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PARTICLES AND FIELDS

# Two- and Four-Level Systems in Magnetic Fields Restricted in Time

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**Abstract** We describe some new exact solutions for two- and four-level systems. In all the cases, external fields have a restricted behavior in time. First, we consider a method to construct new solutions for one-spin equation and give some explicit examples: One of them is in a external magnetic field that acts during a finite time interval. Then we show how these solutions can be used to solve the two-spin equation problem. A solution for two interacting spins is analyzed in the case when the field difference between the external fields in each spin varies adiabatically, vanishing on the time infinity. The latter system can be identified with a quantum gate realized by two coupled quantum dots. The probability of the Swap operation for such a gate can be explicitly expressed in terms of special functions. Using the obtained expressions, we construct plots for the Swap operation for some parameters of the external magnetic field and interaction function.

**Keywords** Four-level system · Spin equation · Exact solution · Quantum dot

## 1 Introduction

Finite-level systems have always played an important role in quantum physics. In particular, two-level systems possess a wide range of applications, for example, in the semi-classical theory of laser beams [1], optical resonance [2], nuclear induction experiments [3], and so on. The best known physical system that could be identified with a two-level system is a fixed spin-one-half object interacting with a magnetic field. The two-level system has been studied by many authors using different methods, see, for example, [4, 5]. Likewise the four-level systems can be used to describe two interacting one-half spins, e.g., the valence electrons in two coupled semiconductor quantum dots [6]. The most detailed theoretical study of the quantum mechanical equations for two- and four-level systems and their exact solutions are presented in [7, 8]. Recently, two- and four-level systems have attract even more attention, due to their relationship to the problem of quantum computation [9] . In this problem, the computation is performed by the manipulation of the so-called one- and two-qubit gates [10]. The one-qubit gate can be identified with a two-level system, and two-qubit gates can be identified with a four-level system. For these reasons, two- and four-level systems are crucial elements of possible quantum computers, which are supposed to efficiently solve problems that are considered

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intractable by classical computers [11, 12]. For physical applications, it is very important to have explicit exact solutions of two- and four-level system equations. In [13, 14], exact solutions of a four-level system are used to describe the theoretical construction of a universal quantum XOR gate using two-coupled quantum dots. This work shows how the exact solutions can be used to establish all the necessary conditions on the external fields needed for the implementation of the gate.

In the present work, we describe some new exact solutions for two- and four-level systems that were not represented in our previous works [7, 8]. These solutions are found for external fields that have a restricted behavior in time, for example, the first solution for two-level system in external field that acts along a finite time interval. In Section 2, we describe a general method to construct exact solutions for two-level systems with external fields restricted in time, and also in this section, we use this method to obtain two explicit external fields and the respective exact solutions. In Section 3, we show how these results can be applied to construct new exact solutions for the four-level system. This system is identified with two interacting spins, and we made a detailed study of the important case when the difference between the external fields in each spin varies adiabatically (vanishes with time). This system can be identified with a quantum gate realized by two coupled quantum dots. The probability of the Swap operation for such a gate can be explicitly expressed in terms of special functions. Using the obtained expressions, we construct plots for the Swap operation for some parameters of the external magnetic field and interaction functions.

## 2 Two-Level Systems

### 2.1 General

We recall to the reader that two-level systems are described by the so-called one-spin equation

$$i\frac{\mathrm{d}v}{\mathrm{d}t} = (\boldsymbol{\sigma}\mathbf{F})\,v,\ \boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)\,,\ \mathbf{F} = (F_1, F_2, F_3)\,, \tag{1}$$

where $v = v\,(t)$ is a two-component spinor, $\sigma_k$ $(k = 1, 2, 3)$ are Pauli matrices, and $F_k = F_k\,(t)$ are components of external field strength, see [7]. The general solution of the spin equation reads

