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STATISTICAL

Infinite Ergodic Theory and Non-extensive Entropies

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Abstract We recapitulate results from the infinite ergodic theory that are relevant to the theory of non-extensive entropies. In particular, we recall that the Lyapunov exponent of the corresponding systems is zero and that the deviation between neighboring trajectories does not necessarily grow polynomially. Nonetheless, as we show, no single quantity can describe this subexponential growth, the generalized q-exponential $\exp_q$ being, in particular, ruled out. We also revisit a number of dynamical systems preserving nonfinite ergodic measure.

Keywords Infinite ergodic theory · Lyapunov exponent · Infinite measure · Non-extensive entropies

1 Introduction

Non-extensive entropies have been much studied in recent years, and the activity has raised a number of interesting questions. Among them is a proposal bringing attention to the dynamics of systems that have zero Lyapunov exponent and are weakly sensitive to initial conditions [1]. According to that proposal, the sensitiveness could be described by the q-generalized Lyapunov exponent λ_q, for some q, in such a way that the average distance between two points after n-iterates would be of the order of $\exp_q(\lambda_q n)$, where $\exp_q(t) = [1 + (1 - q)t]^{\frac{1}{1-q}}$.

In particular, Tsallis et al. [2] conjectured that a version of Pesin's theorem for subexponential instability would relate the q-entropy to the q-generalized Lyapunov exponent, in the same way that Pesin's identity relates the Kolmogorov–Sinai entropy to the Lyapunov exponent. More precisely, the entropy and the exponent would coincide if $\lambda_q > 0$ and $q < 1$, a case in which the average distance increases polynomially.

With this in mind, we recapitulate certain results in infinite ergodic theory, i.e., the ergodic theory of systems preserving a non-finite measure. We are interested in such systems because (a) their Lyapunov exponent is zero and (b) they may exhibit subexponential instability (see Theorem 1 and Remark 1, below). They can therefore be analyzed in the framework of non-extensive entropies.

In this paper, we propose to show that the following assertions hold for some systems preserving an infinite measure (and therefore having zero Lyapunov exponent):

1. No unique quantity describes subexponential instabilities.
2. Subexponential rates can grow faster than polynomial ones.

From this, a twofold conclusion follows: *No single quantity characterizes subexponential growth, and polynomials may be inappropriate to describe it.*

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Our presentation is divided in four steps:

1. In Section 2.1, we introduce a class of examples with infinite invariant measure.
2. Next, we explain why systems with infinite invariant measure exhibit zero Lyapunov exponents (see Theorem 1 and Remark 1).
3. We then show how ergodic theorems can be recovered for infinite invariant measure and how the time averages become intrinsically random, making it difficult to find a unique "generalized Lyapunov" quantity (see Theorem 2 and Remark 3).
4. Finally, in Section 2.3, we show that such systems display subexponential instabilities and offer examples of nonpolynomial subexponential instability.

Our presentation follows [3, 4]. This note establishes no new theorems for infinite ergodic theory; we simply present results that have already been proved elsewhere and recast them to underscore their relevance in the theory of non-extensive statistical mechanics. We expect the theory of infinite measures to shed new light upon certain results concerning non-extensive entropies.

2 Infinite Measure

Given a map $T : X \to X$ acting on a phase space X, its action can lead to very complicated (chaotic) dynamics. In particular, it becomes impossible to predict the exact moment when a relevant event will occur. Ergodic theory can be seen as a quantitative theory of dynamical systems enabling us to deal rigorously with such situations. For example, Birkhoff's ergodic theorem tells us quite precisely how often an event will occur for typical initial states. In fact, a rich quantitative theory is available for systems possessing an invariant finite measure μ, i.e., for systems such that $\mu \circ T^{-1} = \mu$. For smooth systems, moreover, Birkhoff's ergodic theorem characterizes the rate of mixing of a system for regular observables.

