq 0$. This was implemented by an extension of the Poisson structure on the cotangent space such that the brackets satisfy $\left\{x^k,x^\ell\right\} \neq 0$. Upon quantisation, the corresponding operators should then also be noncommutative. A particle moving in an affine space $\mathbf{A}^N$, has its configuration, in a fixed reference frame, given by an element $\{x^k\}$ of the translation group: $Q = \mathbf{R}^N$ with cotangent bundle $T^\star(Q) = \mathbf{R}^N \times \mathbf{R}^N$. In [1], we examined such an extension of the canonical symplectic two-form $\omega_0 = dx^i \wedge dp_i \to \Omega = \omega_0 + \omega_F + \omega_B$:

$$\omega_F = \frac{1}{2} F_{ij}(x)\, dx^i \wedge dx^j \ , \ \omega_B = \frac{1}{2} B^{k\ell}(p)\, dp_k \wedge dp_\ell \qquad (1.1)$$

This extension is form-invariant under a change of the reference frame lifted to the cotangent bundle:

$$T^\star(Q) \to T^\star(Q) : \left(x^i,p_k\right) \to \left(x'^i = A^i{}_j x^j + a^k \, , \, p'_k = p_\ell \left(A^{-1}\right)^\ell{}_k\right) \qquad (1.2)$$

$$\begin{aligned} \Omega \ \to \ \Omega' &= dx'^i \wedge dp'_i + \frac{1}{2} F'_{ij}(x')\, dx'^i \wedge dx'^j + \frac{1}{2} B'^{k\ell}(p')\, dp'_k \wedge dp'_\ell \\ F'_{ij}(x') &= F_{k\ell}(x) \left(A^{-1}\right)^k{}_i \left(A^{-1}\right)^\ell{}_j \, , \, B'^{k\ell}(p') = A^k{}_i A^\ell{}_j B^{ij}(p) \end{aligned} \qquad (1.3)$$

For a general configuration space Q, a diffeomorphism $\phi : x^i \to x'^i \doteq \phi^i(x)$, when lifted to $T^\star(Q)$, becomes

$$\begin{aligned} \widetilde{\phi} &: \left(x^i,p_k\right) \to \left(x'^i = \phi^i(x), p'_k = p_\ell \frac{\partial(\phi^{-1}(x'))^\ell}{\partial x'^k}\right) \\ F'_{ij}(x') &= F_{k\ell}(x) \frac{\partial(\phi^{-1})^k(x')}{\partial x'^i} \frac{\partial(\phi^{-1})^\ell(x')}{\partial x'^j} \\ B'^{k\ell}(p',x') &= \frac{\partial \phi^k(x)}{\partial x^i} \frac{\partial \phi^\ell(x)}{\partial x^j} B^{ij}(p) \end{aligned}$$

In general $B'^{k\ell}$ is function of both variables $\{p',x'\}$ and no intrinsic meaning can be given to the particular form of the extension Ω in equation (1.1).

In this work, we show that such an extension is achieved when $Q = G$ is a Lie group. This is possible because the cotangent bundle $T^\star(G)$ has two distinguished trivialisations, the left- and right trivialisations [7] implemented respectively by the bases of the left- and right invariant differential forms.

In section **2.**, inspired by the rigid body motion, we use the left trivialisation with left invariant or *body-coordinates* and construct a left invariant two-form. In the case of constant F_{ij} and $B^{k\ell}$ fields the ω_F term arises from a symplectic one-cocycle, as introduced by Souriau [8, 9], and ω_B will be automatically left invariant. The constructed two-form Ω is obviously closed but the non degeneracy condition leads in general to a constrained Hamiltonian system. This is examined in more detail for $SU(2)$ in section **3.**. Final considerations are made in section **4.**. Some elements of Lie algebra cohomology [9, 10] are recalled in the appendix.

2. THE PHASE SPACE $\{\mathcal{M}_0 \equiv T^\star(G), \omega_0\}$

Let $\{g^\alpha, \alpha = 1,2,\cdots,N\}$ be coordinates of a group element $g \in G$. Natural or holonomic coordinates of points $(g,\mathbf{p}_g) \in T^\star(G)$ are obtained using the basis $\{\mathbf{d}g^\mu\}$ of the cotangent space $T^\star_g(G)$. They are given by $(g^\alpha,p_\mu)_{hol}$, where $\mathbf{p}_g = p_\mu\, \mathbf{d}g^\mu$. Given a pair of dual bases $\{\mathbf{e}_\alpha\}$ of the Lie algebra $\mathcal{G} \doteq T_e(G)$ and $\{\varepsilon^\alpha\}$ of its dual $\mathcal{G}^\star$, the differential and pull-back of the left- and right translations (L_g,R_g)

define left- and right invariant vector fields and one forms: $\mathbf{e}^L_\alpha(g) \doteq L_{g*|e}\,\mathbf{e}_\alpha$, $\mathbf{e}^R_\alpha(g) \doteq R_{g*|e}\,\mathbf{e}_\alpha$, $\varepsilon^\alpha_L(g) \doteq L^*_{g^{-1}|g}\,\varepsilon^\alpha$, $\varepsilon^\alpha_R(g) \doteq R^*_{g^{-1}|g}\,\varepsilon^\alpha$. With canonical group coordinates, in terms of $L^\alpha{}_\beta(g,h) \doteq \partial(gh)^\alpha/\partial g^\beta$ and $R^\alpha{}_\beta(g,h) \doteq \partial(hg)^\alpha/\partial g^\beta$, they are explicitely given by:

$$\mathbf{e}^L_\alpha(g) = L^\mu{}_\alpha(g,e)\,\frac{\partial}{\partial g^\mu} \quad , \quad \mathbf{e}^R_\alpha(g) = R^\mu{}_\alpha(g,e)\,\frac{\partial}{\partial g^\mu}$$
$$\varepsilon^\alpha_L(g) = L^\alpha{}_\mu(g^{-1},g)\,\mathbf{d}g^\mu \quad , \quad \varepsilon^\alpha_R(g) = R^\alpha{}_\mu(g^{-1},g)\,\mathbf{d}g^\mu \tag{2.1}$$

These bases implement canonical trivialisations of the tangent and cotangent bundle. For the cotangent bundle, which is the arena of symplectic or Hamiltonian formalism, we have a left and a right trivialisation:

$$\begin{aligned}
\lambda \;:\; & T^\star(G) \to G \times \mathcal{G}^\star : (g, p_g = p_\mu\,\mathbf{d}g^\mu) \to \left(g, \pi^L = L^*_{g|e}\,p_g = \pi^L_\mu\,\varepsilon^\mu\right)\\
& \pi^L_\mu = \langle p_g, \mathbf{e}^L_\mu\rangle = p_\nu\,L^\nu{}_\mu(g,e)\\
\rho \;:\; & T^\star(G) \to G \times \mathcal{G}^\star : (g, p_g = p_\mu\,\mathbf{d}g^\mu) \to \left(g, \pi^R = R^*_{g|e}\,p_g = \pi^R_\mu\,\varepsilon^\mu\right)\\
& \pi^R_\mu = \langle p_g, \mathbf{e}^R_\mu\rangle = p_\nu\,R^\nu{}_\mu(g,e)
\end{aligned}$$