$$v\,(t) = R\,(t)\,v^0, \tag{2}$$

where the $2\times 2$ matrix $R\,(t)$ obeys the same spin equation

$$i\frac{\mathrm{d}R\,(t)}{\mathrm{d}t} = (\boldsymbol{\sigma}\mathbf{F})\,R\,(t)\,, \tag{3}$$

and $v^0$ is an arbitrary constant spinor. If $R\,(t = t_0) = I$, where $I$ is $2\times 2$ unity matrix, then $R\,(t) = \hat{u}\,(t)$, where $\hat{u}\,(t)$ is the evolution operator of the spin equation. In the general case, the evolution operator is constructed by the help of the matrix $R\,(t)$ as follows:

$$\hat{u}\,(t) = R\,(t)\,R_0^{-1},\ R_0 = R\,(t = t_0)\,,\ \hat{u}\,(t = t_0) = I\,. \tag{4}$$

The matrix $R(t)$ can always be represented in the form

$$R\,(t) = Ip_0 - i\,(\boldsymbol{\sigma}\mathbf{p})\,,\ \mathbf{p} = (p_1, p_2, p_3)\,,\ p_s = p_s\,(t)\,,\quad s = 0, 1, 2, 3\,. \tag{5}$$

The functions $p_s(t)$ obey the set of equations

$$\dot{p}_0 + (\mathbf{pF}) = 0,\ \dot{\mathbf{p}} + [\mathbf{p}\times\mathbf{F}] - p_0\mathbf{F} = 0\,, \tag{6}$$

which follows from (3). Equation (6) imply that $\Delta = \det R\,(t) = p_0^2 + \mathbf{p}^2$ is an integral of motion.

Let us suppose that spin equation (3) is self-adjoint, which means that the external field $\mathbf{F}$ is real. In such a case we can chose the functions $p_s\,(t)$ to be real. Without loss of generality, in such a case, we can set $\Delta = 1$, which means that $R\,(t)$ is nonsingular. Under the condition $\Delta = 1$, the functions $p_s\,(t)$ can be expressed via three real parameters $\alpha = \alpha\,(t)$, $\theta = \theta\,(t)$, and $\varphi = \varphi\,(t)$ as follows:

$$p_0 = \cos\frac{\varphi-\alpha}{2}\cos\frac{\theta}{2}\,,\ p_1 = -\sin\frac{\varphi+\alpha}{2}\sin\frac{\theta}{2}\,,$$
$$p_2 = \cos\frac{\varphi+\alpha}{2}\sin\frac{\theta}{2}\,,\ p_3 = \sin\frac{\varphi-\alpha}{2}\cos\frac{\theta}{2}\,. \tag{7}$$

With these functions, the evolution operator (5) assumes the form[1]

$$R = R_3\,(-\varphi)\,R_2\,(\theta)\,R_3\,(\alpha)\,,$$
$$R_i\,[\beta\,(t)] = \exp\left[i\sigma_i\frac{\beta\,(t)}{2}\right]\,. \tag{8}$$

Substituting this expression in (3), we see that the above $R\,(t)$ is the evolution operator for the spin equation in the following external field:

$$F_1 = \frac{\dot{\theta}}{2}\sin\varphi + \frac{\dot{\alpha}}{2}\sin\theta\cos\varphi\,,$$
$$F_2 = \frac{\dot{\alpha}}{2}\sin\theta\sin\varphi - \frac{\dot{\theta}}{2}\cos\varphi\,,$$
$$F_3 = \frac{\dot{\varphi}}{2} - \frac{\dot{\alpha}}{2}\cos\theta\,. \tag{9}$$

[1] The operators $R_i\,(\beta)$ in (8) are rotation of an angle $\beta$ along the $i$-axis and the functions $\varphi, \theta, \alpha$ can be identify with the Euler angles. In a similar manner, the functions $p_i$ in (7) can be identify with the Euler parameters.

For any continuous time-dependent functions $\varphi, \theta, \alpha$ with continuous derivative.