Systems of interest do exist, however, including ones that are by no means exotic, which happen to have an infinite invariant measure, i.e., a measure preserved by T with $\mu(X) = \infty$. The "Infinite Ergodic Theory" focuses such systems and tries to answer the simplest quantitative question concerning the long-term behavior of occupation times

$$S_n(A) := \Sigma_{k=0}^{n-1} 1_A(T^k(x)), \tag{1}$$

where 1_A is the characteristic function of the set A [$1_A(x) = 1$ if $x \in A$; otherwise the value is zero]. The quantity $S_n(A)$ counts the number of visits an orbit pays to A before time n. To be slightly more general, we can also look at ergodic sums

$$S_n(f) := \Sigma_{k=0}^{n-1} f(T^k(x)) \tag{2}$$

of measurable functions f.

2.1 Examples

This section introduces a series of examples of infinite-measure preserving transformations and discusses their dynamical and physical relevance:

1. Boole maps, $T : \mathbb{R} \setminus \{0\} \to \mathbb{R}$, $T(x) = x - \frac{1}{x}$. The invariant measure is the Lebesgue measure on the real line [5, 6].
2. Pomeau–Manneville maps, $T : [0, 1] \to [0, 1]$, $T(x) = x + cx^p mod(1)$, in which zero is a parabolic fixed point [i.e., $T'(0) = 1$]. With $p \geq 2$ ($c > 0$), the invariant measure has support in [0, 1] but gives infinite measure to that interval [7, 8].
3. Polynomial and rational maps on $\mathbb{C}$ (quotient of polynomials acting on $\mathbb{C}$) in the Julia set, with parabolic fixed points (points where the derivative has unitary modulus) or with no critical points. The invariant measure is a h-conformal measure concentrated in the Julia set, h being the Hausdorff dimension of the Julia set [9].
4. Some quadratic unimodal maps (or logistic-type maps) in which the invariant measure is absolutely continuous and giving infinite measure to the domain [10–12].
5. Horocycle flows on infinite regular covers of compact hyperbolic surfaces. The invariant measure is the classical volume measure [13].
6. Two-dimensional version of the Boole's map, $T : \mathbb{R}^2 \setminus \{0\} \times \mathbb{R} \to \mathbb{R}^2$, $T(x, y) = (x - \frac{1}{y}, x + y - \frac{1}{y})$. The invariant measure is the Lebesgue measure on the whole two-dimensional plane [14].
7. Semi-dispersing billiards with an infinite cusp. Lenci [15] showed that such billiard tables exhibit an infinite invariant measure. We recall that Sinai started to study billiards as early as 1963, in connection with the Boltzmann hypothesis in statistical mechanics.
8. Cylinder maps [16, 17], which are volume preserving transformations acting on the infinite cylinder $\mathbb{R} \times \mathbb{T}^1$ ($\mathbb{T}^1$ denotes the circle) given by $T(x, y) = (x + \phi(y), y + \alpha)$, where ϕ is a smooth map with zero integral and α is an irrational number. In this case, the σ-finite invariant measure is the Lebesgue

measure on the cylinder [16, 18–22] (Cirilo et al., private communication).

The first two systems are conjugated to the classical doubling function acting on the interval [0, 1], and so their dynamics can be described by the symbolic shift acting on the space of sequences of two symbols. In particular, they are topologically mixing and have infinitely many periodic orbits. In this sense, from a topological point of view, they can be considered as prototypes of one-dimensional chaotic dynamics, even though they subexponentially sensitive to initial data.

The Pomeau–Manneville maps were introduced to model intermittent behavior in fluid dynamics [8]. For those systems, the resulting behavior is an alternation of chaotic (when the orbit stays far from the parabolic point, in which case the map is similar to the doubling angle map) and regular maps when the orbit is trapped near the parabolic point. On the other hand, due to the parabolic fixed point, the Lyapunov exponent of such systems is zero and the invariant measure, infinite. In fact, the same type of behavior is expected for one-dimensional maps without critical point (non-zero derivative) and with a parabolic periodic point. Moreover, parabolic periodic points lead to dynamics with anomalous statistical behavior. For this reason, they have modeled many an interesting physical system.

Polynomial and rational maps are the typical examples of dynamics acting on the complex plane; the classical example is given by the family of quadratical polynomials $P_\mu(z) = z^2 + \mu$. Their study dates back to Fatou, and the analysis of the parameter space introduces the so-called Mandelbrot set. Recall that, in the case of polynomials, the Julia set is defined as the boundary of the set of points with trajectories that do not escape to infinity and hence concentrate all the complexity of the dynamics (see, for instance, Milnor [23]).