They can be viewed as a change of coordinates of a point (g,p_g) in $T^\star(G)$:

$$(g,\mathbf{p}_g) \leftrightarrow (g^\alpha, p_\mu)_{hol} \leftrightarrow (g^\alpha, \pi^L_\mu)_{\mathbf{B}} \leftrightarrow (g^\alpha, \pi^R_\mu)_{\mathbf{S}} \tag{2.2}$$

In rigid body theory, the coordinates of the left trivialisation are the "body" coordinates, whence the subscript $(\,,)_{\mathbf{B}}$. The right trivialisation yields "space" coordinates with subscript $(\,,)_{\mathbf{S}}$. Both are related through the coadjoint representation of G in $\mathcal{G}^\star$:

$$\pi^R_\mu = \mathbf{K}_\mu{}^\nu(g)\,\pi^L_\nu = \mathbf{Ad}^\nu{}_\mu(g^{-1})\,\pi^L_\nu \tag{2.3}$$

Lifting the left multiplication in G to the cotangent bundle yields a group action: $\widetilde{L}_a : T^\star(G) \to T^\star(G) : x = (g,p_g) \to y = (ag, p'_{ag} = L^\star_{a^{-1}|ag}\,p_g)$. In body coordinates: $\left(\widetilde{L}_a\right)_{\mathbf{B}}$: $(g^\alpha, \pi^L_\mu)_{\mathbf{B}} \to ((ag)^\alpha, \pi^L_\mu)_{\mathbf{B}}$. The pull-back of the cotangent projection $\kappa : T^\star(G) \to G : x \doteq (g,p_g) \to g$, acting on the $\{\varepsilon^\alpha(g)\}$ yield $\widetilde{L}_a$ invariant one forms on $T^\star(G) : \langle\varepsilon^\alpha_L(x)| = \kappa^\star_x\,\varepsilon^\alpha_L(\kappa(x))$ and the differentials of the left invariant functions π^L_μ on $T^\star(G)$ also yield $\widetilde{L}_a$ invariant one forms on $T^\star(G)$. Together they provide a left invariant basis of the cotangent space at $x = (g^\alpha, \pi^L_\mu)_{\mathbf{B}} \in T^\star(G)$:

$$\left\{\langle\varepsilon^\alpha_L| \doteq L^\alpha{}_\mu(g^{-1},g)\,\langle\mathbf{d}g^\mu|\;,\;\langle\varepsilon^L_\mu| \doteq \langle\mathbf{d}\pi^L_\mu|\right\} \tag{2.4}$$

Its dual basis in the tangent space $T_x(T^\star(G))$ is given by

$$\left\{|\mathbf{e}^L_\alpha\rangle \doteq |\partial/\partial g^\mu\rangle\,L^\mu{}_\alpha(g,e)\;,\;|\mathbf{e}^\mu_L\rangle \doteq |\partial/\partial\pi^L_\mu\rangle\right\} \tag{2.5}$$

The canonical Liouville one-form $\langle\theta_0| = p_\alpha\,\langle dg^\alpha|$ and its associated symplectic two-form $\omega_0 = -\mathbf{d}\theta_0 = \langle\mathbf{d}g^\alpha| \wedge \langle\mathbf{d}p_\alpha|$, are obtained as:

$$\langle\theta_0| = \pi^L_\mu\,\langle\varepsilon^\mu_L|\;,\;\omega_0 = \langle\varepsilon^\mu_L| \wedge \langle\varepsilon^L_\mu| + \frac{1}{2}\,\pi^L_\mu\,\mathbf{f}^\mu{}_{\alpha\beta}\,\langle\varepsilon^\alpha_L| \wedge \langle\varepsilon^\beta_L| \tag{2.6}$$

The Hamiltonian vector field associated to a function $A(g,\pi^L)$ on phase space $\mathcal{M}_0 \equiv T^\star(G)$, is defined by: $\imath_{\mathbf{X}}\,\omega_0 = \langle\mathbf{d}A|$. Its components are:

$$\begin{aligned}
X^\mu &\doteq \langle\varepsilon^\mu_L|\mathbf{X}\rangle = \langle\mathbf{d}A|\mathbf{e}^\mu_L\rangle\\
X_\alpha &\doteq \langle\varepsilon^L_\alpha|\mathbf{X}\rangle = -\langle\mathbf{d}A|\mathbf{e}^L_\alpha\rangle - \pi^L_\mu\,\mathbf{f}^\mu{}_{\alpha\beta}\,\langle\mathbf{d}A|\mathbf{e}^\beta_L\rangle
\end{aligned} \tag{2.7}$$

With $\imath_{\mathbf{Y}}\,\omega_0 = \langle\mathbf{d}B|$, the Poisson bracket of dynamical variables: $\{A,B\}_0 \doteq \omega_0\,(\mathbf{X},\mathbf{Y})$, is obtained explicitely in (g^α, π^L_μ) variables as:

$$\{A,B\}_0 = \langle\mathbf{d}A|\mathbf{e}^L_\alpha\rangle\,\frac{\partial B}{\partial\pi^L_\alpha} - \frac{\partial A}{\partial\pi^L_\alpha}\,\langle\mathbf{d}B|\mathbf{e}^L_\alpha\rangle - \frac{\partial A}{\partial\pi^L_\alpha}\,\pi^L_\mu\,\mathbf{f}^\mu{}_{\alpha\beta}\,\frac{\partial B}{\partial\pi^L_\beta} \tag{2.8}$$

In particular, the basic Poisson brackets are:

$$\begin{aligned}
\left\{g^\alpha, g^\beta\right\}_0 &= 0 \quad , \quad \left\{g^\alpha, \pi^L_\nu\right\}_0 = L^\alpha{}_\nu(g,e)\\
\left\{\pi^L_\mu, g^\beta\right\}_0 &= -L^\beta{}_\mu(g,e) \quad , \quad \left\{\pi^L_\mu, \pi^L_\nu\right\}_0 = -\,\pi^L_\kappa\,\mathbf{f}^\kappa{}_{\mu\nu}
\end{aligned} \tag{2.9}$$

