PARTICLES AND FIELDS

On Special Requantization of a Black Hole

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Abstract Quantized expressions for the gravitational energy and momentum are derived from a linearized theory of teleparallel gravity. The derivation relies on a second-quantization procedure that constructs annihilation and creation operators for the graviton. The resulting gravitational field is a collection of gravitons, each of which has precise energy and momentum. On the basis of the weak-field approximation of Schwarzschild's solution, a new form for the quantization of the mass of a black hole is derived.

Keywords Teleparallelism · Quantum gravity · Second quantization

1 Introduction

One of the most challenging problems in theoretical physics is the development of a quantum theory of gravitation. General relativity successfully describes classical gravitational fields; yet, it is not a completely satisfactory theory, since it is unable to deal with certain issues, with dark energy, dark matter, and the nature of the singularity inside a black hole, for instance [1].

By the correspondence principle, a quantum theory of gravity should reproduce general relativity in the same way that quantum electrodynamics reproduces Maxwell's equations. It is believed that such a theory

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will play an important role at Planck's scale and in a grand unification scheme, given the extraordinary advance achieved by unifying three of the four fundamental forces.

An early attempt to quantize gravity is the path-integral method applied to the Einstein-Hilbert Lagrangian density. In this approach, only in specific configurations is it possible to make statements about features of the system; thus, in the case of a black hole the entropy is found to be proportional to its event-horizon area, which can be measured by a procedure avoiding space-time singularities [2]. The interaction between particles inside and outside the event horizon will cause the evaporation of the black hole; this phenomenon, known as Hawking's radiation [3], has motivated recent experimental investigations [4]. Whether such an interaction offers a microscopic explanation for the black-hole entropy is the subject of debate that has fueled intense research.

The hamiltonian formulation of general relativity is constrained, hence very difficult to deal with. Dirac developed a method to quantize such systems, which gave rise to a family of theories known as canonical quantum gravity [5, 6]. Among them, Regge calculus [7, 8] and loop quantum gravity (LQG) [9, 10] stand out, an alternative approach being addressed by string theory.

Every attempt to construct a quantum theory of gravity has to face its non-renormalizable feature [11] and deal with the problem of time [12]. Dirac's quantization method applied to teleparallel gravity could provide an interesting approach to that goal, since the energy constraint of its Hamiltonian formulation has shown excellent results when applied to different systems over the years [13, 14].

LQG is the most prominent theory of quantum gravity. It tries to quantize gravity in a nonpertubative way, independently of the chosen spatial background metric [9, 15, 16]. The concept of spin foams being essential, the volume and area operators give rise to "grains of space-time". Loop quantum cosmology stems from LQG; here, however, serious conceptual problems have to be addressed, such as the interpretation of the wave function of the Universe and the avoidance of the Big Bang singularity [17, 18].

In gravitation, the definition of the energy-momentum vector is very controversial. Most attempts lead to coordinate-dependent expressions or limited ones such as the Arnowitt–Deser–Misner (ADM) energy, valid only on the asymptotic space-time [19]. In the context of teleparallel gravity, by contrast, a well defined vector arises naturally from the field equations [13]. This immediately suggests that second-quantization methods be applied to the gravitational energy-momentum tensor in a linearized approximation, an approach adopted in this paper. The gravitational-field properties are then analogous to those of the electromagnetic field. In particular, the gravitational energy-momentum tensor being quadratic in the tetrad field, annihilation and creation operators are easily identified. This approach contrasts with attempts to quantize matter fields in a semi-classical approach rooted in Einstein's equations and does not depend on Dirac's algorithm to identify quantum features of the gravitational field.

In the following, the greek letters μ, ν, ... represent space-time indices, while the latin letters a, b, ..., denote SO(3,1) indices. In both cases, the indices run from 0 to 3. Time and space indices are indicated according to the convention $\mu = 0, i, a = (0), (i)$. The determinant of the tetrad field is denoted $e = \det(e^a{}_\mu)$, and units are such that $G = c = h = 1$.

2 Teleparallel Gravity

This section presents a brief review of teleparallel gravity. In this theory the tetrad field, not the metric tensor, is the dynamical field variable. The tetrad field is related to the metric tensor by the expressions

$$g^{\mu\nu} = e^{a\mu}e_a{}^\nu;$$
$$\eta^{ab} = e^{a\mu}e^b{}_\mu. \tag{1}$$

where $\eta^{ab} = diag(-+++)$ is the metric tensor of Minkowski space-time.