The quadratic family is the classical example of one-dimensional real dynamics with critical points exhibiting chaotic dynamics. Hofbaue et al. [10], Bruin [11], and Al-Khal et al. [12] show that, for certain parameters, the associated map has an infinite absolutely continuous invariant measure. On the other hand, Mayoral and Robledo [24] studied the set of parameter with infinitely renormalizable dynamics [25], which dynamics displays an attracting Cantor set, a situation usually called "the edge-of-chaos attractor" for quadratic maps [26, 27]. Although it is unknown whether that dynamics has infinite invariant measure, the above-mentioned papers show that the q-generalized Lyapunov coefficient is not unique, even if q itself remains unchanged. The situation may be more complex when averages are taken into account, and the results depend on the choice of average (see also Theorem 3 below and the subsequent discussion).

The two-dimensional version of the Boole's map introduced by Henon to model certain problem in celestial mechanics [28, 29] is conjugated to the Baker map acting on a two-dimensional rectangle [14]. Therefore, from a topological point of view, the above-introduced Henon's map has all the relevant topological properties of typical two-dimensional chaotic dynamics.

The Horocycle flow on compact hyperbolic surfaces is the most classical example of minimal and ergodic dynamic respect to finite volume measure (fact proved by Hedlund in 1930). They are associated to the classical geodesic flow on hyperbolic surfaces (free motion on hyperbolic surfaces). Nonetheless, the measure become infinite when infinite covers are considered [13]. To understand the significance of those models, recall that geodesic flows are the geometrical interpretation of a mechanical system free from potential forces.

With respect to the seventh example, above, we recall that billiards were originally studied as a simple model of a Lorentz gas and that Sinai proved that the billiards map of a system in a two-dimensional torus with finitely many convex obstacles is a chaotic ergodic system. The noncompact cusps in the billiards study in Lenci [15] yields a dynamical system with an infinite invariant measure.

The above-discussed cylinder maps can be obtained as the limit of systems conjugated to rigid irrational rotations. For certain types of such systems (the ones with the so-called Diophantine rotations), the typical KAM theory can be applied. They are, therefore, not ergodic at all. Nonetheless, when the rotation is "Liouville," one can construct ergodic examples preserving the Lebesgue measure on the cylinder. Some of those systems have been treated in the context of non-extensive entropies, see for instance Tsallis et al. [2] and Costa et al. [30] for the case of the quadratic family.

2.2 Ergodic Theorem for Infinite Measures

In this subsection, we present an ergodic theorem for infinite measure. First, in Section 2.2.1, we show that, for infinite measure, the average occupation time and the average Birkhoff sum are always zero (see Theorem 1) and explain that time cannot be reparametrized to yield the right average rate (see Theorem 2). In Section 2.2.2, we present an ergodic theorem for non-invertible maps, which can be applied to the first four examples in our list. In Section 2.2.3, we consider the invertible case.

2.2.1 Zero Average

In the context of infinite measure, the first ergodic theorem for recurrent ergodic measure transformations (namely almost every point is recurrent and any invariant set or its complement has zero measure) has the following statement:

Theorem 1 (See Theorem 1.14 in Walters [31]). *Let T be a recurrent ergodic measure transformation on the infinite measure space $(X; A; \mu)$. Then if $f \in L^1(\mu)$,*

$$\frac{1}{n} S_n(f) \to 0. \tag{3}$$

If A is a set with finite measure, then

$$\frac{1}{n} S_n(A) \to 0. \tag{4}$$

Notice that the situation is quite different for finite ergodic measure: The classical Birkhoff's theorem states that $\frac{1}{n} S_n(f) \to \int f$ and $\frac{1}{n} S_n(A) \to \mu(A)$.

