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Modeling long-memory processes by stochastic difference equations and superstatistical approach

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It is shown that the Poissonian-like process with slowly diffusing-like time-dependent average interevent time may be represented as the superstatistical one and exhibits $1/f$ noise. The distribution of the Poissonian-like interevent time may be expressed as q-exponential distribution of the Nonextensive Statistical Mechanics.

Keywords: $1/f$ noise, q-distributions, Point processes, Power-law distributions, Nonlinear stochastic equations

1. INTRODUCTION

The widespread occurrence of processes exhibiting power-law distributions, $1/f$ noise, as well as scaling behavior in general (see [1–4] and references herein), suggests that a generic, at least mathematical, explanation of such phenomena might exist.

The non-Gaussianity is often taken as a signature of fluctuator's interaction [5]. Nevertheless, statistically independent and noninteracting fluctuators may exhibit non-Gaussian noise, as well [6], especially when the fluctuations are strong [7, 8].

We have proposed stochastic models of $1/f^{\beta}$ noise, with $0.5 < \beta < 2$, based on the simple point process models [7] and on the nonlinear stochastic differential equations [4, 9].

On the other hand, the non-Gaussian, power-law and long-range processes may be modeled as superstatistical schemes (see, e.g., [10] and references herein). Superstatistical processes from the superposition of Brownian processes with Lorentzian spectra and a proper distribution of relaxation times [7], from the superposition of stochastic sequences of different size pulses [11], generated by the stochastic differential equation with fluctuating relaxation rate [12] and by the driven Poisson processes [13] are long-range models with the power-law distributions. They may be useful for analysis of traffic [11], financial [13, 14] and other systems.

Here we show that the Poissonian-like point process with slowly diffusing time-dependent average interevent time, i.e., a special case of the non-homogeneous Poisson process, may be represented as the superstatistical one. The distribution of the Poissonian-like interevent time may be expressed as q-exponential distribution of the Nonextensive Statistical Mechanics [15–17].

2. POINT PROCESS MODEL

Point process model of $1/f$ noise was introduced ten years ago [18, 19]. Later on, it was generalized for $1/f^{\beta}$ noise with $0.5 < \beta < 2$ [7]. The signal, flow, current etc. in this approach are represented as a sequence of correlated pulses or series of events

$$x(t) = \bar{a}\sum_{k}\delta(t-t_k). \quad (1)$$

Here $\delta(t)$ is the Dirac δ-function and $\bar{a}$ is an average contribution to the signal $x(t)$ of one pulse at the time moment t_k.

The model [7] is based on the generic multiplicative process for the interevent time $\tau_k = t_{k+1} - t_k$,

$$\tau_{k+1} = \tau_k + \gamma\tau_k^{2\mu-1} + \sigma\tau_k^{\mu}\varepsilon_k \quad (2)$$

generating the power-law distributed

$$P_k(\tau_k) \sim \tau_k^{\alpha}, \qquad \alpha = \frac{2\gamma}{\sigma^2} - 2\mu \quad (3)$$

sequence of the interevent times τ_k in k-space and $1/f^{\beta}$ power spectral density of the signal (1),

$$S(f) \sim \frac{1}{f^{\beta}}, \qquad \beta = 1 + \frac{\alpha}{3-2\mu}. \quad (4)$$

In this approach the (average) interevent time τ_k fluctuates due to the random perturbations by a sequence of uncorrelated normally distributed random variables $\{\varepsilon_k\}$ with zero expectation and unit variance; σ is the standard deviation of the white noise and γ is a coefficient of the nonlinear damping.

Interpreting k as a continuous variable we can transform the difference equation Eq. (2) to the Itô stochastic differential equation in k-space,

$$\mathrm{d}\tau_k = \gamma\tau_k^{2\mu-1}\mathrm{d}k + \sigma\tau_k^{\mu}\mathrm{d}W_k. \quad (5)$$

Here W_k is the Wiener process in k-space.

Transition from the occurrence number k to the actual time t in equation (5) according to the relation $\mathrm{d}t = \tau_k \mathrm{d}k$ yields the differential equation in the actual time t. For the derivation of such equation we rewrite Eq. (5) in the difference form,

$$\Delta\tau = \gamma\tau^{2\mu-1}\Delta k + \sigma\tau^{\mu}\sqrt{\Delta k}\varepsilon_k. \quad (6)$$

Introduction into Eq. (6) of the relation $\Delta k = \Delta t/\tau$ yields Itô stochastic differential equation for the variable $\tau(t)$ as a function of the actual time t,

$$\mathrm{d}\tau = \gamma\tau^{2\mu-2}\mathrm{d}t + \sigma\tau^{\mu-1/2}\mathrm{d}W_t, \quad (7)$$

where W_t is a standard Wiener process.

Equation (7) generates the stochastic variable τ, power-law distributed,

$$P_t(\tau) = \frac{P_k(\tau)}{\langle\tau_k\rangle}\tau \sim \tau^{\alpha+1}, \quad (8)$$

in the actual time t. Here $\langle\tau_k\rangle$ is the average interevent time.