The flow of a particular observable, the Hamiltonian $H(g,\pi^L)$, determines the time evolution of any observable $A(g,\pi^L)$ by the equation: $dA/dt = \{A,H\}_0$. We assume a Hamiltonian is of the form $H(g,\pi^L) = K(\pi^L) + V(g)$.
Here, as in rigid body mechanics, the *kinetic energy* is given by

$$K \doteq \frac{1}{2}\,I^{\alpha\beta}\,\pi^L_\alpha\,\pi^L_\beta \tag{2.10}$$

where $I^{\alpha\beta}$ is the inverse of a constant, positive definite, *inertia tensor* $I_{\mu\nu}$ in the "body" frame. The *potential energy* is a function V defined on the group manifold. The Euler equations of motion read:

$$\langle \varepsilon_L^\alpha | dg/dt \rangle = L^\alpha{}_\beta(g^{-1}, g)\, \frac{dg^\beta}{dt} = \frac{\partial K}{\partial \pi_\alpha^L} \tag{2.11}$$

$$\langle \varepsilon_\mu^L | d\pi^L/dt \rangle = \frac{d\pi_\mu^L}{dt} = -\frac{\partial V}{\partial g^\alpha} L^\alpha{}_\mu(g,e) + \frac{\partial K}{\partial \pi_\nu^L} \pi_\alpha^L \mathbf{f}^\alpha{}_{\nu\mu} \tag{2.12}$$

The first of these equations **(2.11)** relates the angular momentum π_α^L with the angular velocity in the body frame Ω_L^μ:

$$\Omega_L^\alpha \doteq L^\alpha{}_\beta(g^{-1}, g)\, \frac{dg^\beta}{dt} = I^{\alpha\mu}\pi_\mu^L \; ; \; \pi_\mu^L = I_{\mu\nu}\,\Omega_L^\nu \tag{2.13}$$

while the second **(2.12)** takes the classical form

$$\frac{d\pi_\mu^L}{dt} + \pi_\kappa^L \mathbf{f}^\kappa{}_{\mu\nu}\,\Omega_L^\nu = -\frac{\partial V}{\partial g^\alpha} L^\alpha{}_\mu(g,e) \tag{2.14}$$

An example of $V(g)$ is given by a *gravitational potential energy* as follows. Let $\mathbf{L} = \mathbf{e}_\alpha L^\alpha$ be a constant vector in $\mathcal{G}$ (the position of the centre of mass in the body frame) and $\gamma = \gamma_\alpha \varepsilon^\alpha$ a constant vector in $\mathcal{G}^\star$ (the gravitational force in the space fixed frame). The potential energy is defined as:

$$V(g) \doteq -(\gamma | \mathbf{Ad}(g)\,\mathbf{L}) = -\left(\mathbf{K}(g^{-1})\gamma | \mathbf{L}\right) \tag{2.15}$$

where $(\,|\,)$ denotes the canonical pairing between $\mathcal{G}$ and its dual $\mathcal{G}^\star$. To compute $\langle \mathbf{d}V | \mathbf{e}_\mu^L \rangle$ we use the representation of the Maurer-Cartan form:

$$D(g^{-1})\,\mathbf{d}D(g) = D'(g^{-1}\,\mathbf{d}g)$$

where D is any representation D of G, with derived representation D' of $\mathcal{G}$. In particular, $\mathbf{dAd}(g) = \mathbf{Ad}(g)\,\mathbf{ad}(\mathbf{e}_\mu)\,\varepsilon_L^\mu(g)$ and $\mathbf{dK}(g) = \mathbf{K}(g)\,\mathbf{k}(\mathbf{e}_\mu)\,\varepsilon_L^\mu(g)$. This yields:

$$\langle \mathbf{d}V | \mathbf{e}_\mu^L \rangle(g) = -\left(\mathbf{K}(g^{-1})\gamma | \mathbf{ad}(\mathbf{e}_\mu)\,\mathbf{L}\right) = -\left(\Gamma(g)\,|\,\mathbf{ad}(\mathbf{e}_\mu)\,\mathbf{L}\right) \tag{2.16}$$

where $\Gamma(g) \doteq \mathbf{K}(g^{-1})\gamma$ is the variable gravitational force in the body-fixed frame. Using the above formulae to compute $\mathbf{dK}(g^{-1})$, we obtain:

$$\frac{d\Gamma_\mu}{dt} = (\Gamma\,|\,\mathbf{ad}(\mathbf{e}_\mu)\,\Omega_L) = \Gamma_\alpha \mathbf{f}^\alpha{}_{\mu\beta}\,\Omega_L^\beta \tag{2.17}$$

Equation **(2.14)** reads:

$$\frac{d\pi_\mu^L}{dt} + \pi_\alpha^L\,\mathbf{f}^\alpha{}_{\mu\beta}\,\Omega_L^\beta = (\Gamma\,|\,\mathbf{ad}(\mathbf{e}_\mu)\,\mathbf{L}) = \Gamma_\alpha \mathbf{f}^\alpha{}_{\mu\beta}\,L^\beta \tag{2.18}$$

Together with **(2.13)**,

$$\Omega_L^\alpha \doteq L^\alpha{}_\beta(g^{-1}, g)\,\frac{dg^\beta}{dt} = I^{\alpha\mu}\pi_\mu^L$$

the equations **(2.17)** and **(2.18)** form the so-called Euler-Poisson system.

3. MODIFIED SYMPLECTIC STRUCTURE ON $T^\star(G)$

In appendix **A** it is shown that, if $\Theta = \frac{1}{2}\Theta_{\alpha\beta}\,\varepsilon^\alpha \wedge \varepsilon^\beta \in \Lambda^2(\mathcal{G}^\star)$, obeys the cocycle condition **(A.1)**, then $\Theta_L(g) \doteq (1/2)\,\Theta_{\alpha\beta}\,\varepsilon_L^\alpha(g) \wedge \varepsilon_L^\beta(g)$ is a closed left-invariant two-form on G. Including this closed two-form in the canonical two-form, one obtains another symplectic two-form on $T^\star(G)$, which, furthermore, is $\widetilde{L}_a$ invariant. So we define:

$$\omega_I = \omega_0 - \Theta_L = \langle \varepsilon_L^\mu | \wedge \langle \mathbf{d}\pi_\mu^L | + \frac{1}{2}\left(\pi_\mu^L\,\mathbf{f}^\mu{}_{\alpha\beta} - \Theta_{\alpha\beta}\right) \langle \varepsilon_L^\alpha | \wedge \langle \varepsilon_L^\beta | \tag{3.1}$$

The Poisson brackets are also modified and **(2.8)**, **(2.9)** become:

$$\{A,B\}_I = \frac{\partial A}{\partial g^\mu} L^\mu{}_\alpha(g,e)\,\frac{\partial B}{\partial \pi_\alpha^L} - \frac{\partial B}{\partial g^\mu} L^\mu{}_\alpha(g,e)\,\frac{\partial A}{\partial \pi_\alpha^L} - \left(\pi_\mu^L\,\mathbf{f}^\mu{}_{\alpha\beta} - \Theta_{\alpha\beta}\right) \frac{\partial A}{\partial \pi_\alpha^L}\,\frac{\partial B}{\partial \pi_\beta^L} \tag{3.2}$$

