Brazilian Journal of Physics
ISSN: 0103-9733
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Sociedade Brasileira de Física
Brasil

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Brazilian Journal of Physics, vol. 41, núm. 4-6, 2011, pp. 290-296
Sociedade Brasileira de Física
São Paulo, Brasil

Braz J Phys (2011) 41:290–296
DOI 10.1007/s13538-011-0048-8

GENERAL AND APPLIED PHYSICS

Motion of Charged Particle in Electric and Magnetic Fields in 3D Noncommutative Spaces and Related Problems

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Received: 14 August 2011 / Published online: 29 September 2011

Abstract In three-dimensional noncommutative phase space, the energy spectrum and wave functions for the motion of a charged particle in a magnetic field are derived. Due to the momentum–momentum noncommutativity, the particle feels an effective magnetic field in a new direction. When an external electric field perpendicular to this effective magnetic field is applied, the Hall conductivity can be calculated. To get the Hall conductivity, one should define the electric currents from the probability currents in quantum mechanics rather than extending the classical electric currents to quantum mechanics directly. When the electric field is not perpendicular to the effective magnetic field, it is difficult to define the Hall conductivity.

Keywords Noncommutative phase space · Magnetic field · Hall effect · Electric current

1 Introduction

The idea of space–time noncommutativity has a long history [1]. The original motivation for introducing space–time noncommutativity is to resolve the problem of infinite energies in quantum field theory. Renewed interest in such an idea is mainly due to the recent discoveries in string theory and M theory that effects of noncommutative (NC) spaces may appear near the string scale and at higher energies [2–4]. Recently, a lot of investigations have been done on the theory of NC spaces [5–23] such as the quantum Hall effects (QHE) [7–10], the harmonic oscillator [11–14], the coherent states [15], the thermodynamics [16], the classical–quantum transition [17], etc.. The QHE [7–10] refers to the phenomenon that when a magnetic field perpendicular to the electric current flowing through an electric conductor is applied, there appears the potential difference (Hall voltage) in the direction perpendicular to the current and the magnetic field. For a particle confined to a NC plane $(\widehat{x}_1, \widehat{x}_2)$ with $[\widehat{x}_1, \widehat{x}_2] = \mathrm{i}\mu$, the Hamiltonian for QHE reads

$$\widehat{H}_{2D} = \frac{(\widehat{p}_1 + qB_0\widehat{x}_2/2)^2}{2m} + \frac{(\widehat{p}_2 - qB_0\widehat{x}_1/2)^2}{2m} - q\varepsilon\widehat{x}_1 \qquad (1)$$

where, ε is the strength of the electric field and $(-B_0\widehat{x}_2/2, B_0\widehat{x}_1/2) = \left(\widehat{A}_1, \widehat{A}_2\right)$ is the vector potential of a magnetic field B_0 perpendicular to the plane $(\widehat{x}_1, \widehat{x}_2)$. In (1), the last term $-q\varepsilon\widehat{x}_1 = \widehat{A}_0$ is the scalar potential, which corresponds to the case that the electric field is along the x-axis. Actually, it is not necessary to take the scalar potential as this form. The electric field can be along the y-axis and now the scalar potential is $\widehat{A}_0 = -q\varepsilon\widehat{x}_2$. When the direction of the electric field is arbitrary, the scalar potential takes the form $\widehat{A}_0 = -q(\varepsilon_1\widehat{x}_1 + \varepsilon_1\widehat{x}_2)$ with $(\varepsilon_1, \varepsilon_2)$ being the electric field. No matter what direction of the electric field is, the Hall electric current is perpendicular to the magnetic field and the electric field in the Hall effect.

To discuss QHE, we should define the electric currents. In classical physics, the electric current is $(J_1, J_2, J_3) = n_0 q(dx_1/dt, dx_2/dt, dx_3/dt)$, where n_0 is the particle number density and $(dx_1/dt, dx_2/dt, dx_3/dt)$ is the classical velocity. Extending this definition to quantum mechanics directly,

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one obtains the components of the current operator for the system (1)

$$\widehat{J}_x = n_0 q \frac{d\widehat{x}_1}{dt} = \frac{qn_0}{\mathrm{i}h}\left[\widehat{x}_1, \widehat{H}\right] = n_0 q \frac{\widehat{p}_1 + qB_0\widehat{x}_2/2}{m}\left(1 + \frac{qB_0\mu}{2h}\right) \quad (2a)$$

$$\widehat{J}_y = n_0 q \frac{d\widehat{x}_2}{dt} = \frac{qn_0}{\mathrm{i}h}\left[\widehat{x}_2, \widehat{H}\right] = n_0 q \left[\frac{\widehat{p}_2 - qB_0\widehat{x}_1/2}{m}\left(1 + \frac{qB_0\mu}{2h}\right) + q\varepsilon\frac{\mu}{h}\right] \quad (2b)$$