Theorem 1 shows that smooth systems preserving an infinite measure have *zero Lyapunov exponent*. More precisely,

Remark 1 Observe that if T is a smooth one-dimensional map on the line and $\log T' \in L^1(\mu)$, then it follows that

$$\frac{1}{n} \log T^{n\prime}(x) = \frac{1}{n} S_n\left(\log T'(x)\right) \to 0, \qquad (x \text{ a.e.}), \tag{5}$$

which means that for almost every point, the Lyapunov exponent is zero.

It is natural to ask whether it is possible to find a sequence $\{a_n\}$ of positive normalizing constants such that, for all $A \in \mathcal{A}$, it follows that $\frac{1}{a_n} S_n(1_A) \to \mu(A)$, and for any $f \in L^1(\mu)$, it follows that $\frac{1}{a_n} S_n(f) \to \int f d\mu$. The sequence $\{a_n\}$ could be regarded as an appropriate reparametrization of time, which would yield an extension of the ergodic theorem for spaces with infinite measure. The following theorem shows that this is impossible:

Theorem 2 (See Theorem 2.4.2 in Aaronson [3]) *Let T be a recurrent ergodic measure transformation on the infinite measure space $(X; A; \mu)$, and let any sequence $\{a_n\}_{n \in \mathbb{N}}$. Then for all $f \in L^1(\mu)$ and positive either*

$$\liminf \frac{1}{a_n} S_n(f) = 0 \quad (\textit{almost every } x), \tag{6}$$

or

$$\limsup \frac{1}{a_n} S_n(f) = \infty \quad (\textit{almost every } x). \tag{7}$$

This theorem shows that any potential normalizing sequence either overestimates or underestimates the actual size of ergodic sums. Moreover, it shows that, in infinite-measure systems, the time average of an observation function fluctuates.

2.2.2 Ergodic Theorems for Non-invertible Maps

For certain non-invertible maps, such as the first four examples in Section 2.1, to find the appropriate normalizing sequences a_n (or time rescaling), one has to define the dual operator $\hat{T} : L^1(\mu) \to L^1(\mu)$, given by $\hat{T}(f) = f \circ T^{-1}$, which describes the evolution of measures under the action of T in the level of densities:

Definition 1 (See Section 3.7 in Aaronson [3]) The system is said to be pointwise dual ergodic if there exist constants $a_n = a_n(T)$, $n \in \mathbb{N}$ such that, for any $L^1(\mu)$, it follows that

$$\lim \frac{1}{a_n} \Sigma_{k=0}^{n-1} \hat{T}^k(f) = \int f\, du \tag{8}$$

The sequence $a_n = a_n(T)$ is uniquely determined up to an asymptotic equality and is called the return sequence of T. Moreover, when the map is pointwise dual ergodic, there exists sets $A \in \mathcal{A}$ with $\mu(A) < +\infty$ such that

$$\lim \frac{1}{a_n} \Sigma_{k=0}^{n-1} \hat{T}^k(1_A) = \mu(A). \tag{9}$$

Such sets are called *Darling–Kac (DK) sets*. This means that the measure of the set A is recovered by pulling back the Lebesgue measure in A, averaged by the sequences $\{a_n\}$. The next result shows that the properly normalized time rescaling of an observation function converges in distribution, provided that the return sequence has certain properties.

If the return sequence is regularly varying with index $\alpha \in [0, 1]$ [i.e., $a_n(T) = n^\alpha h(n)$ and for any $c > 0$ $\frac{h(cn)}{h(n)} \to c^\alpha$], the asymptotic behavior of $S_n(f)$ can be described almost surely in distribution as follows, provided that the map is a recurrent ergodic measure transformation:

Theorem 3 (See Theorem 5 in Zweimuller [4] and Theorem 3.6.4 in Aaronson [3]) *Let T be a recurrent ergodic measure transformation on the infinite measure space $(X; A; \mu)$. Assume there is some DK set $A \in \mathcal{A}$. If*

$a_n = a_n(T)$ is regularly varying of index $1-\alpha$ (for some $\alpha \in [0;1]$), then for all $f \in L^1(\mu)$ and all $t > 0$

$$\mu_A\left(\frac{1}{a_n}S_n(f) < t\right) \to Pr\left[M_\alpha(t)\int_X f d\mu\right] \quad as\, n \to \infty. \tag{10}$$