Consider now a manifold endowed with the Cartan connection [20] $\Gamma_{\mu\lambda\nu} = e^a{}_\mu \partial_\lambda e_{a\nu}$, which can equally well be written in the form

$$\Gamma_{\mu\lambda\nu} = {}^0\Gamma_{\mu\lambda\nu} + K_{\mu\lambda\nu}, \tag{2}$$

where ${}^0\Gamma_{\mu\lambda\nu}$ are the Christoffel symbols, and $K_{\mu\lambda\nu}$, which is given by

$$K_{\mu\lambda\nu} = \frac{1}{2}\left(T_{\lambda\mu\nu} + T_{\nu\lambda\mu} + T_{\mu\lambda\nu}\right), \tag{3}$$

is the contortion tensor defined in terms of the torsion tensor constructed from the Cartan connection. Thus the torsion tensor is $T_{\mu\lambda\nu} = e_{a\mu}T^a{}_{\lambda\nu}$ with

$$T^a{}_{\lambda\nu} = \partial_\lambda e^a{}_\nu - \partial_\nu e^a{}_\lambda. \tag{4}$$

The curvature tensor obtained from $\Gamma_{\mu\lambda\nu}$ is identically zero. Equation (2) thus leads to

$$eR(e) \equiv -e\left(\frac{1}{4}T^{abc}T_{abc} + \frac{1}{2}T^{abc}T_{bac} - T^a T_a\right) + 2\partial_\mu\left(eT^\mu\right). \tag{5}$$

We then drop the divergence on the right-hand side to define the teleparallel Lagrangian density

$$\mathfrak{L}\left(e_{a\mu}\right) = -\kappa\, e\left(\frac{1}{4}T^{abc}T_{abc} + \frac{1}{2}T^{abc}T_{bac} - T^a T_a\right) - \mathfrak{L}_M$$
$$\equiv -\kappa\, e\Sigma^{abc}T_{abc} - \mathfrak{L}_M, \tag{6}$$

where $\kappa = 1/(16\pi)$, $\mathfrak{L}_M$ stands for Lagrangian density of matter fields, and Σ^{abc} is given by

$$\Sigma^{abc} = \frac{1}{4}\left(T^{abc} + T^{bac} - T^{cab}\right) + \frac{1}{2}\left(\eta^{ac}T^b - \eta^{ab}T^c\right), \tag{7}$$

with $T^a = T^b{}_b{}^a$. It is important to notice that the Einstein-Hilbert Lagrangian density is equivalent to its teleparallel version (6). Thus, concerning observational data, both theories share the same results.

We then compute the variational derivative of the Lagrangian density with respect to $e^{a\mu}$ to obtain the following field equations:

$$e_{a\lambda}e_{b\mu}\partial_\nu\left(e\Sigma^{b\lambda\nu}\right) - e\left(\Sigma^{b\nu}{}_a T_{b\nu\mu} - \frac{1}{4}e_{a\mu}T_{bcd}\Sigma^{bcd}\right)$$
$$= \frac{1}{4\kappa}eT_{a\mu}, \tag{8}$$

where $T_{a\mu}$ is the energy-momentum tensor of matter fields. Explicit calculation shows that (8) is equivalent to Einstein's equations.

The field equations can be rewritten in the form

$$\partial_\nu \left(e \Sigma^{a\lambda\nu} \right) = \frac{1}{4\kappa} e \, e^a{}_\mu \left(t^{\lambda\mu} + T^{\lambda\mu} \right), \tag{9}$$

where $t^{\lambda\mu}$ is defined by the equality

$$t^{\lambda\mu} = \kappa \left(4\Sigma^{bc\lambda} T_{bc}{}^\mu - g^{\lambda\mu} \Sigma^{bcd} T_{bcd} \right). \tag{10}$$

Since $\Sigma^{a\lambda\nu}$ is skew-symmetric in the last two indices, it follows that

$$\partial_\lambda \partial_\nu \left(e \Sigma^{a\lambda\nu} \right) \equiv 0. \tag{11}$$

We thus see that

$$\partial_\lambda \left(e t^{a\lambda} + e T^{a\lambda} \right) = 0, \tag{12}$$

which yields the continuity equation

$$\frac{d}{dt} \int_V \mathrm{d}^3 x \, e \, e^a{}_\mu \left(t^{0\mu} + T^{0\mu} \right) = -\oint_S dS_j \left[e \, e^a{}_\mu \left(t^{j\mu} + T^{j\mu} \right) \right].$$

Accordingly, $t^{\lambda\mu}$ is interpreted as the energy-momentum tensor of the gravitational field [21], and the total energy-momentum in a three-dimensional volume V of space is

$$P^a = \int_V \mathrm{d}^3 x \, e \, e^a{}_\mu \left(t^{0\mu} + T^{0\mu} \right). \tag{13}$$

The right-hand side remains invariant under coordinate transformations, transforms like a vector under Lorentz transformations and hence displays the expected features of a true energy-momentum vector.