Of course, the above expressions cannot be used to solve the general problem (1) for a an arbitrary general field **F**, due to the difficulty to solve the integral equations involved in to express $(\varphi, \theta, \alpha)$ as a function of $(F_1, F_2, F_3)$. However, these expressions can be used to find exact solutions for external fields with some particular characteristic. For example, we can construct exactly solutions for periodic external fields by setting

$$\theta = \omega t\ ,\ \ \alpha = \Omega t\ ,\ \ \cos\varphi = C\ ,$$

where $\omega$, $\Omega$, and $C$ are constants. Other important kind of external fields are those whose action is restricted in time, once these are the most common fields in experiments. We can find a variety of solutions for restricted in time external fields, just by constructing functions that assume a constant value outside of some interval. In the next section, we give some explicit realization of this kind of fields.

### 2.2 Exact Solutions for Some Restricted in Time External Fields

We can use the results of the above section to construct exact solution of the spin equation for external fields restricted in time. It can be done by constructing continuous time-dependent functions $\varphi, \theta, \alpha$, with continuous derivatives, which assumes a fixed value outside of a certain interval ($\dot{\varphi} = \dot{\theta} = \dot{\alpha} = 0$ for $|t| > T$).

1. Let us chose the functions:

$$\begin{aligned}
&\theta\,(t) = \theta_0\ ,\ \ \varphi\,(t) = \varphi_0,\ \ \alpha\,(t) = \alpha_0\ ,\ \ t \le -T\ ,\\
&\theta\,(t) = \frac{\theta_1 - \theta_0}{2}\sin\frac{\pi t}{2T} + \frac{\theta_1 + \theta_0}{2}\ ,\ \ |t| < T\ ,\\
&\varphi\,(t) = \frac{\varphi_1 - \varphi_0}{2}\sin\frac{\pi t}{2T} + \frac{\varphi_1 + \varphi_0}{2}\ ,\ \ |t| < T\ ,\\
&\alpha\,(t) = \frac{\alpha_1 - \alpha_0}{2}\sin\frac{\pi t}{2T} + \frac{\alpha_1 + \alpha_0}{2}\ ,\ \ |t| < T\ ,\\
&\theta\,(t) = \theta_1\ ,\ \ \varphi\,(t) = \varphi_1\ ,\alpha\,(t) = \alpha_1\ ,\ \ t \ge T\ ,
\end{aligned} \tag{10}$$

where $\alpha_0, \alpha_1, \theta_0, \theta_1, \varphi_0,$ and $\varphi_1$ are arbitrary constants. With this, the external field **F** (9) will be zero at $|t| \ge T$, where $T$ is a constant, and for $|t| < T$ we have

$$\begin{aligned}
F_1\,(t) &= \frac{\pi}{8T}\,(\theta_0 - \theta_1)\cos\frac{\pi t}{2T}\sin\varphi\\
&\quad + \frac{\pi}{8T}\,(\alpha_0 - \alpha_1)\cos\frac{\pi t}{2T}\sin\theta\cos\varphi\ ,\\
F_2\,(t) &= \frac{\pi}{8T}\,(\alpha_0 - \alpha_1)\cos\frac{\pi t}{2T}\sin\theta\sin\varphi\\
&\quad - \frac{\pi}{8T}\,(\theta_0 - \theta_1)\cos\frac{\pi t}{2T}\cos\varphi\ ,\\
F_3\,(t) &= \frac{\pi}{8T}\,(\varphi_1 - \varphi_0)\cos\frac{\pi t}{2T}\\
&\quad - \frac{\pi}{8T}\,(\alpha_0 - \alpha_1)\cos\frac{\pi t}{2T}\cos\theta\ .
\end{aligned}$$

The external field under consideration is not zero only on a finite interval $|t| < T$ and is continuous for all $t$. The exact solution of the equation (3) for such a field is constructed by substituting the function (10) in (7).

