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PARTICLES AND FIELDS

Non-Hermitian Hamiltonians with Real Spectrum in Quantum Mechanics

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Abstract Examples are given of non-Hermitian Hamiltonian operators which have a real spectrum. Some of the investigated operators are expressed in terms of the generators of the Weyl–Heisenberg algebra. It is argued that the existence of an involutive operator $\hat{J}$ which renders the Hamiltonian $\hat{J}$-Hermitian leads to the unambiguous definition of an associated positive definite norm allowing for the standard probabilistic interpretation of quantum mechanics. Non-Hermitian extensions of the Poeschl–Teller Hamiltonian are also considered. Hermitian counterparts obtained by similarity transformations are constructed.

Keywords Non-Hermitian Hamiltonians · Pseudo-Hermiticity · Krein spaces · Indefinite norm · $\mathcal{PT}$-Symmetry

1 Introduction

The interest in the study of non-Hermitian Hamiltonians in physics has been related, in the past, with the interpretation of some properties, such as transfer phenomena, typical of open systems. At present, it is also associated with new kinds of quantum theories characterized by non-Hermitian Hamiltonians with $\mathcal{PT}$-symmetry (the product of parity and time reversal) and real spectra, the recent developments being motivated by field-theoretic models, such as the Lee model. The results obtained originated a consistent extension of the standard quantum mechanics. The notion of $\mathcal{PT}$-symmetry can be placed in a general mathematical context known as *pseudo-Hermiticity*, a concept studied in the Krein space framework.

In non-relativistic quantum mechanics, the Hamiltonian operator is assumed to be Hermitian. It is well-known, however, that some relativistic extensions, such as the Klein–Gordon theory, lead to Hamiltonian operators H which are non-Hermitian, $H \neq H^\dagger$ [1]. The generator of the time evolution (Hamiltonian) of the quantal state of a free spinless relativistic particle of mass M and momentum $\mathbf{p}$ reads

$$H = \begin{pmatrix} \frac{\mathbf{p}^2}{2M} + Mc^2 & \frac{\mathbf{p}^2}{2M} \\ -\frac{\mathbf{p}^2}{2M} & -\frac{\mathbf{p}^2}{2M} - Mc^2 \end{pmatrix},$$

where c denotes the velocity of light. This matrix is not Hermitian but is P-Hermitian for

$$P = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

that is, $PH = H^\dagger P$. The matrix P allows for the definition of a P-norm, according to which the eigenvectors of H are P-orthogonal. The indefiniteness of the P-norm is physically meaningful, since it is related to the possibility of the particles having a positive or negative charge but precludes its use in the probabilistic interpretation of quantum mechanics. However,

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a positive definite matrix Q may be constructed according to which H is Q-Hermitian, $QH = H^\dagger Q$, so that it allows for the conventional interpretation of quantum mechanics.

Generally, in non-Hermitian quantum mechanics, an indefinite norm operator P occurs which renders the Hamiltonian P-Hermitian, that is, $PH = H^\dagger P$. Particularly, P may be the parity operator $\mathcal{P}$ that performs spatial reflection and has the effect $p \to -p$ and $x \to -x$. The indefinite norm operator P does not allow for the usual probabilistic interpretation of quantum mechanics because it is not positive definite. A positive definite operator Q which can play the role of P in the sense that $QH = H^\dagger Q$ is required for the conventional interpretation of quantum mechanics.

Non-Hermitian Hamiltonian operators with a real spectrum have been the object of intense research activity [2–8]. For instance, the Hamiltonian $H = p^2 + x^2 + ix^3$ has been studied by Bender and others [3, 4], who observed that its spectrum is real due to the $\mathcal{PT}$-symmetry being unbroken. That operator is not symmetric under $\mathcal{P}$ or $\mathcal{T}$ separately but is invariant under their combined operation and so it is said to possess space–time symmetry. Here, $\mathcal{T}$ denotes the anti-linear time-reversal operator, which has the effect $p \to -p$, $x \to x$, and $i \to -i$. Following Bender's work, many researchers, as, for instance, González Lopéz et al. [5–7], investigated such non-Hermitian Hamiltonians with real spectra.