However, we can have another definition of the currents. In commutative quantum mechanics, we have the probability density $\rho=\psi^*\times\psi$ and the probability currents

$$\widehat{J}_{xq} = \psi^* \times \frac{\widehat{p}_1 + qB_0\widehat{x}_2/2}{m}\psi, \widehat{J}_{yq} = \psi^* \times \frac{\widehat{p}_2 - qB_0\widehat{x}_1/2}{m}\psi \quad (3)$$

From these probability currents, we can define the electric current operators for QHE

$$\widehat{J}_1 = n_0 q\frac{\widehat{p}_1 + qB_0\widehat{x}_2/2}{m}, \widehat{J}_2 = n_0 q\frac{\widehat{p}_2 - qB_0\widehat{x}_1/2}{m} \quad (4)$$

In commutative space μ=0, we have $\widehat{J}_1 = \widehat{J}_x, \widehat{J}_2 = \widehat{J}_y$ and there is no ambiguity. In NC case, $\left(\widehat{J}_1, \widehat{J}_2\right)$ and $\left(\widehat{J}_x, \widehat{J}_y\right)$ are different obviously. From (2a), (2b), and (4), we can define two different Hall conductivities σ_{cH} and σ_{qH} as follows

$$\left\langle \widehat{J}_x \right\rangle = 0, \left\langle \widehat{J}_y \right\rangle = \sigma_{cH}\varepsilon \quad (5)$$

$$\left\langle \widehat{J}_1 \right\rangle = 0, \left\langle \widehat{J}_2 \right\rangle = \sigma_{qH}\varepsilon \quad (6)$$

The two conductivities σ_{cH} and σ_{qH} have the relation

$$\sigma_{qH} = \left[\sigma_{cH} - n_0 q^2 \varepsilon \frac{\mu}{h}\right] \Big/ \left(1 + \frac{qB_0\mu}{2h}\right) \quad (7)$$

In principle, we should detect which of σ_{cH} and σ_{qH} is correct by experiments. However, such experiments are difficult to carry out at present. We hope that this problem can be judged theoretically by studying a modified system. A simple modification to the 2D system (1) is that the particle is allowed to move along the magnetic field and so we are faced with a three-dimensional (3D) system. The Hamiltonian for this 3D system reads

$$\widehat{H} = \widehat{H}_0 - q(\varepsilon_1\widehat{x}_1 + \varepsilon_2\widehat{x}_2) \quad (8a)$$

$$\widehat{H}_0 = \frac{(\widehat{p}_1 + qB_0\widehat{x}_2/2)^2}{2m} + \frac{(\widehat{p}_2 - qB_0\widehat{x}_1/2)^2}{2m} + \frac{\widehat{p}_3^2}{2m} \quad (8b)$$

The scalar potential is chosen in a more general one. How to choose the electric field (ε_1, ε_2) will be given in the following. For the three-dimensional system (8a) and (8b), the third component of the electric current corresponding to definition (2a) and (2b) is $\widehat{J}_z = n_0 q(d\widehat{x}_3/dt)$ and corresponding to the definition (4) is $\widehat{J}_3 = n_0 q\widehat{p}_3/m$.

Here, we make some remarks about the Hamiltonian (8a) and (8b). Classically, the term $\widehat{p}_3^2/(2m)$ is the kinetic energy for the particle to move along the magnetic field, which becomes operator in quantum mechanics. In commutative spaces, this term commutes with the rest part of the Hamiltonian (8a) and (8b) and does not affect the Hall effect. In NC phase space, this term does not commute with the rest part of the Hamiltonian (8a) and (8b) and may induce new results for the Hall currents, further the Hall conductivity. So, we deal with the system (8a) and (8b) in NC phase space.

When there are no momentum–momentum noncomutativity, it is shown that the quantum Hamiltonian $\widehat{H}\left(\widehat{x}_j, \widehat{p}_j\right)$ (j=1, 2, 3) can be obtained from a classical constrained system $H\left(x_j, p_j\right)$[17]. When $[\widehat{p}_1, \widehat{p}_2] = [\widehat{p}_2, \widehat{p}_3] = [\widehat{p}_3, \widehat{p}_1] = \mathrm{i}\nu \neq 0$, we can make the transformations $\widehat{p}_1 = \widehat{\pi}_1 + \beta\widehat{x}_2$, $\widehat{p}_2 = \widehat{\pi}_2 + \beta\widehat{x}_3$ and $\widehat{p}_3 = \widehat{\pi}_3 + \beta\widehat{x}_1$ with $\beta = (\ h - \sqrt{\ h^2 - 4\mu\nu})/(2\mu)$, so that $\left[\widehat{\pi}_j, \widehat{\pi}_k\right] = 0$ (k=1, 2, 3). Using these transformations, the Hamiltonian $\widehat{H}\left(\widehat{x}_j, \widehat{p}_j\right)$ in NC phase space can be rewritten in the form $\widehat{H}\left(\widehat{x}_j, \widehat{\pi}_j + \beta\widehat{x}_k\right)$. Similar to the discussions in [17], one sees that the Hamiltonian $\widehat{H}\left(\widehat{x}_j, \widehat{\pi}_j + \beta\widehat{x}_k\right)$ can be obtained from the corresponding classical constrained system $H\left(x_j, \pi_j + \beta x_k\right)$. Rewriting $\pi_j+\beta x_k$ as p_j, $\widehat{H}\left(\widehat{x}_j, \widehat{p}_j\right)$ is derived from a corresponding classical one H (x_j, p_j) even in the NC phase space. In another word, the Hamiltonian (8a) and (8b) can be obtained from the corresponding classical one with $\left(\widehat{x}_j, \widehat{p}_j\right)$ replaced by the classical variables (x_j, p_j).