2. Let the functions $\theta = \theta(t)$, $\varphi = \varphi(t)$, and $\alpha\,(t)$ have the form

$$\begin{aligned}
\theta(t) &= \frac{\theta_0 t}{T_1}\exp\left[-\left(\frac{t}{T_1}\right)^2\right] + \theta_1\ ,\\
\varphi(t) &= \frac{\varphi_0 t}{T_2}\exp\left[-\left(\frac{t}{T_2}\right)^2\right] + \varphi_1\ ,\\
\alpha\,(t) &= \frac{\alpha_0 t}{T_3}\exp\left[-\left(\frac{t}{T_3}\right)^2\right] + \alpha_1\ ,
\end{aligned} \tag{11}$$

where $\alpha_1, \theta_0,\ \theta_1,\ \varphi_0,\ \varphi_1,\ \alpha_0,$ and $T_k,\ \ (k = 1, 2, 3)$ are arbitrary constants. With this, the external field **F** (9) becomes

$$\begin{aligned}
F_1\,(t) &= -\frac{\theta_0}{T_1}\left[1 - 2\left(\frac{t}{T_1}\right)^2\right]\exp\left[-\left(\frac{t}{T_1}\right)^2\right]\sin\varphi\\
&\quad - \frac{\alpha_0}{T_3}\left[1 - 2\left(\frac{t}{T_3}\right)^2\right]\exp\left[-\left(\frac{t}{T_3}\right)^2\right]\sin\theta\cos\varphi,\\
F_2\,(t) &= \frac{\theta_0}{T_1}\left[1 - 2\left(\frac{t}{T_1}\right)^2\right]\exp\left[-\left(\frac{t}{T_1}\right)^2\right]\cos\varphi\\
&\quad - \frac{\alpha_0}{T_3}\left[1 - 2\left(\frac{t}{T_3}\right)^2\right]\exp\left[-\left(\frac{t}{T_3}\right)^2\right]\sin\theta\sin\varphi,\\
F_3\,(t) &= \frac{\varphi_0}{T_2}\left[1 - 2\left(\frac{t}{T_2}\right)^2\right]\exp\left[-\left(\frac{t}{T_2}\right)^2\right]\\
&\quad + \frac{\alpha_0}{T_3}\left[1 - 2\left(\frac{t}{T_3}\right)^2\right]\exp\left[-\left(\frac{t}{T_3}\right)^2\right]\cos\theta\ .
\end{aligned} \tag{12}$$

This external field vanishes at $t \to \pm\infty$. The exact solution of the equation (3) for such a field is constructed by substituting the function (11) in (7).

3. By combining the $\varphi, \theta, \alpha$ functions of the two above examples, it is possible to construct 6 fields more in the form (9) that are restricted in time, with the exact solutions given by (5).

## 3 Four-Level Systems

### 3.1 General

We write the Schrödinger equation for a four-level system in the following form ($\hbar = 1$), see [8]:

$$i\frac{d\Psi}{dt} = \hat{H}(\mathbf{G}, \mathbf{F}, J)\,\Psi\,,$$
$$\hat{H} = (\boldsymbol{\rho}\cdot\mathbf{G}) + (\boldsymbol{\Sigma}\cdot\mathbf{F}) + \frac{J}{2}(\boldsymbol{\Sigma}\cdot\boldsymbol{\rho})\,. \tag{13}$$