In particular, the fundamental brackets are:

$$\left\{g^\alpha, g^\beta\right\}_I = 0 \;\;,\;\; \left\{g^\alpha, \pi_\nu^L\right\}_I = L^\alpha{}_\nu(g,e)$$
$$\left\{\pi_\mu^L, g^\beta\right\}_I = -L^\beta{}_\mu(g,e) \;\;,\;\; \left\{\pi_\mu^L, \pi_\nu^L\right\}_I = -\left(\pi_\kappa^L\,\mathbf{f}^\kappa{}_{\mu\nu} - \Theta_{\mu\nu}\right) \tag{3.3}$$

The modified symplectic structure induces an additional interaction and the Euler equations become:

$$\Omega_L^\alpha \doteq L^\alpha{}_\beta(g^{-1}, g)\,\frac{dg^\beta}{dt} = \frac{\partial K}{\partial \pi_\alpha^L} = I^{\alpha\mu}\pi_\mu^L \tag{3.4}$$

$$\frac{d\pi_\mu^L}{dt} = -\langle \mathbf{d}V | \mathbf{e}_\mu^L \rangle + \frac{\partial K}{\partial \pi_\alpha^L}\left(\pi_\kappa^L\,\mathbf{f}^\kappa{}_{\alpha\mu} - \Theta_{\alpha\mu}\right) \tag{3.5}$$

The relation between the velocity in the body frame and the angular momentum **(2.13)** is maintained: $\pi_\mu^L = I_{\mu\nu}\,\Omega_L^\nu$, while the second **(2.14)** takes the interaction into account:

$$\frac{d\pi_\mu^L}{dt} + \pi_\kappa^L\,\mathbf{f}^\kappa{}_{\mu\alpha}\,\Omega_L^\alpha = -\langle \mathbf{d}V | \mathbf{e}_\mu^L \rangle - \Omega_L^\alpha\,\Theta_{\alpha\mu} \tag{3.6}$$

For a semisimple Lie algebra $\mathcal{G}$, we have $\Theta_{\alpha\beta} = -\xi_\mu\,\mathbf{f}^\mu{}_{\alpha\beta}$ and we may define a modified Liouville one-form:

$$\langle \theta_I | = \pi'_\mu\,\langle \varepsilon_L^\mu | \;,\; \pi'_\mu \doteq \pi_\mu^L + \xi_\mu \tag{3.7}$$

and the symplectic two-form reads

$$\omega_I = -\mathbf{d}\langle \theta_I | = \langle \varepsilon_L^\mu | \wedge \langle \mathbf{d}\pi'_\mu | + \frac{1}{2}\pi'_\mu\,\mathbf{f}^\mu{}_{\alpha\beta}\,\langle \varepsilon_L^\alpha | \wedge \langle \varepsilon_L^\beta | \tag{3.8}$$

This means that such that $\{g^\alpha, p'_\mu = p_\mu + \xi_\beta L^\beta{}_\mu(g^{-1}; g)\}$ are Darboux coordinates:

$$\langle \theta_I | = p'_\mu\,\langle \mathbf{d}g^\mu | \;,\; \omega_I \doteq -\mathbf{d}\langle \theta_I | = \langle \mathbf{d}g^\mu | \wedge \langle \mathbf{d}p'_\mu | \tag{3.9}$$

In $\left(g^\alpha, \pi'_\mu\right)$ coordinates, the Hamiltonian reads

$$H' = K'(\pi') + V(g) = \frac{1}{2} I^{\mu\nu}\,(\pi'_\mu - \xi_\mu)\,(\pi'_\nu - \xi_\nu) + V(g) \tag{3.10}$$

and the Euler equations read:

$$L^\alpha{}_\beta(g^{-1}, g)\,\frac{dg^\beta}{dt} = \frac{\partial K'}{\partial \pi'_\alpha} = I^{\alpha\mu}\,(\pi'_\mu - \xi_\mu) \tag{3.11}$$

$$\frac{d\pi'_\mu}{dt} = -\langle \mathbf{d}V | \mathbf{e}_\mu^L \rangle + \frac{\partial K'}{\partial \pi'_\alpha}\left(\pi'_\kappa\,\mathbf{f}^\kappa{}_{\alpha\mu}\right) \tag{3.12}$$

which, obviously are equivalent to **(3.4)** and **(3.12)**.

4. THE CLOSED TWO-FORM ω_L

Configuration space coordinates which do not Poisson commute, are obtained through the addition of a left-invariant and closed two-form to **(3.1)**:

$$\Upsilon^L \doteq \frac{1}{2}\Upsilon^{\mu\nu}\langle \mathbf{d}\pi^L_\mu| \wedge \langle \mathbf{d}\pi^L_\nu| \tag{4.1}$$

$$\begin{aligned}\omega_L \doteq \omega_0 - \Theta_L + \Upsilon^L \; &= \; \langle \varepsilon^\mu_L| \wedge \langle \mathbf{d}\pi^L_\mu| + \frac{1}{2}\left(\pi^L_\mu\, \mathbf{f}^\mu{}_{\alpha\beta} - \Theta_{\alpha\beta}\right)\langle \varepsilon^\alpha_L| \wedge \langle \varepsilon^\beta_L| \\ &\quad + \frac{1}{2}\Upsilon^{\mu\nu}\langle \mathbf{d}\pi^L_\mu| \wedge \langle \mathbf{d}\pi^L_\nu| \end{aligned} \tag{4.2}$$

With the notation $S_{\alpha\beta} \equiv \left(\pi^L_\mu\, \mathbf{f}^\mu{}_{\alpha\beta} - \Theta_{\alpha\beta}\right)$, we wite ω_L in matrix form:

$$\omega_L \equiv \frac{1}{2}\left(\langle\varepsilon^\alpha_L| \quad \langle \mathbf{d}\pi^L_\mu|\right) \wedge \begin{pmatrix} S_{\alpha\beta} & \delta_\alpha{}^\nu \\ -\delta^\mu{}_\beta & \Upsilon^{\mu\nu} \end{pmatrix} \begin{pmatrix} \langle \varepsilon^\beta_L| \\ \langle \mathbf{d}\pi^L_\nu| \end{pmatrix} \tag{4.3}$$