Here, μ_A denotes the measure μ restricted to the set A and can be replaced by any probability absolutely continuous respect to μ, and $M_\alpha, \alpha \in [0;1]$ denotes a non-negative real random variable distributed according to the (normalized) Mittag–Leffler distribution of order α, which can be characterized by its moments

$$\mathbb{E}[M_\alpha^r] = r!\frac{\Gamma(1+\alpha)^r}{\Gamma(1+r\alpha)} \quad (r \geq 0). \tag{11}$$

Even though Theorem 3 shows that normalized time averages only converge to distributions, it is natural to wonder if the double average of the weighted Birkhoff sums converge. In fact, the following theorem shows how the expected value of normalized time averages also has a limit:

Theorem 4 (See Theorem 1 in Aaronson et al. [32]) *Let T be a pointwise dual ergodic map and assume that $a_n = n^\alpha h(n)$ is regularly varying with index $\alpha \in (0,1]$. Then for any function $f \in L^1(u)$*

$$\lim \frac{1}{\log N}\Sigma\frac{1}{na_n}S_n(f) = \int f\, du \tag{12}$$

in measure.

2.2.3 Ergodic Theorems for Some Invertible Maps

In the case of invertible maps, such as the last four examples in Section 2.1, a different approach is considered:

Definition 2 (See Section 3.3 in Aaronson [3]) It is said that the system is rationally ergodic, if there exists a set of positive and finite measure A such that

$$\int S_n(A)^2 \leq C\left(\int S_n(A)\right)^2 \tag{13}$$

for some positive constant C.

For such maps, a special ergodic theorem can be established:

Theorem 5 (See Theorem 3.3.1 in Aaronson [3]) *If T is rationally ergodic, there exists a sequence $\{a_n\}$ such that, for any $m_l \to \infty$, there exists $n_k = m_{l_k}$ ensuring that, for any function $f \in L^1(\mu)$,*

$$\lim \frac{1}{N}\Sigma\frac{1}{a_{n_k}}S_{n_k}(f) = \int f\, d\mu \tag{14}$$

in measure. Moreover, the sequence $\{a_n\}$ is given by

$$a_n = a_n(A) = \frac{1}{\mu(A)^2}\Sigma_{k=1}^n \mu\left(A \cap T^{-k}(A)\right). \tag{15}$$

Ledrappier et al. [13] prove that the Horocycle flow is rationally ergodic. Lenci [15] proves the same for semi-dispersing billiards and Aaronson and Keane [22] and Cirilo et al. (private communication) for some cylinder maps.

2.3 Consequences for the Examples

We now go back to the problem of finding the subexponential Lyapunov exponents. Theorem 5 shows that, in certain cases, it is impossible to find a unique quantity for almost every point, even in the case of ergodic systems. Actually, the "Lyapunov exponent" behaves as a random variable. In particular, assuming that the transformation T is one-dimensional and smooth, if $f = \log(T')$, then the hypothesis of Theorem 3 holds and $\alpha \neq 0$. It follows that, given three points $t_1 < t_2 < t_3$, there is a set of positive measure of initial conditions such that $\exp(a_n t_1) < (T^n)'(x) < \exp(a_n t_2)$ (provided that n is large) and a set of positive measure of initial conditions such that $\exp(a_n t_2) < (T^n)'(x) < \exp(a_n t_3)$.

However, Theorem 3 gives the range of fluctuation of the Lyapunov exponent, and up to a constant dependent on set of initial conditions, the sequences a_n determine the rate of separation of trajectories. In fact, for almost every point x, there exists $t(x)$ such that, for any $\epsilon > 0$ and sufficiently large n,

$$\exp\left(a_n\left(t(x)-\epsilon\right)\right) < (T^n)'(x) < \exp\left(a_n\left(t(x)+\epsilon\right)\right). \tag{16}$$

For a few of the maps described above, we can now explicitly write the normalizing sequences $a_n = a_n(T)$ and apply Theorem 3:

1. For the Pomeau–Manneville maps, $a_n = n^{\frac{1}{p-1}}$ if $p > 2$, $a_n = n/\log n$ if $p = 2$.
2. For the Boole map, $a_n = \sqrt{n}$.
3. For rational maps on $\mathbb{C}$ with parabolic points in the Julia set, $a_n = n^{\beta-1}$ for $1 < \beta < 4$, and $a_n = n/\log n$ if $\beta = 4$, with $\beta = hp/(p+1)$. Here p is the first integer larger than unity such that the derivative at the parabolic fixed point does not vanish and h is the Hausdorff dimension of the Julia set J.