Here $\Psi$ is a four-component column; in the general case, the interaction function $J$ as well as the external fields (the vectors $\mathbf{G}$ and $\mathbf{F}$) are time dependent; and the $4 \times 4$ matrices $\boldsymbol{\rho}$ and $\boldsymbol{\Sigma}$ have the forms

$$\boldsymbol{\Sigma} = I \otimes \boldsymbol{\sigma}\,,\ \ \boldsymbol{\rho} = \boldsymbol{\sigma} \otimes I\,, (\boldsymbol{\Sigma}\cdot\boldsymbol{\rho}) = \boldsymbol{\sigma}\otimes\boldsymbol{\sigma} = \sum_{i=1}^{3}\sigma_i \otimes \sigma_i\,,$$

where $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ are the Pauli matrices and $I$ is the $2 \times 2$ identity matrices. The Hamiltonian matrix reads

$$\hat{H} = \begin{pmatrix} F_3+G_3+\frac{J}{2} & F_1-iF_2 & G_1-iG_2 & 0 \\ F_1+iF_2 & G_3-F_3-\frac{J}{2} & J & G_1-iG_2 \\ G_1+iG_2 & J & F_3-G_3-\frac{J}{2} & F_1-iF_2 \\ 0 & G_1+iG_2 & F_1+iF_2 & \frac{J}{2}-G_3-F_3 \end{pmatrix}. \tag{14}$$

Such a model is used to describe two spins subjected to the external magnetic fields $\mathbf{F}$ and $\mathbf{G}$ and interacting with each other through a spherically symmetric Heisenberg interaction whose intensity is given by the interaction function $J$. In particular, this model was used to describe two coupled quantum dots [14]. In our work [8], a series of exact solution of equation (13) for different choices of the interaction function and the external fields are found for the first time.

### 3.2 Reduction to the Two-Level System Case

For a special case of two spins subjected to *parallel* external magnetic fields, which we write as

$$\mathbf{G} = (0, 0, \mu_B g_1 B_1)\,,\ \mathbf{F} = (0, 0, \mu_B g_2 B_2)\,,\ B_{1,2} = B_{1,2}(t)\,, \tag{15}$$

where $\mu_B$ is the Bohr magneton and $g_1$ and $g_2$ are effective $g$-factors for the corresponding spins (see, for example, [15]), one can show that the evolution operator $\hat{U}(t)$ for the equation (13) can be reduced to an evolution operator $\hat{u}(t)$ (4) for the Schrödinger equation of a two-level system [8]. Such a reduction is given by the equation

$$\hat{U}(t) = \exp\left(-\frac{i}{2}\left[(\Sigma_3 + \rho_3)\,\Gamma(t) + \Sigma_3\rho_3\Phi(t)\right]\right) M(t)\,,$$
$$\Gamma(t) = \int_0^t B_+(\tau)\,d\tau\,,\ \ B_+ = \mu_B(g_1 B_1 + g_2 B_2)\,,$$
$$\Phi(t) = \int_0^t J(t)\,d\tau\,,\ \ M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & u_{11} & u_{12} & 0 \\ 0 & u_{21} & u_{22} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{16}$$

where $\hat{u}(t) = ||u_{ij}||$ obeys the Schrödinger equation for the following two-level system (see [7])

$$i\frac{d\hat{u}}{dt} = (\boldsymbol{\sigma}\cdot\mathbf{K})\,\hat{u}\,,\ \ \hat{u}(0) = I\,,$$
$$\mathbf{K}(t) = (J(t), 0, B_-(t))\,,\ \ B_- = \mu_B(g_1 B_1 - g_2 B_2)\,. \tag{17}$$

Thus, in the case under consideration, the four-level system problem is reduced to solve the two-level system problem (17) with an effective magnetic field $\mathbf{K}$.

We can now use the expression (5) to construct exact solution for the four-level system (13). For an external field $\mathbf{K}$ in the form of (17), we have from (9),

$$\frac{\dot{\alpha}}{2}\sin\theta\sin\varphi - \frac{\dot{\theta}}{2}\cos\varphi = 0\,,$$

and a solution in the form (5) can be construct for an external field in the form

$$K_1 = J = \frac{\dot{\theta}}{2\sin\varphi}\,,\ \ K_3 = B_- = -\frac{\dot{\eta}}{2\tan\varphi}\,,$$
$$\eta = \ln(\cos\varphi\sin\theta)\,, \tag{18}$$

with $\theta$ and $\alpha$ any continuous time-dependent functions with continuous derivative. So we can construct restricted in time interactions $J$ or fields difference $B_-$ by choosing functions $\theta$ and $\eta$ that assume a constant value outside of a desired interval. Also in this case, explicit examples can be constructed with the functions given in the preceding section.