The degeneracy of (ω_L) is examined comsidering the equation

$$\imath_{|\mathbf{X}\rangle}\omega_L = \langle \mathbf{d}A| \tag{4.4}$$

In the bases **(2.4)**, **(2.5)**: $X^\alpha \doteq \langle \varepsilon^\alpha_L|\mathbf{X}\rangle$, $X_\mu \doteq \langle \varepsilon^L_\mu|\mathbf{X}\rangle$ and **(4.4)** reads:

$$\begin{aligned} X^\alpha\, \Phi_\alpha{}^\nu &= \langle \mathbf{d}A|\mathbf{e}^\nu_L\rangle + \langle \mathbf{d}A|\mathbf{e}^L_\mu\rangle \Upsilon^{\mu\nu}, \\ X_\mu\, \Psi^\mu{}_\beta &= -\langle \mathbf{d}A|\mathbf{e}^L_\beta\rangle + \langle \mathbf{d}A|\mathbf{e}^\alpha_L\rangle S_{\alpha\beta} \end{aligned} \tag{4.5}$$

where we introduced the matrices, linear in the momenta:

$$\Phi_\alpha{}^\nu \doteq \delta_\alpha{}^\nu + S_{\alpha\mu}\Upsilon^{\mu\nu} \;,\; \Psi^\mu{}_\beta \doteq \delta^\mu{}_\beta + \Upsilon^{\mu\nu} S_{\nu\beta} \tag{4.6}$$

They are mutually transposed and the products $\Phi S = S\Psi$, $\Upsilon\Phi = \Psi\Upsilon$ are antisymmetric. The fundamental equation **(4.4)**, defining Hamiltonian vector fields, has a solution if Φ and Ψ have inverses, i.e. if

$$\Delta \doteq \det\Phi \equiv \det\Psi \neq 0 \tag{4.7}$$

The matrices $\Upsilon\Phi^{-1} = \Psi^{-1}\Upsilon$ and $\Phi^{-1}S = S\Psi^{-1}$ are then also antisymmetric. The Hamiltonian vector fields are obtained as:

$$\begin{aligned} X^\alpha &= (\Psi^{-1})^\alpha{}_\mu \left(\langle \mathbf{d}A|\mathbf{e}^\mu_L\rangle - \Upsilon^{\mu\nu}\langle \mathbf{d}A|\mathbf{e}^L_\nu\rangle\right) \\ &= \left(\langle \mathbf{d}A|\mathbf{e}^\nu_L\rangle + \langle \mathbf{d}A|\mathbf{e}^L_\mu\rangle\Upsilon^{\mu\nu}\right)(\Phi^{-1})_\nu{}^\alpha \\ X_\mu &= (\Phi^{-1})_\mu{}^\alpha \left(-\langle \mathbf{d}A|\mathbf{e}^L_\alpha\rangle - S_{\alpha\beta}\langle \mathbf{d}A|\mathbf{e}^\beta_L\rangle\right) \\ &= \left(-\langle \mathbf{d}A|\mathbf{e}^L_\beta\rangle + \langle \mathbf{d}A|\mathbf{e}^\alpha_L\rangle S_{\alpha\beta}\right)(\Psi^{-1})^\beta{}_\mu \end{aligned} \tag{4.8}$$

The Poisson brackets between the basic dynamical variables are:

$$\begin{aligned} \left\{g^\alpha, g^\beta\right\}_L &= -L^\alpha{}_\kappa(g,e)\, L^\beta{}_\lambda(g,e)\, \Upsilon^{\kappa\mu}\,(\Phi^{-1})_\mu{}^\lambda \\ \left\{g^\alpha, \pi^L_\nu\right\}_L &= L^\alpha{}_\kappa(g,e)\,(\Psi^{-1})^\kappa{}_\nu\,, \\ \left\{\pi^L_\mu, g^\beta\right\}_L &= -L^\beta{}_\kappa(g,e)\,(\Psi^{-1})^\kappa{}_\mu \\ \left\{\pi^L_\mu, \pi^L_\nu\right\}_L &= -S_{\mu\kappa}\,(\Psi^{-1})^\kappa{}_\nu \end{aligned} \tag{4.9}$$

For a Hamiltonian $H = K + V$, the equations of motion are:

$$\begin{aligned} \Omega^\alpha_L \doteq L^\alpha{}_\beta(g^{-1},g)\,\frac{dg^\beta}{dt} &= \left(\frac{\partial K}{\partial \pi^L_\nu} + \langle \mathbf{d}V|\mathbf{e}^L_\mu\rangle\,\Upsilon^{\mu\nu}\right)(\Phi^{-1})_\nu{}^\alpha \\ \frac{d\pi^L_\mu}{dt} &= \left(-\langle \mathbf{d}V|\mathbf{e}^L_\beta\rangle + \frac{\partial K}{\partial \pi^L_\alpha}\, S_{\alpha\beta}\right)(\Psi^{-1})^\beta{}_\mu \end{aligned}$$

Since Φ, Ψ are linear in π^L, Δ is a polynomial in π^L of degree at most equal to N, the dimension of the Lie group. It defines an algebraic variety in $\mathcal{G}^\star$:

$$\Pi_1 \doteq \{(g,\pi^L)|\Delta(\pi^L) = 0\} \tag{4.10}$$

and its complement $\mathcal{V}_\Delta \doteq \mathcal{G}^\star \backslash \Pi_1$ defines a manifold

$$\mathcal{M}'_0 \doteq G \times \mathcal{V}_\Delta \tag{4.11}$$

with symplectic structure given by ω_L, restricted to $\mathcal{M}'_0$. If it happens that Π_1 itself is an algebraic manifold, an imbedded submanifold is obtained:

$$\mathcal{M}_1 \doteq G \times \Pi_1 \tag{4.12}$$

with imbedding in $\mathcal{M}_0 \doteq G \times \mathcal{G}^\star$: $j_1 : \mathcal{M}_1 \hookrightarrow \mathcal{M}_0$. The system is then constrained to $\mathcal{M}_1$ and we may look for solutions of **(4.4)** restricted to $\mathcal{M}_1$. Such solutions may exist if further conditions are imposed on the Hamiltonian. To proceed systematically, we follow the algorithm of Gotay, Nester and Hinds [11]. To keep things simple, this will be done in the next section for the semi-simple group $SU(2)$.