Compared to the scalar and vector potentials in (1), an extra condition that the component $\widehat{A}_3$ of the vector potential along the magnetic field is zero is used in (8a) and (8b). Choosing the symmetric gauge for the magnetic field, this result is natural. Such a conclusion should be derived from the noncommutative Maxwell equations too.

To get the NC electrodynamics, the Seiberg–Witten map [2, 24, 25] is usually used. In such a theory, the field strength is related to the potential through the relation $\widehat{F}_{\mu\nu} = \partial_\mu \widehat{A}_\nu - \partial_\nu \widehat{A}_\mu - \mathrm{ig}\left(\widehat{A}_\mu * \widehat{A}_\nu - \widehat{A}_\nu * \widehat{A}_\mu\right)$, where * means the star or Moyal product. The quantities $\left(\widehat{F}_{12}, \widehat{F}_{23}, \widehat{F}_{31}\right)$ give the magnetic field and $\left(\widehat{F}_{01}, \widehat{F}_{02}, \widehat{F}_{03}\right)$ is the electric field. The configuration of the magnetic and electric fields in (8a) and (8b) corresponds to $\widehat{F}_{12} = B_0$, $\widehat{F}_{23} = \widehat{F}_{31} = 0$, $\widehat{F}_{01} = \varepsilon_1, \widehat{F}_{02} = \varepsilon_2$ and $\widehat{F}_{03} = 0$. In such a case, it is found that $\widehat{A}_3$ is really zero and $\left(\widehat{A}_0, \widehat{A}_1, \widehat{A}_2\right)$ are functions of the coordinates (x_2, x_2).

In the next section, we derive the energy spectrum and wave functions for the charged particle moving in a background magnetic field. Meanwhile, the quantum Hall effect is discussed and the Hall conductivity is given. It is found that the definition (4) is more reasonable. The third section is the conclusion.

2 QHE in Three-Dimensional Noncommutative Spaces

The coordinate and momentum operators obey the commutation relations in 3D noncommutative phase space

$$\begin{aligned} &[\widehat{x}_1, \widehat{x}_2] = [\widehat{x}_2, \widehat{x}_3] = [\widehat{x}_3, \widehat{x}_1] = \mathrm{i}\mu \\ &[\widehat{p}_1, \widehat{p}_2] = [\widehat{p}_2, \widehat{p}_3] = [\widehat{p}_3, \widehat{p}_1] = \mathrm{i}\nu \\ &[\widehat{x}_1, \widehat{p}_1] = [\widehat{x}_2, \widehat{p}_2] = [\widehat{x}_3, \widehat{p}_3] = \mathrm{i}h \end{aligned} \tag{9}$$

with μ and ν being the noncommutative parameters. To solve the stationary Schrödinger equation $\widehat{H}|\psi\rangle = E|\psi\rangle$, we diagonalize the Hamiltonian H_0 first. In commutative quantum mechanics, the operator $\widehat{p}_3$ commutes with the Hamiltonian H in (8a) and (8b) and so the momentum along the z-axis is conserved. Replacing $\widehat{p}_3$ by its eigenvalue, the three-dimensional problem is reduced to that charged particle moving on a plane. Thus, the Hamiltonian is diagonalized easily. In noncommutative phases, the operator $\widehat{p}_3$ does not commute with H_0 due to the commutation relations (9) and the momentum along the z-axis is not conserved.

To diagonalize the Hamiltonian H_0, we first have a look at the properties of the operators

$$\widehat{\upsilon}_1 = \frac{\widehat{p}_1 + qB_0\widehat{x}_2/2}{m}, \widehat{\upsilon}_2 = \frac{\widehat{p}_2 - qB_0\widehat{x}_1/2}{m}, \widehat{\upsilon}_3 = \frac{\widehat{p}_3}{m} \tag{10}$$

which are named velocity operators in commutative space. These velocity operators are the mechanical momentum operators $(\widehat{p}_1 + qB_0\widehat{x}_2/2,\ \widehat{p}_2 - qB_0\widehat{x}_1/2,\ \widehat{p}_3)$over the mass of the particle. These operators satisfy the following commutation relations

$$[\widehat{\upsilon}_1, \widehat{\upsilon}_2] = \frac{\mathrm{i}h}{m}\omega_{\mathrm{eff}}, \quad [\widehat{\upsilon}_2, \widehat{\upsilon}_3] = [\widehat{\upsilon}_3, \widehat{\upsilon}_1] = \frac{\mathrm{i}h}{m}\alpha \tag{11}$$

where the parameters α and ω_{eff} take the forms

$$\alpha = \frac{\nu}{mh}, \omega_{eff} = \frac{qB_0}{m}\left(1 + \frac{qB_0\mu}{4h} + \frac{\nu}{qB_0 h}\right) \tag{12}$$