### 3.3 An Adiabatic Variation of the Field Difference in Each Spin

Although the expression (18) allows to construct a variety of external fields with some particular characteristic, the solution for a general field can hardly be constructed in this manner. In this section, we will analyze a special case for a specific external parallel field (15) restricted in time. Consider a four-level system in which the field difference $B_-$ (17) varies adiabatically with time, i.e., a variation that met the adiabaticity criterion [16], while the interaction function is constant. Namely, we chose

$$J = a\,,\ \ B_-(t) = c/\cosh\omega t\,, \tag{19}$$

where $a$, $c$, and $\omega$ are real constants. In practical application, the pulse applied to the system (e.g., two coupled quantum dots) needs to be shorter than the decoherence time of the system. But such fast pulse can cause a transition of the system to higher energy levels, and consequently, its dynamic can no longer be described by the Hamiltonian (14). The $c/\cosh\omega t$ dependence is the most adequate kind of a variation to avoid this higher energy level transition [16]. In addition, it is reasonable to assume that if the only quantity that varies is $B_-$ and $B_+ \gg B_-$, the interaction function will remain constant [14] . With regard to the variation of $B_-$, there are some proposals for the application of localized magnetic fields [17] and some techniques that permit the manipulation of the $g$-factor by changing the size of the dots or by the application of external electromagnetic fields [15, 18].

From the previous section, we know that the evolution operator (16) of a four-level system with the parameters (19) is expressed via an evolution operator of a two-level system with effective field

$$\mathbf{K}(t) = (a, 0, c/\cosh\omega t)\ . \tag{20}$$

The exact solution for the evolution operator with such a field can be constructed using our previous results [7]. It has the form

$$\hat{u}(t) = \frac{1}{\left|G_1^0\right|^2 + \left|G_2^0\right|^2}\begin{pmatrix} G_1(z) & -\bar{G}_2(z) \\ G_2(z) & \bar{G}_1(z)\end{pmatrix} \times \begin{pmatrix} \bar{G}_1^0 & \bar{G}_2^0 \\ -G_2^0 & G_1^0\end{pmatrix}, \tag{21}$$

where

$$G_1(z) = i(2c+\omega)\, z^{\mu}(1-z)^{\nu}\, F(\alpha,\nu;\gamma;z)\ ,$$
$$G_2(z) = 2az^{\mu+1/2}(1-z)^{\nu}\, F(\alpha,\nu+1;\gamma+1;z)\ ,$$
$$G_{1,2}^0 = G_{1,2}(-1)\ ,\ z = \left(\frac{e^{\varphi}+i}{e^{\varphi}-i}\right)^2\ ,\ \varphi = \omega t\,,$$
$$\alpha = \gamma+\nu\,,\ \mu = \frac{c}{2\omega},\ \nu = i\frac{|a|}{\omega}\ ,\ \gamma = \frac{1}{2}+2\mu\,, \tag{22}$$

$F(\alpha, \beta, \gamma, z)$ is the Gauss hypergeometric function, and complex conjugate quantities are designated by a bar above.

Substituting (22) into (17), we obtain

$$\hat{R}(t) = \exp\left(\frac{iat}{2}\right) \times \begin{pmatrix} \exp\left[-i(at+\Gamma(t))\right] & 0\ 0 & 0 \\ 0 & \hat{u}(t) & 0 \\ 0 & & 0 \\ 0 & 0\ 0 & \exp\left[-i(at-\Gamma(t))\right]\end{pmatrix},$$

with $\hat{u}(t)$ given in (21).