5. A CASE STUDY: $SU(2)$

The dynamical variables are functions on $\mathcal{M}_0 \doteq SU(2) \times su(2)^\star$. A basis $\{\mathbf{e}_\alpha\}$ of the Lie algebra $su(2)$ may be chosen such that its structure constants are the Kronecker symbols $[\mathbf{e}_\alpha, \mathbf{e}_\beta] = \mathbf{e}_\mu\, \varepsilon^\mu{}_{\alpha\beta}$. The Killing metric $\eta_{\alpha\beta} \doteq \varepsilon^\mu{}_{\alpha\nu}\,\varepsilon^\nu{}_{\beta\mu} = -2\,\delta_{\alpha\beta}$, provides an isomorphism between $su(2)$ and $su(2)^\star$. The metric $\delta_{\alpha\beta}$ with inverse $\delta^{\mu\nu}$ will be freely used to raise or to lower indices. Θ_L is written in terms of a *magnetic field* ξ_μ as $\Theta_{\alpha\beta} = -\xi_\kappa\,\varepsilon^\kappa{}_{\alpha\beta}$ and any antisymmetric Υ can be written in terms of τ^λ, a *dual magnetic field in momentum space*, as $Y^{\mu\nu} = \tau^\lambda \varepsilon_\lambda{}^{\mu\nu}$. Defining $\pi'_\kappa \doteq \pi^L_\kappa + \xi_\kappa$, ω_L reads:

$$\omega_L \equiv \frac{1}{2}\left(\langle\varepsilon^\alpha_L| \quad \langle\mathbf{d}\pi^L_\mu|\right) \wedge \begin{pmatrix} \pi'_\kappa \varepsilon^\kappa{}_{\alpha\beta} & \delta_\alpha{}^\nu \\ -\delta^\mu{}_\beta & \tau^\lambda \varepsilon_\lambda{}^{\mu\nu} \end{pmatrix} \begin{pmatrix} \langle\varepsilon^\beta_L| \\ \langle\mathbf{d}\pi^L_\nu| \end{pmatrix} \tag{5.1}$$

The fundamental equation **(4.4)**: $\iota_{|\mathbf{X}\rangle}\, \omega_L = \langle\mathbf{d}H|$ becomes:

$$X^\alpha \pi'_\kappa \varepsilon^\kappa{}_{\alpha\beta} - X_\beta = H_\beta\ ,\ X^\nu + X_\mu \tau^\lambda \varepsilon_\lambda{}^{\mu\nu} = H^\nu$$

where $H_\beta \doteq (\partial H/\partial g^\alpha)\, L^\alpha{}_\beta(g,e)$, $H^\nu \doteq \left(\partial H/\partial \pi^L_\nu\right)$. The matrices **(4.6)** are given explicitely by $\Phi_\alpha{}^\nu \doteq C_1\,\delta_\alpha{}^\nu + \tau_\alpha \pi'^\nu$ and $\Psi^\mu{}_\beta \doteq C_1\,\delta^\mu{}_\beta + \pi'^\mu \tau_\beta$, where $C_1 \doteq (1-\pi'\cdot\tau)$. They obey $\Phi_\alpha{}^\nu \left(\delta_\nu{}^\beta - \tau_\nu \pi'^\beta\right) = C_1\,\delta_\alpha{}^\beta$ and $\Psi^\mu{}_\beta\left(\delta^\beta{}_\nu - \pi'^\beta \tau_\nu\right) = C_1\,\delta^\mu{}_\nu$. It follows that **(4.5)** implies:

$$X^\alpha (1-\pi'\cdot\tau) \;=\; H^\alpha - \pi'^\alpha(\tau_\beta H^\beta) - \varepsilon^{\alpha\mu}{}_\nu\, H_\mu\, \tau^\nu \tag{5.2}$$
$$X_\mu (1-\pi'\cdot\tau) \;=\; -H_\mu + \tau_\mu(\pi'^\nu H_\nu) - \varepsilon_{\mu\alpha}{}^\beta\, H^\alpha\, \pi'_\beta \tag{5.3}$$

5.1. The non degenerate case

The determinant of the matrices Φ and Ψ is given by $\Delta = (C_1)^2$. Obviously the plane $\Pi_1 \doteq \{(g,\pi^L)|(1-\pi'\cdot\tau)=0\}$ is an algebraic manifold in $\mathcal{G}^\star$. Its complement $\mathcal{V}_\Delta \doteq \mathcal{G}^\star \backslash \Pi_1$ defines a manifold $\mathcal{M}'_0 \doteq G \times \mathcal{V}_\Delta$ with symplectic structure ω_L, retricted to $\mathcal{M}'_0$. On $\mathcal{M}'_0$, Φ and Ψ have inverses:

$$(\Psi^{-1})^\beta{}_\nu = (C_1)^{-1}\left(\delta^\beta{}_\nu - \pi'^\beta \tau_\nu\right)\ ,$$
$$(\Phi^{-1})_\nu{}^\beta = (C_1)^{-1}\left(\delta_\nu{}^\beta - \tau_\nu \pi'^\beta\right) \tag{5.4}$$

For a Hamiltonian $H = K(\pi^L) + V(g)$, the Hamiltonian vector fields are read off from **(5.2)** and **(5.3)** with ensuing equations of motion:

$$\Omega^\alpha_L \;\doteq\; L^\alpha{}_\beta(g^{-1},g)\,\frac{dg^\beta}{dt} = \left(\frac{\partial K}{\partial \pi^L_\nu} + \langle\mathbf{d}V|\mathbf{e}^L_\mu\rangle\,\tau^\lambda \varepsilon_\lambda{}^{\mu\nu}\right)(\Phi^{-1})_\nu{}^\alpha$$
$$\frac{d\pi^L_\mu}{dt} \;=\; \left(-\langle\mathbf{d}V|\mathbf{e}^L_\beta\rangle + \frac{\partial K}{\partial \pi^L_\alpha}\,\pi'_\kappa \varepsilon^\kappa{}_{\alpha\beta}\right)(\Psi^{-1})^\beta{}_\mu \tag{5.5}$$

For a purely kinetic Hamiltonian, we obtain:

$$\Omega^\alpha_L = \frac{\partial K}{\partial \pi^L_\mu}\,(\Phi^{-1})_\mu{}^\alpha\ ,\ \frac{d\pi^L_\mu}{dt} = \Omega^\alpha_L\, \pi'_\beta\, \varepsilon^\beta{}_{\alpha\mu} \tag{5.6}$$

5.2. The degenerate case

The equation $C_1 \equiv (1-\pi'\cdot\tau) = 0$ defines a two dimensional plane Π_1 in $su(2)^\star \cong \mathbf{R}^3$. The *primary constrained manifold*, defined by $\mathcal{M}_1 \doteq SU(2)\times\Pi_1$, is imbedded in $\mathcal{M}_0 \doteq SU(2)\times su(2)^\star$. On $\mathcal{M}_1$, the closed two-form ω_L is degenerate and the pairing of $\pi' \in su(2)^\star$ with $\tau \in su(2)$ equals 1. So $|\tau\rangle \neq 0$ and, without loss of generality, we take $\{\tau^\alpha\} = \{0,0,\tau\}$. In what follows, greek indices $\{\alpha,\beta,\mu,\nu,\cdots\}$ shall vary in $\{1,2,3\}$, while latin indices $\{a,b,m,n,\cdots\}$ assume only the values $\{1,2\}$. The imbedding is given by:

$$a\,j_1 \;:\; \mathcal{M}_1 \hookrightarrow \mathcal{M}_0 :$$
$$x_1 \equiv (g^\alpha,\pi^L_m) \to x_0 = j_1(x_1) \equiv (g^\alpha,\pi^L_m,\pi^L_3 = 1/\tau - \xi_3) \tag{5.7}$$

with its differential or push-forward:

$$j_{1\star} : T\mathcal{M}_1 \to T\mathcal{M}_0 : (x_1;X^\alpha,X_m) \to (x_0;X^\alpha,X_m,X_3=0) \tag{5.8}$$