The construction of positive norm operators required by the quantum mechanical probabilistic interpretation of non-Hermitian Hamiltonians of this type is a topic of current interest. The problem of non-uniqueness of the metric was addressed by Scholtz et al. [2] who resolved it by considering an irreducible set of observables. Bender et al. [3, 4], in the context of $\mathcal{PT}$-invariant theories, defined a new operator $\mathcal{C}$ and the $\mathcal{CPT}$ scalar product, which is positive definite. However, to construct $\mathcal{C}$, the eigenvalues and eigenvectors of the Hamiltonian have to be determined, which can be done explicitly for soluble models but in general situations only a perturbative expansion of $\mathcal{C}$ is available (for extensive bibliography, see the references cited in [8]). Let the non-Hermitian Hamiltonian H acting on the Hilbert space $\mathcal{H}$ have real eigenvalues λ_j and let the corresponding right and left eigenvectors, denoted by $|\phi_j\rangle$ and $\langle\psi_j|$, respectively, form two complete systems. We have

$$H|\phi_j\rangle = \lambda_j|\phi_j\rangle, \quad \langle\psi_j|H = \lambda_j\langle\psi|.$$

The completeness of the eigenvectors $|\phi_i\rangle$ means that any $|\xi\rangle \in \mathcal{H}$ may be expanded as $|\xi\rangle = \sum_{i=1}^n |\phi_i\rangle c_i$, for certain $c_i \in \mathbf{C}$. Mostafazadeh [9] has shown that there exists positive definite Hermitian operators Q such that QH is Hermitian, $QH = H^\dagger Q$, and under the similarity transformation

$$\tilde{H} = Q^{\frac{1}{2}} H Q^{-\frac{1}{2}},$$

a Hermitian operator $\tilde{H}$ is obtained.

Mostafazadeh's result is easily understood in the finite dimensional case. Let $\mathcal{H} = \mathbf{C}^n$ and denote by $\langle\psi|\phi\rangle$ the inner product of the vectors $|\phi\rangle, |\psi\rangle \in \mathbf{C}^n$. Dirac's *bra-ket* notation is used throughout. Let A be an $n \times n$ non-Hermitian matrix, that is, $A \neq A^\dagger$, where $A^\dagger$ denotes the adjoint matrix. Let us also assume that the eigenvalues λ_j of A are real and that the right and left eigenvectors, denoted, respectively, by $|\phi_j\rangle$ and $\langle\psi_j|$, form two complete systems,

$$A|\phi_j\rangle = \lambda_j|\phi_j\rangle, \quad \langle\psi_j|A = \lambda_j\langle\psi_j|.$$

Assuming for simplicity that the eigenvalues are distinct, it follows that $\langle\psi_j|\phi_k\rangle = 0$ if $j \neq k$. If the eigenvalues are conveniently normalized, we have moreover

$$\langle\psi_j|\phi_k\rangle = \delta_{jk}.$$

Let Q be the operator such that $|\psi_j\rangle = Q|\phi_j\rangle$. This operator is Hermitian and positive definite (notice that $\langle\psi_j|\phi_k\rangle = \langle\phi_j|\psi_k\rangle$) and has the property that QA is Hermitian, $QA = A^\dagger Q$. These assertions may be easily verified. Thus, under the similarity transformation

$$\tilde{A} = Q^{\frac{1}{2}} A Q^{-\frac{1}{2}},$$

a Hermitian operator is obtained, in agreement with Mostafazadeh's result [9]. The operator Q induces the inner product

$$\langle\psi|Q|\phi\rangle,$$

satisfying $\langle\psi|Q|\psi\rangle > 0$, for $|\psi\rangle \neq 0$. Given an arbitrary vector $|\xi\rangle \in \mathbf{C}^n$ such that $\langle\xi|Q|\xi\rangle = 1$, according to the rules of quantum mechanics, the component $c_j = \langle\phi_j|Q|\xi\rangle = \langle\psi_j|\xi\rangle$ of the vector $|\xi\rangle = \sum c_j|\phi_j\rangle$ may be understood as the amplitude (square root) of the probability for the result of a measurement of the observable A to be λ_j. However, Q must be fixed by some supplementary requirement. Physical theories involving non-Hermitian Hamiltonians with real eigenvalues also prescribe certain Hermitian involutive operators J, $J^2 = I$, such that JH is Hermitian, $JH = H^\dagger J$, so that the eigenvectors $|\phi_j\rangle$ of H may be normalized according to $\langle\phi_i|J|\phi_j\rangle = \eta_i\delta_{ij}$, $\eta_i = \pm 1$. Since $\langle\phi_j|J$ is a left eigenvector, i.e., $\langle\phi_j|JH = \lambda_j\langle\phi_j|J$, it is natural to chose $\langle\psi_j| = \eta_j\langle\phi_j|J$ as the desired left eigenvector satisfying $\langle\psi_j|\phi_j\rangle = 1$ and to define Q by the relation $|\psi_j\rangle = \eta_j J|\phi_j\rangle = Q|\phi_j\rangle$. By this prescription, Q is fixed unambiguously and the relation $JQJ = Q^{-1}$ is satisfied, as the following argument shows. Indeed, let us consider the left eigenvectors of H, $\langle\psi_i| = \langle\phi_i|Q = \eta_i\langle\phi_i|J$. Since

$$|\psi_i\rangle = \eta_i J|\phi_i\rangle = \eta_i J Q^{-1}|\psi_i\rangle = Q|\phi_i\rangle = Q(\eta_i J|\psi_i\rangle) \\ = \eta_i QJ|\psi_i\rangle,$$

we have $QJ|\psi_i\rangle = JQ^{-1}|\psi_i\rangle$. Since, moreover, the $|\psi_i\rangle$ constitute a complete system of vectors, the result follows. This conclusion is in agreement with the criterium proposed by Bender and collaborators [10] for the complete specification of a suitable norm operator Q, namely $J \log QJ = -\log Q$.