Thus, any transition amplitude for the four-level system can be calculated with the help of the evolution operator. Let us, for example, calculate the transition amplitude between the states $|\uparrow\downarrow\rangle$ and $|\downarrow\uparrow\rangle$, which have the form

$$|\uparrow\downarrow\rangle = |\uparrow\rangle\otimes|\downarrow\rangle = \begin{pmatrix}0\\1\\0\\0\end{pmatrix},$$
$$|\downarrow\uparrow\rangle = |\downarrow\rangle\otimes|\uparrow\rangle = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}. \tag{23}$$

The transition between these states represents, in quantum computation, the so-called Swap operation and can be experimentally measured [19] . From the general expression (16), we see that

$$\langle\uparrow\downarrow|\,\hat{R}\,|\downarrow\uparrow\rangle = \langle\uparrow|\,\hat{u}\,|\downarrow\rangle\ ,\ \langle\downarrow\uparrow|\,\hat{R}\,|\uparrow\downarrow\rangle = \langle\downarrow|\,\hat{u}\,|\uparrow\rangle\ ,$$
$$|\uparrow\rangle = \begin{pmatrix}1\\0\end{pmatrix},\ |\downarrow\rangle = \begin{pmatrix}0\\1\end{pmatrix}. \tag{24}$$

Therefore, in the case of the Swap operation between the states (23), we need only to calculate matrix elements of the two-level system evolution operator. One has also to stress that in this case the Swap operation does not depend on the fields' sum $B_+$.

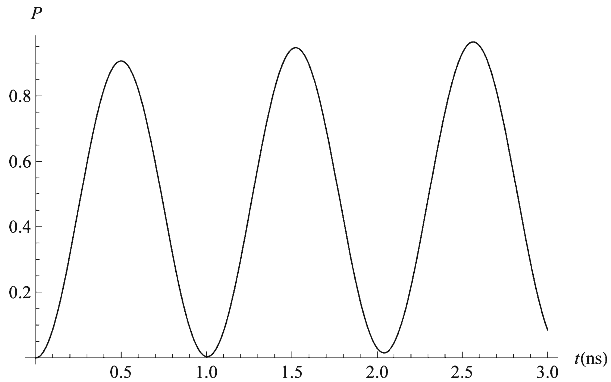


**Fig. 1** Probability of the Swap operation as a function of time for $J = 2 \times 10^{-3}$ eV, $\omega = 1$ GHz, and $B_- = 11 \times 10^{-3}$ T

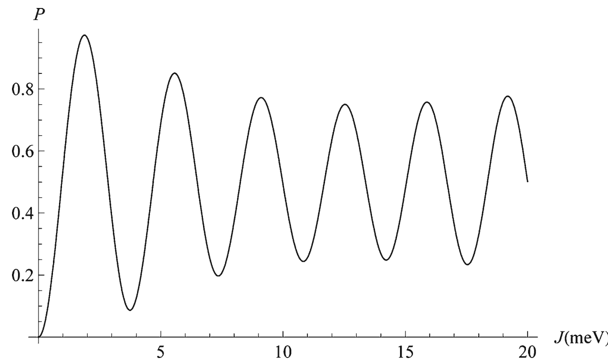


**Fig. 3** Probability as a function of the interaction $J$ for $c = 30$, $t = 1$ ns, and $\omega = 15$ GHz

Using (21), we calculate the probability amplitude for the Swap operation with the adiabatic variation (19),

$$\left|\langle\downarrow|\,\hat{u}\,|\uparrow\rangle\right|^2 = \frac{\left|G_2(z)\,\bar{G}_1^0 - \bar{G}_1(z)\,G_2^0\right|^2}{\left(\left|G_2^0\right|^2 + \left|G_1^0\right|^2\right)^2} \,.$$