The pull-back transforms forms on $\mathcal{M}_0$ into forms on $\mathcal{M}_1$:

$$j_1{}^\star : \bigwedge{}^\bullet(T^\star\mathcal{M}_0) \to \bigwedge{}^\bullet(T^\star\mathcal{M}_1) \tag{5.9}$$

In particular the pull-back of ω_L to the five dimensional manifold $\mathcal{M}_1$ is

$$\widetilde{\omega}_{L|1} \doteq j_1{}^\star(\omega_L) \tag{5.10}$$

The restriction of ω_L to $\mathcal{M}_1$, not to be confused with its pull-back, is denoted by $\omega_{L|1} \doteq \omega_L \circ j_1$. In matrix representation:

$$\omega_{L|1} \;=\; \frac{1}{2}\left(\langle\varepsilon^\alpha_L| \quad \langle\mathbf{d}\pi^L_\mu|\right) \wedge \begin{pmatrix} 0 & 1/\tau & -\pi'_2 & 1 & 0 & 0 \\ -1/\tau & 0 & \pi'_1 & 0 & 1 & 0 \\ \pi'_2 & -\pi'_1 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & \tau & 0 \\ 0 & -1 & 0 & -\tau & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \langle\varepsilon^\beta_L| \\ \\ \langle\mathbf{d}\pi^L_\nu| \end{pmatrix} \tag{5.11}$$

Let $(T\mathcal{M}_0)_{|1} \doteq \{(x,\mathbf{X}) \in T\mathcal{M}_0\,|\,x \in \mathcal{M}_1\}$ be the subbundle of $T\mathcal{M}_0$ restricted to $\mathcal{M}_1$. Following the GNH algorithm [11], we look for a vector field $|\mathbf{X}\rangle$ in $(T\mathcal{M}_0)_{|1}$, tangent to $\mathcal{M}_1$ and solution of $\iota_{|\mathbf{X}\rangle}\omega_{L|1} = \langle\mathbf{d}H| \circ j_1$.

Explicitely :

$$-(1/\tau)X2 + \pi'_2 X3 - X_1 \;=\; \langle\mathbf{d}V|\mathbf{e}^L_1\rangle$$
$$+(1/\tau)X1 - \pi'_1 X3 - X_2 \;=\; \langle\mathbf{d}V|\mathbf{e}^L_2\rangle$$
$$-\pi'_2 X1 + \pi'_1 X2 - X_3 \;=\; \langle\mathbf{d}V|\mathbf{e}^L_3\rangle$$

$$\begin{aligned} X1 - \tau X_2 &= \partial K/\partial \pi^L_1 \\ X2 + \tau X_1 &= \partial K/\partial \pi^L_2 \\ X3 &= \partial K/\partial \pi^L_3 \end{aligned}$$

Two independent null vectors of $\omega_{L|1}$, solution of $\imath_{|\mathbf{Z}\rangle}\omega_{L|1} = 0$, are given by:

$$\begin{aligned} |\mathbf{Z}^1\rangle &= |\mathbf{e}^L_1\rangle + (1/\tau)\,|\partial/\partial\pi^L_2\rangle - \pi'_2\,|\partial/\partial\pi^L_3\rangle \\ |\mathbf{Z}^2\rangle &= |\mathbf{e}^L_2\rangle - (1/\tau)\,|\partial/\partial\pi^L_1\rangle + \pi'_1\,|\partial/\partial\pi^L_3\rangle \end{aligned} \quad (5.12)$$

Consistency requires $\{\langle \mathbf{d}H|\mathbf{Z}^a\rangle = 0\}$ for $(a = 1,2)$ and $\pi'_3 = 1/\tau$.

$$\begin{aligned} C_{21} \equiv \pi'_2\,(\partial K/\partial\pi^L_3) - \pi'_3\,(\partial K/\partial\pi^L_2) - \langle \mathbf{d}V|\mathbf{e}^L_1\rangle &= 0 \\ C_{22} \equiv \pi'_3\,(\partial K/\partial\pi^L_1) - \pi'_1\,(\partial K/\partial\pi^L_3) - \langle \mathbf{d}V|\mathbf{e}^L_2\rangle &= 0 \end{aligned} \quad (5.13)$$

These two equations define a secondary constrained manifold $\mathcal{M}_2 \subset \mathcal{M}_1$, on which a particular solution of **(??)** is

$$|\mathbf{X}_P\rangle = |\mathbf{e}^L_1\rangle\,\partial K/\partial\pi^L_1 + |\mathbf{e}^L_2\rangle\,\partial K/\partial\pi^L_2 + |\mathbf{e}^L_3\rangle\,\partial K/\partial\pi^L_3 + |\partial/\partial\pi^L_3\rangle\, C_{23} \quad (5.14)$$

where $C_{23} \equiv \pi'_1\,(\partial K/\partial\pi^L_2) - \pi'_2\,(\partial K/\partial\pi^L_1) - \langle \mathbf{d}V|\mathbf{e}^L_3\rangle$. The general solution $|\mathbf{X}_G\rangle$ of **(??)**, <u>on $\mathcal{M}_2$</u> , still contains two arbitrary functions ζ_1 and ζ_2:

$$(X_G) = \zeta_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1/\tau \\ -\pi'_2 \end{pmatrix} + \zeta_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1/\tau \\ 0 \\ +\pi'_1 \end{pmatrix} + \begin{pmatrix} \partial K/\partial\pi^L_1 \\ \partial K/\partial\pi^L_2 \\ \partial K/\partial\pi^L_3 \\ 0 \\ 0 \\ C_{23} \end{pmatrix} \quad (5.15)$$

This vector must be tangent to $\mathcal{M}_1$ and $\mathcal{M}_2$. This leads to three equations

$$\langle \mathbf{d}C_1\,|\,\mathbf{X}_G\rangle = 0;\ \langle \mathbf{d}C_{21}\,|\,\mathbf{X}_G\rangle = 0;\ \langle \mathbf{d}C_{22}\,|\,\mathbf{X}_G\rangle = 0 \quad (5.16)$$

If these three equations determine or not the two arbitrary functions ζ_1 and ζ_2, will depend on the kinetic energy $K(\pi^L)$ and on the particular form of the potential $V(g)$. If they do so, the system will have a solution. If not, they will define a tertiary constraint manifold $\mathcal{M}_3$ and the analysis must proceed.