In noncommutative phase space, is there an operator which commutes with the Hamiltonian (8a) and (8b) and reduces to $\widehat{\upsilon}_3$ or $\widehat{p}_3$ in the commutative limit? The answer is yes. We write such an operator as

$$\widehat{V} = (\widehat{\upsilon}_1 + \widehat{\upsilon}_2)\frac{\sin\theta}{\sqrt{2}} + \widehat{\upsilon}_3\cos\theta \tag{13}$$

Commutation of this operator with H_0 gives

$$\begin{aligned} &\cos\theta = \omega_{eff}/\omega_0, \sin\theta = \sqrt{2}\alpha/\omega_0, \omega_0 \\ &\quad = \sqrt{\omega_{eff}^2 + 2\alpha^2} \end{aligned} \tag{14}$$

In the commutative limit, α=0 and so $\widehat{V} = \widehat{\upsilon}_3$. In commutative quantum mechanics, the Hamiltonian can be diagonalized by introducing the annihilation and creation operators

$$\begin{aligned} \widehat{a} &= \sqrt{\frac{m}{h\omega_c}}\left[\frac{\widehat{\upsilon}_2 - \widehat{\upsilon}_1}{2} - \mathrm{i}\frac{\widehat{\upsilon}_2 + \widehat{\upsilon}_1}{2}\right], \widehat{a}^\dagger \\ &= \sqrt{\frac{m}{h\omega_c}}\left[\frac{\widehat{\upsilon}_2 - \widehat{\upsilon}_1}{2} + \mathrm{i}\frac{\widehat{\upsilon}_2 + \widehat{\upsilon}_1}{2}\right] \end{aligned} \tag{15}$$

where ω_c=qB_0/m. The Hamiltonian H_0 is cast as $\widehat{H}_0 = h\omega_c\left(\widehat{a}^\dagger\widehat{a} + 1/2\right) + m\widehat{\upsilon}_3^2/2$. Through some investigations, it is found that the annihilation and creation operators in noncommutative spaces are deformations of (15)

$$\widehat{A} = \sqrt{\frac{m}{h\omega_0}}\left[\frac{\widehat{\upsilon}_2 - \widehat{\upsilon}_1}{2} - \mathrm{i}\frac{\widehat{\upsilon}_2 + \widehat{\upsilon}_1}{2}\cos\theta + \mathrm{i}\widehat{\upsilon}_3\frac{\sin\theta}{\sqrt{2}}\right] \tag{16}$$

$$\widehat{A}^\dagger = \sqrt{\frac{m}{h\omega_0}}\left[\frac{\widehat{\upsilon}_2 - \widehat{\upsilon}_1}{2} + \mathrm{i}\frac{\widehat{\upsilon}_2 + \widehat{\upsilon}_1}{2}\cos\theta - \mathrm{i}\widehat{\upsilon}_3\frac{\sin\theta}{\sqrt{2}}\right] \tag{17}$$

One can prove the commutation relations

$$\left[\widehat{A}, \widehat{A}^\dagger\right] = 1, \left[\widehat{V}, \widehat{A}\right] = \left[\widehat{V}, \widehat{A}^\dagger\right] = 0 \tag{18}$$

Using (13), (16), and (17), the Hamiltonian $\hat{H}_0$ is rewritten as

$$\widehat{H}_0 = h\omega_0\left(\widehat{A}^\dagger\widehat{A} + 1/2\right) + m\widehat{V}^2/2 \quad (19)$$

which is diagonalized as the operator $\widehat{V}$ commutes with $\widehat{A}$ and $\widehat{A}^\dagger$. We write the common eigenstates of $\widehat{V}$ and $\widehat{A}^\dagger\widehat{A}$ as $|\eta, n_A\rangle$ with

$$\widehat{V}|\eta, n_A\rangle = \eta|\eta, n_A\rangle, \widehat{A}^\dagger\widehat{A}|\eta, n_A\rangle = n_A|\eta, n_A\rangle \quad (20)$$

From (13), one sees that the operator $\widehat{V}$ is Hermitian and so η is a real quantity. The integer n_A takes the values 0, 1, 2, 3, $\cdots$. Eigenvalues of the Hamiltonian $\hat{H}_0$ are

$$E_0 = h\omega_0(n_A + 1/2) + m\eta^2/2 \quad (21)$$

When η=0, the energy spectrum is discrete with equal spacing $\hbar\omega_0$. The operator $\widehat{V}$ in (13) can be written as the scalar product of two vectors $(\widehat{\upsilon}_1, \widehat{\upsilon}_2, \widehat{\upsilon}_3)$ and $\vec{n} = \left(\sin\theta/\sqrt{2}, \sin\theta/\sqrt{2}, \cos\theta\right)$. The above results show that the mechanical momentum along the direction $\vec{n}$ is conserved. From the Heisenberg equations of motion, we can show that $\vec{n}$ is the direction of an effective magnetic field.