In order to use the adiabatic pulse to implement some quantum operations (like the Swap or the XOR gate), the duration of the pulse needs to be shorter than the dephasing time of the system. For example, in GaAs quantum dots, this time is about 10 ns [19], which correspond to $\omega \simeq 1$ GHz. In typical experimental conditions, we have fields of about 5 T and $J = 2 \times 10^{-3}$ eV, and to satisfy the condition $B_+ \gg B_-$, we can set the amplitude $|B_-| = 11$ mT. So, some characteristic values for our system are

$$\frac{|a|}{\omega} = \frac{|J|}{\hbar\omega} \simeq 3\,, \quad \frac{c}{\omega} = \frac{\mu_B\,|B_-|}{\hbar\omega} \simeq 1\,.$$

In Fig. 1, we have plots of the probability as a function of time for the above values of the parameters. The first maximum occurs at $t = 0.5$ ns with a probability of $P = 90\%$. For larger time, as the $\cosh^{-1}$ approaches to zero, this probability varies as $A_1 \sin^2(at) + A_2$ where $A_i = A_i(\omega, a, c)$. The amplitude $A_1$ decreases as $c$ increases while the shift $A_2$ increases. The functions $A_i$ change significantly with $\omega$ only for $c > 10a$.

The dependence of the probability on the parameter $\omega$ become noticeable for $c > 4a$. In Fig. 2, we plot this dependence for $a/c = 2$ and $a/c = 1/6$. The parameter $\omega$ can be used to significantly attenuate the Swap transition for values of $c > 10a$.

A numerical study shows a strong dependence of the maximum values on the parameter $a$. This fact can be used to measure the interaction $J$. In Fig. 3, we plot the dependence of the probability on $J$. The attenuation of the second maximum can be achieved by increasing the ratio $c/a$.

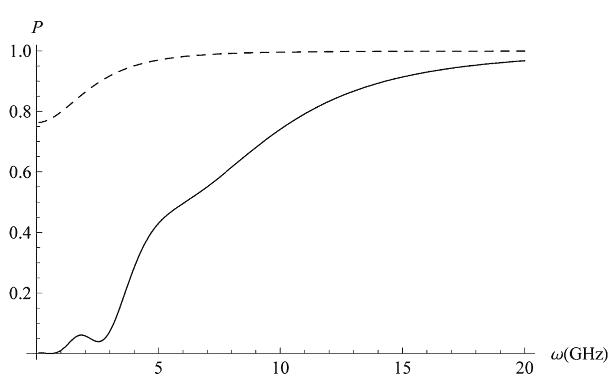


**Fig. 2** Probability as a function of $\omega$ for the values $c = 1$ (*dashed line*) and $c = 12$ (*solid line*) in $t = 0.8$ ns and $a = 2$

## 4 Final Remarks

We have described a method to construct some classes of exact solutions of the one-spin equation and their respective external field configuration. Although this method cannot be used to solve the one-spin equation for a general external field, it is very powerful in constructing solutions whose external fields have some desired characteristic. As an application, we explicitly construct the exact solutions for some restricted in time external fields. These are a very important kind of fields, once the fields used in practical application usually act in a finite time interval. After that, we show how these results can be applied in the problem of two interacting spins subjected to different magnetic fields, i.e., to solve the two-spin equation for parallel fields. In this case, our method can be used to control not only the characteristics of the external magnetic fields but also the behavior of the interaction function between the spins. In a general manner, the exact solutions of the two-spin equation have a wide application in the description of two coupled quantum dots and, especially, in the construction of quantum gates. In order to clarify this point, we show how an arbitrary exact solution, not only the ones obtained with the method described here, can be used to obtain the operational characteristics of a Swap operation. In this analysis, we chose the very important case of a restricted in time adiabatic variation of the external magnetic field and showed explicitly the dependence of the operational characteristics on the parameters of the external field. Besides, we describe the behavior of the system using explicit values of the parameters that can be obtained in experimental conditions. A graphical analysis of this behavior is presented, from where we can see that the gate can be better controlled if the experimental setup was assembled respecting some specific relation between the parameters.

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