3 Quantized Energy and Requantization of Matter

We now turn our attention to a linearized theory of teleparallelism. In this approximation the tetrad field can be written in the form

$$e^a{}_\mu \approx \delta^a_\mu + \psi^a{}_\mu, \tag{14}$$

and the metric tensor has a similar expansion:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}.$$

At this point, the energy-momentum of matter fields is zero, and the energy-momentum vector P^a depends on $t^{a\mu}$ only. Simple algebraic manipulations convert $t^{a\mu}$ to the form

$$t^{0a} \approx -\kappa \frac{1}{4} \delta^a_\mu \left(\partial^\nu \psi^{c0} - \partial^0 \psi^{c\nu} \right) \partial^\mu \psi_{c\nu} - \kappa \frac{1}{2} \delta^a_\mu \eta^{0\mu}$$
$$\times \left(\left(\partial_\nu \psi^c{}_\gamma \right) \left(\partial^\gamma \psi_c{}^\nu \right) - \left(\partial^\lambda \psi^c{}_\lambda \right) \left(\partial_\mu \psi_c{}^\mu \right) \right). \tag{15}$$

Consider now the Fourier expansion of $\psi_{a\mu}$ in a three-dimensional normalized volume,

$$\psi_{a\mu} = \frac{1}{\sqrt{2\omega}} \sum_{\mathbf{k}} \left(A_{a\mu} \left(\mathbf{k}, t \right) e^{i\mathbf{k}\cdot\mathbf{x}} + A^\dagger_{a\mu} \left(\mathbf{k}, t \right) e^{-i\mathbf{k}\cdot\mathbf{x}} \right), \tag{16}$$

where the time dependence of the coefficients $A_{a\mu}(\mathbf{k}, t)$ is of the form $e^{-i\omega t}$. For the purposes of second quantization, it is sufficient to consider terms containing $A^{a\mu}(\mathbf{k}, t) A^\dagger_{a\mu}(\mathbf{k}, t)$ and analogous combinations, no generality being lost, since quadratic terms with $A^{a\mu}(\mathbf{k}, t)$ and $A^\dagger_{a\mu}(\mathbf{k}, t)$ do not contribute to the hamiltonian operator in the context of annihilation and creation operators. Notice, moreover, that the time average of these quadratic terms over a period vanishes. Thus, in the transversal gauge $k^\mu \psi_{a\mu} = 0$, where $k^\mu = (\omega, \mathbf{k})$ with $k^\mu k_\mu = 0$, the time average of the energy in (13) reads

$$P^{(0)} \approx \kappa \sum_{\mathbf{k}} \frac{1}{2}\omega \left(A^{a\mu} \left(\mathbf{k} \right) A^\dagger_{a\mu} \left(\mathbf{k} \right) + A^{\dagger a\mu} \left(\mathbf{k} \right) A_{a\mu} \left(\mathbf{k} \right) \right). \tag{17}$$

The coefficients in (16) assumed to be operators, the constraint $[A^{\dagger a\mu}(\mathbf{k}), A_{b\nu}(\mathbf{k}')] = \delta_{\mathbf{k}\mathbf{k}'} \delta^a_b \delta^\mu_\nu$ leads to the hamiltonian operator

$$\hat{H} \approx \kappa \sum_{\mathbf{k}} \frac{1}{4}\omega \left(\hat{A}^{\dagger a\mu} \left(\mathbf{k} \right) \hat{A}_{a\mu} \left(\mathbf{k} \right) + \frac{1}{2} \right), \tag{18}$$

where the $\hat{A}^{\dagger a\mu}(\mathbf{k})$ are creation and the $\hat{A}_{a\mu}(\mathbf{k})$ are annihilation operators. A similar analysis starting from (13) leads to the following expression for the gravitational momentum in terms of creation and annihilation operators:

$$\hat{\mathbf{P}} \approx \kappa \sum_{\mathbf{k}} \frac{1}{4}\mathbf{k} \left(\hat{A}^{\dagger a\mu} \left(\mathbf{k} \right) \hat{A}_{a\mu} \left(\mathbf{k} \right) + \frac{1}{2} \right). \tag{19}$$