4. Horocycle flows on periodic hyperbolic surfaces, $a(t) = t/\ln(t)^k$ with $k \in \frac{1}{2}$ depending on the surface [33].
5. For the cylinder maps studied in Aaronson and Keane [22], the rate is $n/\log n$.

While the rescaling of time in such cases is polynomial, quite a different situation arises when the asymptotic growth of the derivative is considered. We then recall that, in one dimension, $\log(T^{n\prime}(x)) = \Sigma_{j=0}^{n-1} \log\left(T'\left(T^j(x)\right)\right)$, take advantage of Theorem 3 and calculate explicitly the sequences a_n to determine the rate of subexponential instability:

1. *Subexponential Lyapunov exponents for Pomeau–Manneville maps:* Given any pair of positive numbers $t_1 < t_2$, in the case that $p > 1$, it follows that for large n there is a positive set of initial conditions such that $\exp(t_1 n^{\frac{1}{p}}) < (T^n)'(x) < \exp(t_2 n^{\frac{1}{p}})$. If $p = 1$, it follows that for large n there is a positive set of initial conditions such that $\exp(t_1 \frac{n}{\ln(n)}) < (T^n)'(x) < \exp(t_2 \frac{n}{\ln(n)})$.
2. *Subexponential Lyapunov exponents for maps on $\mathbb{C}$ with parabolic points in the Julia set*: This is similar to the Pomeau–Manneville maps.
3. *Subexponential Lyapunov exponents for Boole maps:* There is a positive set of initial conditions for which $\exp(t_1\sqrt{n}) < (T^n)'(x) < \exp(t_2\sqrt{n})$.

In any case, it follows that *the subexponential growth of the derivative by iteration is larger than the growth of any polynomial and hence cannot be described by any* $\exp_q$ *for any* q.

For quadratic families, no explicit calculation has been presented. Nonetheless, a vast range of different subinstabilities can be expected. This is discussed further at the end of the present section.

We notice that Theorem 4 covers the case in which T is a one-dimensional smooth map (recall Remark 1) and yields the expected random fluctuation of the subexponential rate of separation. Explicitly,

$$\lim \frac{1}{\log N} \Sigma \frac{1}{na_n} (T^n)'(x) = \int \ln(T')(x)\, du. \tag{17}$$

It is also important to point out that in certain cases (e.g., the Boole and Pomeau–Manneville maps), the rates a_n are related to the *induced (or return) map*: Given a set A with $\mu(A) < \infty$ and assuming that almost every return point (which is the case in the examples we have considered and in the hypotheses of the theorems), then for almost every point x one can define $n(x) = \min\{n \geq 1 : T^n(x) \in A\}$ and then the map $x \to T^n(x)$. It turns out that, when it is restricted to A, the measure μ is ergodic and finite. And examples of unimodal maps (quadratic-type maps) with infinite measure can be obtained through a return-map construction, usually called *tower construction*, which imposes a type of non-integrability condition for the return times (see Al-Khal et al. [12] for details). These analyses would provide precise descriptions of the subexponential instability. And yet other approaches are known. In particular, quantitative recurrence in systems with infinite-invariant measure has been studied, precisely and in detail, in Galatolo et al. [34]. And on the basis of a series of examples, Bonanno et al. [35] presented a detailed investigation of the relationship between quantitative recurrence indicators and the algorithmic complexity of orbits in weakly chaotic dynamical systems.

In the case of the Pomeau–Manneville maps, the search for a Pesin-type formula relating the "subexponential Lyapunov exponent" with some generalized entropy was successfully concluded in Korabel and Barkai [36]. In this paper, we have related the quantity defined by (17) with the entropy introduced by Krengel [37], which is the normalized Kolmogorov–Sinai entropy for the first return map defined above.

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