6. CONCLUSIONS

In this work, we analysed the consistency of a modification of the symplectic two-form on the cotangent bundle of a group manifold. This was done in order to obtain classical, i.e. Poisson, noncommuting configuration (group) coordinates. This was achieved in the non degenerate case, with the closed two-form ω_L which is then symplectic. We do not address here the general quantization problem of such a system and refer e.g. to [12] for a general review on quantization methods. It should be stressed that, whatever the quantisation scheme, any such obtained framework has little to do with *non commutative geometry*, either in the sense of A.Connes or as a quantum field theory on non-commutative spaces.

APPENDIX A: THE SYMPLECTIC ONE-COCYCLE

A one-cochain θ on $\mathcal{G}$ with values in $\mathcal{G}^\star$, on which $\mathcal{G}$ acts with the coadjoint representation $\mathbf{k}$, $\theta \in C^1(\mathcal{G},\mathcal{G}^\star,\mathbf{k})$, is a linear map $\theta : \mathcal{G} \to \mathcal{G}^\star : \mathbf{u} \to \theta(\mathbf{u})$. Its components are $\theta_{\alpha,\mu} \doteq \langle\theta(\mathbf{e}_\mu)|\mathbf{e}_\alpha\rangle$. It is a one-cocycle, $\theta \in Z^1(\mathcal{G},\mathcal{G}^\star,\mathbf{k})$, if its coboundary, $(\delta_1\theta)(\mathbf{u},\mathbf{v}) \doteq \mathbf{k}(\mathbf{u})\theta(\mathbf{v}) - \mathbf{k}(\mathbf{v})\theta(\mathbf{u}) - \theta([\mathbf{u},\mathbf{v}])$, vanishes.

$$\begin{aligned} \langle(\delta_1\theta)(\mathbf{u},\mathbf{v})|\mathbf{w}\rangle &\doteq -\langle\theta(\mathbf{v})|[\mathbf{u},\mathbf{w}]\rangle + \langle\theta(\mathbf{u})|[\mathbf{v},\mathbf{w}]\rangle - \langle\theta([\mathbf{u},\mathbf{v}])|\mathbf{w}\rangle = 0 \\ \langle(\delta_1\theta)(\mathbf{e}_\mu,\mathbf{e}_\nu)|\mathbf{e}_\alpha\rangle &\doteq -\theta_{\kappa,\nu}\,\mathbf{f}^\kappa{}_{\mu\alpha} + \theta_{\kappa,\mu}\,\mathbf{f}^\kappa{}_{\nu\alpha} - \theta_{\kappa,\alpha}\,\mathbf{f}^\kappa{}_{\mu\nu} = 0 \end{aligned}$$

The one-cocycle σ is called symplectic if $\Sigma(\mathbf{u},\mathbf{v}) \doteq \langle\sigma(\mathbf{u})|\mathbf{v}\rangle$ is antisymmetric, $\Sigma(\mathbf{u},\mathbf{v}) = -\Sigma(\mathbf{v},\mathbf{u})$ or $\Sigma_{[\alpha\mu]} \doteq \sigma_{\alpha,\mu} = -\sigma_{\mu,\alpha}$. Any antisymmetric Θ defined in terms of $\theta \in C^1(\mathcal{G},\mathcal{G}^\star,\mathbf{k})$ as $\Theta_{[\alpha\beta]} = \theta_{\alpha,\beta}$ is actually a 2-cochain on $\mathcal{G}$ with values in $\mathbf{R}$ and trivial representation: $\Theta \in C^2(\mathcal{G},\mathbf{R},\mathbf{0})$. Furthermore, when $\theta \in Z^1(\mathcal{G},\mathcal{G}^\star,\mathbf{k})$, Θ is a 2-cocycle of $Z^2(\mathcal{G},\mathbf{R},\mathbf{0})$:

$$(\delta_2\Theta)(\mathbf{u},\mathbf{v},\mathbf{w}) \doteq -\Theta([\mathbf{u},\mathbf{v}],\mathbf{w}) + \Theta([\mathbf{u},\mathbf{w}],\mathbf{v}) - \Theta([\mathbf{v},\mathbf{w}],\mathbf{u}) = 0$$

$$(\delta_2\Theta)(\mathbf{e}_\alpha,\mathbf{e}_\beta,\mathbf{e}_\gamma) \doteq -\Theta_{\kappa\gamma}\mathbf{f}^\kappa{}_{\alpha\beta} + \Theta_{\kappa\beta}\,\mathbf{f}^\kappa{}_{\alpha\gamma} - \Theta_{\kappa\alpha}\,\mathbf{f}^\kappa{}_{\beta\gamma} = 0 \quad (A.1)$$

In general let $\Theta = \frac{1}{2}\Theta_{\alpha\beta}\,\varepsilon^\alpha \wedge \varepsilon^\beta \in \Lambda^2(\mathcal{G}^\star)$, obey the cocycle condition **(A.1)**. Acting with $L^\star{}_{g^{-1}|g}$ yields the left-invariant two form:

$$\Theta_L(g) \doteq L^\star{}_{g^{-1}|g}\,\Theta = \frac{1}{2}\,\Theta_{\alpha\beta}\,\varepsilon^\alpha_L(g) \wedge \varepsilon^\beta_L(g) \quad (A.2)$$

Using the cocycle relation and the Maurer-Cartan structure equations, it is seen that $\Theta_L(g)$ is a closed left-invariant two-form on G.

When $\mathcal{G}$ is semisimple, Θ is exact. Indeed, the Whitehead lemmas state that $H^1(\mathcal{G},\mathbf{R},\mathbf{0}) = 0$ and $H^2(\mathcal{G},\mathbf{R},\mathbf{0}) = 0$. In particular, $\Theta \in B^2(\mathcal{G},\mathbf{R},\mathbf{0})$ is a coboundary and there exists an element ξ of $C^1(\mathcal{G},\mathbf{R},\mathbf{0}) \equiv \mathcal{G}^\star$ such that $\Theta(\mathbf{u},\mathbf{v}) = (\delta_1(\xi))(\mathbf{u},\mathbf{v}) = -\xi([\mathbf{u},\mathbf{v}])$ or

$$\Theta_{\alpha\beta} = -\xi_\mu\,\mathbf{f}^\mu{}_{\alpha\beta} \quad (A.3)$$

The constant vector $\xi \in T^\star(\mathcal{G})$ is the analogue of a magnetic field in the abelian case $G \equiv \mathbf{R}^3$.

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