The signal averaged over the time interval τ_k according to Eq. (1) is $I = \bar{a}/\tau_k$. Therefore, the distribution density of the intensity of the point process (1) averaged over the time interval τ_k is [7]

$$P(I) = \frac{\bar{a}\bar{I}}{I^3} P_k\left(\frac{\bar{a}}{I}\right) \sim \frac{1}{I^\lambda}, \quad \lambda = 3+\alpha. \tag{9}$$

Motivations for the introduction of the model (1)–(9) are presented in papers [4, 7].

On the other hand, stochastic point processes arise in different fields, such as physics, economics, cosmology, ecology, neurology, seismology, traffic flow, and Internet (see, e.g., [20–28] and references herein).

The proposed point process model [7, 18, 19] can been modified [4, 9] and useful for the modeling and analysis of self-organized systems, atmospheric variability, $1/f$ noise observed financial markets [13], cognitive experiments, time intervals production in tapping and oscillatory motion of the hand [29] and other systems.

Here we generalize the point process model of $1/f$ noise and relate it to q-exponential distribution.

3. RELATION TO SUPERSTATISTICS AND NONEXTENSIVE STATISTICAL MECHANICS

Due to divergence of the power-law distribution and the requirement of the stationarity of the process, the stochastic difference equation (2) should be analyzed together with the appropriate restrictions of the diffusion of the interevent time τ_k in some finite interval $\tau_{\rm min} < \tau_k < \tau_{\rm max}$. By analogy with Ref. [9] we improve Eq. (2) as

$$\begin{aligned} \tau_{k+1} &= \tau_k + \sigma^2\left[\mu + \frac{\alpha}{2} + \frac{1}{2}\left(\frac{\tau_{\rm min}}{\tau_k} - \frac{\tau_k}{\tau_{\rm max}}\right)\right]\tau_k^{2\mu-1} \\ &\quad + \sigma\tau_k^{\mu}\varepsilon_k. \end{aligned} \tag{10}$$

The associated Fokker-Planck equation gives the steady-state distribution

$$P_k(\tau_k) = C\exp\left\{-\frac{\tau_k}{\tau_{\rm max}} - \frac{\tau_{\rm min}}{\tau_k}\right\}\tau_k^{\alpha}, \qquad \alpha > -1. \tag{11}$$

The constant C is defined from the normalization

$$\begin{aligned} C^{-1} &= 2(\tau_{\rm min}\tau_{\rm max})^{\frac{\alpha+1}{2}} K_{\alpha+1}\left(2\sqrt{\frac{\tau_{\rm min}}{\tau_{\rm max}}}\right) \\ &\simeq \Gamma(\alpha+1)\tau_{\rm max}^{\alpha+1}, \qquad \tau_{\rm min} \ll \tau_{\rm max}. \end{aligned} \tag{12}$$

Here $K_\nu(z)$ is the modified Bessel function of the second kind.

As examples, in figure 1 we show distribution density, $P(\tau)$, of the interevent time, τ, and spectrum, $S(f)$, of the point process defined by Eq. (10) coinciding with analytical results.

Further we assume that τ_k is a slowly diffusing time-dependent average interevent time of the Poissonian-like process with the time-dependent rate. Within this assumption the actual interevent time τ_j is given by the conditional probability

$$\varphi(\tau_j|\tau_k) = \frac{1}{\tau_k}e^{-\tau_j/\tau_k}, \tag{13}$$

similar to the non-homogeneous Poisson process.

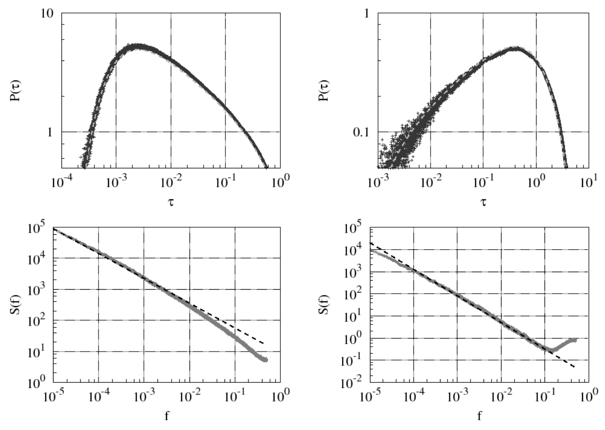

FIG. 1: Distribution density, $P(\tau)$, of the interevent time, τ, and the power spectral density, $S(f)$, of the point process defined by Eq. (10) with the parameters $\tau_{\rm min} = 10^{-3}$, $\tau_{\rm max} = 1$, $\mu = 0.5$, $\sigma = 10^{-2}$; whereas $\alpha = -0.4$, left column, and $\alpha = 0.4$, right column, in accordance with the theoretical results by Eqs. (11) and (4), respectively, dashed lines.

This additional stochasticity of the actual interevent time τ_j by randomization (13) of the concrete occurrence times does not influence the low frequency power spectra of the signal.

The generalized model (10)-(13) represents, however, a more realistic situation, because the concrete event occurs at random time (like in the Poisson case) controlled, nevertheless, by the Poissonian distribution with the slowly modulated according to equation (10) average interevent time.