Equations (18) and (19) are remarkably similar to the QED expressions for the electromagnetic energy and momentum, respectively. If a set of orthonormal wave functions obeying $\hat{H}\Psi = \mathcal{E}\Psi$ is postulated, then the eigenvalue of $\hat{H}$ is

$$\mathcal{E} \approx \kappa \sum_{\mathbf{k}} \frac{1}{4}\omega N_{\mathbf{k}}, \tag{20}$$

where $N_{\mathbf{k}}$ is the number of gravitons, and the constant second term within parentheses on the right-hand side of (18) was dropped to avoid a trivial divergence. In this context, the gravitational field can be regarded as a collection of gravitons playing the role of photons in an electromagnetic field.

In order to make a prediction, let us consider the space-time around a black hole, which is described by the line element

$$ds^2 = -(1 - 2M/r)\, dt^2 + (1 - 2M/r)^{-1}\, dr^2$$
$$+ r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2, \tag{21}$$

found in Schwarzschild's solution [22]. Under the weak field condition $M/r << 1$, (21) fits our linearized theory.

For stationary observers, (13) is reduced to the form

$$\mathcal{E} = \frac{1}{8\pi} \int_S d\theta\, d\phi\, (eT^1),$$

where

$$eT^1 = 2r\sin\theta \left(1 - \sqrt{\left(1 - \frac{2M}{r}\right)}\right).$$

If we then let $S \to \infty$, to cover the whole space-time, we find that the energy of the Schwarzschild space-time is $\mathcal{E} = M$, a result that is well known in the context of asymptotic expressions such as the ADM energy (see for instance [23]) and holds equally well in the weak-field approximation. We can therefore extract the mass of the black hole from (20), which yields the expression

$$M \approx \kappa \sum_{\mathbf{k}} \frac{1}{4} \omega N_{\mathbf{k}}. \tag{22}$$

This identification is valid only for inertial frames, since it depends on our derivation of the energy-momentum vector from the expansion (14), and the chosen gauge. Notice should be taken that (22) was derived in an empty space-time and relates the number of gravitons to the mass of the black hole. That result tells us that mass is quantized in a new sense, other than the concept of particle or atom. For this reason, we refer to it as a requantization of a black hole.

Since the mass of the black hole on the left-hand side of (22) is finite, the right-hand side is expected to be finite as well. The sum should therefore be truncated at some point, and the mass of the black hole, given by

$$M \approx m_0 + m_1 + \ldots + m_N$$

where each m is a quantum of mass dependent on a specific frequency.

That the gravitons are bosons is shown by the commutation relations between the annihilation and creation operators and by the eigenvalue of the hamiltonian operator. Although the preceding analysis has led to no conclusion concerning spin, preliminary studies extending the teleparallel-gravity procedure to the gravitational angular momentum suggest that the graviton has spin 2.

4 Conclusion

The results in this paper can be summarized as follows. Teleparallel gravity yields an energy-momentum vector that is independent of coordinate transformations. In a linearized theory, $\psi_{a\mu}$ can be expanded in a Fourier series, and this leads to second-quantized expressions for the energy and momentum. For inertial frames, the energy of Schwarzschild's space-time is given by M, a result that is valid even in the weak-field approximation and identifies a new form of quantization, which is associated with the number of gravitons.

It should be noticed that the requantization encompasses only the gravitational energy-momentum; no matter fields are involved. The resulting new quantization of the font of gravitation contrasts with what loop quantum gravity asserts with its discrete space-time.

The definition of the ADM energy shows that the mass of a black hole has to be associated with a particle of integer spin—the graviton. Matter, however, is constituted of fermions. The resulting apparent contradiction constitutes a serious obstacle obstructing the development of quantum theories of gravity.

The next challenge, thus, is to develop a general theory for the requantization of mass, one that identifies general creation and annihilation operators. A general equation for quantum gravity in the teleparallel case could be analogous to the Wheeler-DeWitt equation [24–26], and hence bring to light the quantum nature of matter. The pursuit of a quantum theory of gravity should not treat matter and space-time on different footings. The inextricable connection between the two entities should emerge from the (re)quantization of matter itself.

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