By some calculations, the Heisenberg equations of motion for the velocity vector operator $(\widehat{\upsilon}_1, \widehat{\upsilon}_2, \widehat{\upsilon}_3) = \widehat{\upsilon}$ are

$$m\frac{d\widehat{\upsilon}}{dt} = q\vec{B}_\alpha \times \widehat{\upsilon} \quad (22)$$

where $\vec{B}_\alpha = (q\omega_0/m)\vec{n}$ is an effective magnetic field induced by the noncommutativity of the spaces. In another word, the particle feels a magnetic field in a new direction $\vec{n}$. This new direction is caused by the momentum–momentum noncommutivity as the momentum–momentum noncommutivity or nonzero ν makes sin$\theta \neq 0$.

For the 2D case, the parameter α=0 and so sinθ=0. Now, the effective magnetic field is still along the original magnetic field B_0.

Though the energy spectrum of $\hat{H}_0$ is found, the states of the system cannot be described by $|\eta, n_A\rangle$ completely. For a three-dimensional motion, we should use three quantum numbers to describe the states of the system. In another word, we should find another operator which commutes with the operators $\widehat{V}$ and $\widehat{A}^\dagger\widehat{A}$. The process to find such an operator in noncommutative 3D phase space is a little lengthy. At first, we define the following operators

$$\widehat{u}_1 = \frac{\widehat{p}_1 - \sigma\widehat{x}_2}{m},\ \widehat{u}_2 = \frac{\widehat{p}_2 + \sigma\widehat{x}_1}{m} \quad (23)$$

In commutative quantum mechanics, setting $\sigma = qB_0/2$ and replacing $(\widehat{\upsilon}_1, \widehat{\upsilon}_2)$ by $(\widehat{u}_1, \widehat{u}_2)$ in (15), one gets new creation and annihilation operators which commutes $\widehat{\upsilon}_3$ and $\widehat{a}^\dagger\widehat{a}$. In noncommutative 3D spaces, we construct the following two operators

$$\widehat{B}_1 = \frac{1}{\sqrt{\Lambda}}\left[\frac{\widehat{u}_2 - \widehat{u}_1}{2} + \mathrm{i}\frac{\widehat{u}_2 + \widehat{u}_1}{2}\cos\theta - \mathrm{i}\widehat{\upsilon}_3\rho\frac{\sin\theta}{\sqrt{2}}\right] \quad (24)$$

$$\widehat{B}_1^\dagger = \frac{1}{\sqrt{\Lambda}}\left[\frac{\widehat{u}_2 - \widehat{u}_1}{2} - \mathrm{i}\frac{\widehat{u}_2 + \widehat{u}_1}{2}\cos\theta + \mathrm{i}\widehat{\upsilon}_3\rho\frac{\sin\theta}{\sqrt{2}}\right] \quad (25)$$

where the parameter Λ is chosen as

$$\Lambda = \frac{2h}{m\omega_0}\left[\frac{\sigma}{m}\left(1 - \frac{\nu}{2\sigma h} - \frac{\sigma\mu}{2h}\right)\omega_{eff} - \rho\alpha^2\right] \quad (26)$$

to ensure that the two operators (24) and (25) obey $\left[\widehat{B}_1, \widehat{B}_1^\dagger\right] = 1$. The parameters σ and ρ are determined by the commutation of the operators (24) and (25) with $\widehat{V}$ and $\widehat{A}^\dagger\widehat{A}$.

By some calculations, we have

$$\left[\widehat{V}, \widehat{B}_1\right] = \frac{ih\alpha}{m^2\omega_0\sqrt{\Lambda}}\left[m\omega_{eff} - \frac{\nu}{h} - \frac{qB_0}{2} + \sigma\left(1 + \frac{qB_0\mu}{2h}\right)\right] = 0 \quad (27)$$

When ν=0, the parameter α=0 and (27) automatically holds. From (14), one further sees that sinθ=0 and the operator $\widehat{\upsilon}_3$ disappears from (24) and (25). In this case, the parameter ρ is unnecessary. Commutation of the operators $\left(\widehat{B}_1, \widehat{B}_1^\dagger\right)$ with $\widehat{A}^\dagger\widehat{A}$ gives

$$\sigma = \frac{qB_0/2}{1 + qB_0\mu/(2h)} \quad (28)$$

Using $\left(\widehat{B}_1, \widehat{B}_1^\dagger\right)$, one can form Hermitian operators such as $\widehat{B}_1 + \widehat{B}_1^\dagger$, $\widehat{B}_1^\dagger\widehat{B}_1$. States of the system are described by the common eigenstates of $\widehat{V}$, $\widehat{A}^\dagger\widehat{A}$ and the Hermitian operator formed by $\left(\widehat{B}_1, \widehat{B}_1^\dagger\right)$. When $\nu \neq 0$, the situation becomes more complex. Equation (27) determines the parameter σ as

$$\sigma = \frac{\nu/h + qB_0/2 - m\omega_{\mathrm{eff}}}{1 + qB_0\mu/(2h)} \quad (29)$$