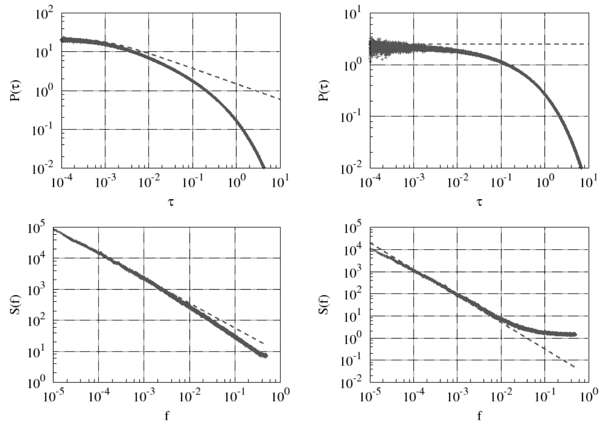

FIG. 2: Distribution density, $P(\tau)$, of the interevent time, τ, and the spectrum, $S(f)$, of the Poissonian-like point process defined by Eqs. (10) and (13) with the parameters as in figure 1 in accordance with the theoretical results by Eqs. (18), (19) and (4), respectively, dashed lines. The analytical dotted curve according to the exact Eq. (15) coincides with the numerical calculations.

In such a case, the distribution of the actual interevent time τ_j is expressed analogically to the superstatistical schemes [10, 13],

$$P_j(\tau_j) = \int \varphi(\tau_j|\tau_k)P_k(\tau_k)d\tau_k. \tag{14}$$

Introduction of Eqs. (11) and (13) into Eq. (14) yields

$$P_j(\tau_j) = C\int_0^\infty \exp\left\{-\frac{\tau_k}{\tau_{\max}} - \frac{\tau_{\min}+\tau_j}{\tau_k}\right\}\tau_k^{\alpha-1}d\tau_k$$
$$= \frac{2}{\tau_{\max}\Gamma(\alpha+1)}\left(\frac{\tau_{\min}+\tau_j}{\tau_{\max}}\right)^{\alpha/2} K_\alpha\left(2\sqrt{\frac{\tau_{\min}+\tau_j}{\tau_{\max}}}\right). \quad (15)$$

It has been shown [7, 18, 19] that the small interevent times and clustering of the events [4] make the greatest contribution to $1/f^\beta$ noise and the exhibition of the long-range scaled features. Expansion of Eq. (15) for $\tau_{\min}+\tau_j \ll \tau_{\max}$, yields

$$P_j(\tau_j) = \frac{\Gamma(|\alpha|)}{\Gamma(\alpha+1)}\frac{1}{\tau_{\max}}\left(\frac{\tau_{\min}+\tau_j}{\tau_{\max}}\right)^{\frac{\alpha-|\alpha|}{2}}, \quad \alpha \neq 0, \quad (16)$$

and

$$P_j(\tau_j) = \mathrm{const} - \frac{1}{2}\ln\left(\frac{\tau_{\min}+\tau_j}{\tau_{\max}}\right), \qquad \alpha = 0. \quad (17)$$

From Eq. (16) for $\alpha > 0$ we have

$$P_j(\tau_j) \simeq \frac{1}{\alpha\tau_{\max}} = \mathrm{const}, \qquad \alpha > 0. \quad (18)$$

Therefore, the power-law distribution (3) for $\alpha \geqslant 0$ as a result of introduction along with the superstatistical scheme of the second stochastic (non-homogeneous Poisson) process, according to Eqs. (17) and (18), reduces to the flat, $P_j(\tau_j) \sim$ const, distribution.

For $\alpha < 0$, however, Eq. (16) yields the q-exponential distribution,

$$P_j(\tau_j) = \frac{\Gamma(-\alpha)}{\Gamma(1+\alpha)}\frac{1}{\tau_{\max}}\left(\frac{\tau_{\min}}{\tau_{\max}}\right)^{\alpha} e_q^{-|\alpha|\frac{\tau_j}{\tau_{\min}}},$$
$$-1 < \alpha < 0, \quad (19)$$

with the index $q = 1 + 1/|\alpha|$.

In figure 2 we demonstrate that the distribution density, $P(\tau)$, of the interevent time, τ, of the superstatistical Poissonian-like point process defined by Eqs. (10) and (13) looks quite different from the distribution density of the autoregressive point process (10). The power spectral density, $S(f)$, of such non-homogeneous Poissonian-like process, however, is almost the same as of the simple autoregressive point process (10), i.e., the superimposition of the Poissonian distribution does not influence the low-frequency spectrum.

Note, that the special nonlinear stochastic differential equations [14] may generate q-Gaussian distributed signals with $1/f^\beta$ power spectrum, exhibiting bursts, similar to the crackling processes [30] and observable in long-term memory time series [31, 32].

4. CONCLUSION

The special nonlinear stochastic difference and differential equations generating power-law distributed signals and $1/f$ noise, due to the appropriate restriction of the diffusion-like motion of the stochastic variable, may result in q-exponential or q-Gaussian distributions of the variable, preserving exhibition of $1/f$ noise.

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