Commutation of $\widehat{B}_1$ with $\widehat{A}$ gives a relation to determine the parameter ρ

$$\sigma - \frac{qB_0}{2} = (1 + \rho)\frac{m\alpha\sin\theta}{\sqrt{2}\cos\theta} \quad (30)$$

Unfortunately, $\widehat{B}_1$ does not commute with $\widehat{A}^\dagger$

$$\left[\widehat{A}^\dagger, \widehat{B}_1\right] = \theta_0 \qquad \theta_0 = (\rho - 1)\frac{h\alpha^2}{\omega_0\sqrt{h\omega_0 m\Lambda}} \tag{31}$$

However, using $\left(\widehat{B}_1, \widehat{B}_1^\dagger\right)$ and $\left(\widehat{A}, \widehat{A}^\dagger\right)$, we can construct two set of independent annihilation and creation operators

$$\widehat{A}_1 = \frac{\widehat{A}+\widehat{B}_1}{\sqrt{2(1-\theta_0)}}, \; \widehat{A}_1^\dagger = \frac{\widehat{A}^\dagger+\widehat{B}_1^\dagger}{\sqrt{2(1-\theta_0)}} \qquad \widehat{A}_2 = \frac{\widehat{A}-\widehat{B}_1}{\sqrt{2(1+\theta_0)}}, \; \widehat{A}_2^\dagger = \frac{\widehat{A}^\dagger-\widehat{B}_1^\dagger}{\sqrt{2(1+\theta_0)}} \tag{32}$$

It is not difficult to prove the following commutation relations

$$\left[\widehat{A}_1, \widehat{A}_2\right] = \left[\widehat{A}_1^\dagger, \widehat{A}_2^\dagger\right] = 0 \qquad \left[\widehat{A}_j, \widehat{A}_k^\dagger\right] = \delta_{jk} \tag{33}$$

with j, k=1, 2. The operators $\left(\widehat{A}, \widehat{A}^\dagger\right)$ can be rewritten as

$$\widehat{A} = \frac{1}{\sqrt{2}}\left[\widehat{A}_1(1-\theta_0) + \widehat{A}_2(1+\theta_0)\right] \qquad \widehat{A}^\dagger = \frac{1}{\sqrt{2}}\left[\widehat{A}_1^\dagger(1-\theta_0) + \widehat{A}_2^\dagger(1+\theta_0)\right] \tag{34}$$

Through careful inspection, one may notice that $\left(\widehat{A}, \widehat{A}^\dagger\right)$ commute with the operators

$$\widehat{B} = \frac{1}{\sqrt{2}}\left[\widehat{A}_1(1+\theta_0) - \widehat{A}_2(1-\theta_0)\right] \qquad \widehat{B}^\dagger = \frac{1}{\sqrt{2}}\left[\widehat{A}_1^\dagger(1+\theta_0) - \widehat{A}_2^\dagger(1-\theta_0)\right] \tag{35}$$

which satisfy $\left[\widehat{B}, \widehat{B}^\dagger\right] = 1$. So in noncommutative phase space, the states for a charged particle moving in a magnetic field should be described by the common eigenstates of $\widehat{V}$, $\widehat{A}^\dagger\widehat{A}$ and the Hermitian operator formed by $\left(\widehat{B}, \widehat{B}^\dagger\right)$. For example, using $\left(\widehat{B}, \widehat{B}^\dagger\right)$ one may form a Hermitian operator $\widehat{B}^\dagger\widehat{B}$. Eigenvalues of this operator are integers and do not appear in the energy spectrum (21), which means the degenerate degree of the states is infinite. Up to now, the energy spectrum and quantum states for a charged particle moving in a magnetic field in 3D noncommutative phase space are derived.

From (13), (16), and (17), the operators $\widehat{\upsilon}_j$(j=1, 2, 3) can be rewritten as

$$\widehat{\upsilon}_1 = \widehat{V}\frac{\sin\theta}{\sqrt{2}} - \frac{1}{2}\sqrt{\frac{h\omega_0}{m}}\left[\widehat{A}(1-\mathrm{i}\cos\theta) + \widehat{A}^\dagger(1+\mathrm{i}\cos\theta)\right]$$
$$\widehat{\upsilon}_2 = \widehat{V}\frac{\sin\theta}{\sqrt{2}} + \frac{1}{2}\sqrt{\frac{h\omega_0}{m}}\left[\widehat{A}(1+\mathrm{i}\cos\theta) + \widehat{A}^\dagger(1-\mathrm{i}\cos\theta)\right]$$
$$\widehat{\upsilon}_3 = \widehat{V}\cos\theta - \frac{\sin\theta}{\sqrt{2}}\sqrt{\frac{h\omega_0}{m}}\left[\mathrm{i}\left(\widehat{A} - \widehat{A}^\dagger\right)\right] \tag{36}$$

Recalling that $\vec{n} = \left(\sin\theta/\sqrt{2}, \sin\theta/\sqrt{2}, \cos\theta\right)$ is the direction of the effective magnetic field $\vec{B}_\alpha$, we let the electric field be perpendicular to this direction to discuss QHE. For example, we can apply the electric field in the direction $\vec{n}_{12} = \left(1/\sqrt{2}, -1/\sqrt{2}, 0\right)$ or $(\varepsilon_1, \varepsilon_2) = \varepsilon(1/\sqrt{2}, -1/\sqrt{2})$. In this case, the Hamiltonian (8a) and (8b) becomes

$$\widehat{H} = \widehat{H}_0 - \frac{q\varepsilon}{\sqrt{2}}(\widehat{x}_1 - \widehat{x}_2) \tag{37}$$

After diagonalization, the Hamiltonian (37) becomes

$$\begin{aligned}\widehat{H} &= h\omega_0\left(\widehat{A}^\dagger\widehat{A} + 1/2\right) + m\widehat{V}^2/2 \\ &- \frac{q\varepsilon m}{\sqrt{2}(\sigma+qB_0/2)}\left[\frac{\theta_0\sqrt{\Lambda}}{\left(1+\theta_0^2\right)\cos\theta} - \left(\rho\sin^2\theta + \cos\theta\right)\sqrt{\frac{h\omega_0}{m}}\right]\mathrm{i}\left(\widehat{A} - \widehat{A}^\dagger\right) \\ &- \frac{q\varepsilon m}{\sqrt{2}(\sigma+qB_0/2)}\left\{\frac{\sqrt{\Lambda\left(1-\theta_0^2\right)}}{\left(1+\theta_0^2\right)\cos\theta}\mathrm{i}\left(\widehat{B}^\dagger - \widehat{B}\right) + \left(\rho\frac{\sin 2\theta}{\sqrt{2}} - \sqrt{2}\sin\theta\right)\widehat{V}\right\}\end{aligned} \tag{38}$$

Writing the eigenstates of $\hat{H}$ as $D(\chi)|\eta, n, \xi\rangle$, we find that the states $|\eta, n, \xi\rangle$ satisfy

$$\begin{aligned}&\widehat{V}|\eta, n, \xi\rangle = \eta|\eta, n, \xi\rangle \\ &\mathrm{i}\left(\widehat{B}^\dagger - \widehat{B}\right)|\eta, n, \xi\rangle = \xi|\eta, n, \xi\rangle \\ &\widehat{A}^\dagger\widehat{A}|\eta, n, \xi\rangle = n|\eta, n, \xi\rangle\end{aligned} \tag{39}$$

where $D(\chi) = \exp\left(\chi\widehat{A}^\dagger - \chi^*\widehat{A}\right)$ is a displacement operator. The parameter χ is

$$\chi = \frac{-\mathrm{i}q\varepsilon m}{\sqrt{2}h\omega_0(\sigma + qB_0/2)} \times \left[\frac{\theta_0\sqrt{\Lambda}}{\left(1+\theta_0^2\right)\cos\theta} - \left(\rho\sin^2\theta + \cos\theta\right)\sqrt{\frac{h\omega_0}{m}}\right] \tag{40}$$

The direction which is perpendicular to the effective magnetic field $\vec{n}$ and the electric field $\vec{n}_{12}$ is $\vec{n} = \left(\cos\theta/\sqrt{2}, \cos\theta/\sqrt{2}, -\sin\theta\right)$. From the definition (4), the current operators in the direction $\vec{n}_{12}$ and $\vec{n}_{123}$ are respectively

$$\widehat{J}_{12} = \left(\widehat{J}_1 - \widehat{J}_2\right)/\sqrt{2} = n_0 q(\widehat{\upsilon}_1 - \widehat{\upsilon}_2)/\sqrt{2} \\ = n_0 q\sqrt{\tfrac{h\omega_0}{2m}}\left(\widehat{A} + \widehat{A}^{\dagger}\right) \tag{41}$$

$$\widehat{J}_{123} = \left(\widehat{J}_1 + \widehat{J}_2\right)\cos\theta/\sqrt{2} - \widehat{J}_3 \sin\theta \\ = n_0 q\left[(\widehat{\upsilon}_1 + \widehat{\upsilon}_2)\cos\theta/\sqrt{2} - \widehat{\upsilon}_3 \sin\theta\right] \\ = n_0 q\sqrt{\tfrac{h\omega_0}{2m}}\mathrm{i}\left(\widehat{A} - \widehat{A}^{\dagger}\right) \tag{42}$$

The operator $\widehat{V}$ now disappears from these current operators. Under the states $D(\chi)|\eta, n, \xi\rangle$, we have the mean values

$$\left\langle \widehat{J}_{12} \right\rangle = 0 \\ \left\langle \widehat{J}_{123} \right\rangle = qn_0\sqrt{\tfrac{h\omega_0}{2m}}\mathrm{i}(\chi - \chi^*) \\ = qn_0 \tfrac{q\varepsilon}{(\sigma + qB_0/2)} \sqrt{\tfrac{m}{h\omega_0}}\left[\tfrac{\theta_0\sqrt{\Lambda}}{(1+\theta_0^2)\cos\theta} - \left(\rho \sin^2\theta + \cos\theta\right)\sqrt{\tfrac{h\omega_0}{m}}\right] \tag{43}$$

Setting $\left\langle \widehat{J}_{123} \right\rangle = \sigma_{qH}\varepsilon$, we obtain the Hall conductivity

$$\sigma_{qH} = qn_0 \frac{q}{(\sigma + qB_0/2)}\sqrt{\frac{m}{h\omega_0}} \\ \times\left[\frac{\theta_0\sqrt{\Lambda}}{(1+\theta_0^2)\cos\theta} - \left(\rho \sin^2\theta + \cos\theta\right)\sqrt{\frac{h\omega_0}{m}}\right] \tag{44}$$

When ν=0, by similar calculations we have

$$\sigma_{qH} = -qn_0 \frac{q}{(\sigma + qB_0/2)} \tag{45}$$

where σ is given by (28). Further setting μ=0, we get $\sigma_{qH} = -qn_0/B_0$, which is just the result in commutative space.

If we use definition (2a) and (2b), we will meet difficulties to find the Hall conductivity. Now, the components of the currents are

$$\widehat{J}_x = n_0 q\frac{d\widehat{x}_1}{dt} = n_0 q\left[\widehat{\upsilon}_1\left(1 + \frac{qB_0\mu}{2h}\right) + \frac{q\varepsilon\mu}{\sqrt{2}h}\right] \tag{46a}$$

$$\widehat{J}_y = n_0 q\frac{d\widehat{x}_2}{dt} = n_0 q\left[\widehat{\upsilon}_2\left(1 + \frac{qB_0\mu}{2h}\right) + \frac{q\varepsilon\mu}{\sqrt{2}h}\right] \tag{46b}$$

$$\widehat{J}_z = qn_0\frac{d\widehat{x}_3}{dt} = \frac{qn_0}{\mathrm{i}h}\left[\widehat{x}_3,\ \widehat{H}\right] \\ = qn_0\left[\widehat{\upsilon}_3 - \frac{qB_0\mu}{2h}(\widehat{\upsilon}_1 + \widehat{\upsilon}_2)\right] \tag{46c}$$

From these component currents, the electric current perpendicular to the magnetic and electric fields is

$$\widehat{J}_{123} = \left(\widehat{J}_x + \widehat{J}_y\right)\cos\theta/\sqrt{2} - \widehat{J}_z \sin\theta = \\ = n_0 q\tfrac{q\varepsilon\mu}{h}\cos\theta + n_0 q\widehat{V}\left[\left(1 + \tfrac{qB_0\mu}{2h}\right)\tfrac{\sin 2\theta}{2} + \tfrac{\sqrt{2}qB_0\mu}{2h}\sin^2\theta - \cos\theta\right] \\ +\mathrm{i}\left(\widehat{A} - \widehat{A}^{\dagger}\right)n_0 q\sqrt{\tfrac{h\omega_0}{2m}}\left[\left(1 + \tfrac{qB_0\mu}{2h}\right)\tfrac{\cos^2\theta}{\sqrt{2}} + \tfrac{qB_0\mu}{4h}\sin 2\theta + \tfrac{\sin\theta}{\sqrt{2}}\right] \tag{47}$$

Its expectation value is

$$\left\langle \widehat{J}_{123} \right\rangle = n_0 q\tfrac{q\varepsilon\mu}{h}\cos\theta + n_0 q\eta\left[\left(1 + \tfrac{qB_0\mu}{2h}\right)\tfrac{\sin 2\theta}{2} + \tfrac{\sqrt{2}qB_0\mu}{2h}\sin^2\theta - \cos\theta\right] \\ +2\mathrm{i}\chi n_0 q\sqrt{\tfrac{h\omega_0}{2m}}\left[\left(1 + \tfrac{qB_0\mu}{2h}\right)\tfrac{\cos^2\theta}{\sqrt{2}} + \tfrac{qB_0\mu}{4h}\sin 2\theta + \tfrac{\sin\theta}{\sqrt{2}}\right] \tag{48}$$

The second term is independent of the electric field and so the expectation value $\left\langle \widehat{J}_{123} \right\rangle$ is not proportional to the electric field, which results in that the Hall conductivity cannot be defined though the relation $\left\langle \widehat{J}_{123} \right\rangle = \sigma_H\varepsilon$.

Calculations also show that when the electric field is not perpendicular to the effective magnetic field $\vec{B}_\alpha$, the Hall conductivity is difficult to define, no matter the electric currents (2a) or (2b) or (4) are used. We see that the 3D problem contains new contents compared to the 2D case.

3 Conclusions

The motion of a charged particle in a magnetic field was studied in three-dimensional noncommutative phase space. In such a case, the particle feels an effective magnetic field in a new direction due to the momentum–momentum noncommutativity. The energy spectrum is a discrete one with equal spacing embedded in a continuous one as shown in (14), which is similar to that in commutative space. Compared with the energy spectrum, the wave functions are much more complex. To find the operators $\left(\widehat{B}, \widehat{B}^{\dagger}\right)$ in (35) proved to be the key step to get the states of the system. From the discussions of the Hall conductivity, it was shown that in three-dimensional noncommutative phase space we should define the electric current from the probability current in quantum mechanics.

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