We investigate simple examples of non-Hermitian Hamiltonians which have real spectra and may be diagonalized with the help of algebraic methods. Moreover, these operators have complete systems of right and left eigenvectors.

2 The Harmonic Oscillator

For the sake of completeness, let us consider the harmonic oscillator Hamiltonian

$$H = \frac{1}{2}(p^2 + x^2), \tag{1}$$

which acts on the space L^2 of square integrable differentiable functions of the real variable x endowed with the usual inner product

$$\langle\phi|\psi\rangle = \int \mathrm{d}x\, \phi(x)^*\psi(x).$$

As it is well-known, $p : L^2 \to L^2$ is the differential operator $f(x) \to -i(\mathrm{d}f/\mathrm{d}x)$ and $x : L^2 \to L^2$ is the multiplicative operator $f(x) \to xf(x)$. These operators satisfy the *quantum condition* $[p, x] = -i$ and the harmonic oscillator Hamiltonian is Hermitian. Following the well-known Dirac's approach, its spectrum is determined with the help of the Weil–Heisenberg algebra generated by the *creation* and *annihilation operators* $a^\dagger$ and a, respectively, defined by the linear combinations

$$a = \frac{1}{\sqrt{2}}(x + ip), \quad a^\dagger = \frac{1}{\sqrt{2}}(x - ip). \tag{2}$$

As the notation conveys, $a^\dagger$ is the adjoint of a and these operators satisfy the commutation relation $[a, a^\dagger] = 1$. One easily finds

$$H = a^\dagger a + \frac{1}{2}.$$

Moreover, for the vector $|\phi_0\rangle$ such that $a|\phi_0\rangle = 0$ and $|\phi_n\rangle = a^{\dagger n}|\phi_0\rangle$, we have

$$H|\phi_n\rangle = \left(n + \frac{1}{2}\right)|\phi_n\rangle, \quad n = 0, 1, 2, \cdots .$$

The vector $|\phi_0\rangle$ stands, in Dirac's notation, for nothing else than a solution $\phi_0(x)$ of the differential equation

$$\left(x + \frac{\mathrm{d}}{\mathrm{d}x}\right)\phi_0(x) = 0, \quad \phi_0(x) = K_0 \mathrm{e}^{-\frac{x^2}{2}}.$$

On the other hand, $|\phi_n\rangle$ is identified with the function

$$\phi_n(x) = K_n\left(x - \frac{\mathrm{d}}{\mathrm{d}x}\right)^n \mathrm{e}^{-\frac{x^2}{2}}, \tag{3}$$

where K_n is a normalization constant.

3 Non-Hermitian Operators with a Real Spectrum

We give simple examples of non-Hermitian operators with a real spectrum and a bi-orthogonal set of eigenvectors. In each case, we show that there exists an involutive operator $\mathcal{J}$ which renders the Hamiltonian $\mathcal{J}$-Hermitian and allows for the unambiguous definition of a positive definite norm operator suitable for the conventional quantum mechanical interpretation.

3.1 The Extended Harmonic Oscillator

We begin by considering the operator

$$H_\beta = \frac{\beta}{2}(p^2 + x^2) + i\sqrt{2}p \\ = \beta a^\dagger a + (a - a^\dagger) + \frac{\beta}{2}, \quad \beta > 0. \tag{4}$$

Although H_β is non-Hermitian and it is even not $\mathcal{PT}$-symmetric, it is nevertheless $\mathcal{P}$-Hermitian, i.e., $\mathcal{P}H_\beta = H_\beta^\dagger\mathcal{P}$. In order to determine the spectrum of H_β, we write

$$H_\beta = \beta\left(a^\dagger + \frac{1}{\beta}\right)\left(a - \frac{1}{\beta}\right) + \frac{1}{\beta} + \frac{\beta}{2}.$$

The spectrum and the eigenvectors of H_β are easily determined by the usual technique due to Dirac. Although the operators $(a^\dagger + 1/\beta)$ and $(a - 1/\beta)$ are not the adjoint of each other, they generate a Weil–Heisenberg algebra, so we still have

$$\left[\left(a^\dagger + \frac{1}{\beta}\right)\left(a - \frac{1}{\beta}\right), \left(a^\dagger + \frac{1}{\beta}\right)^n\right] = n\left(a^\dagger + \frac{1}{\beta}\right)^n,$$

$$\left[\left(a^\dagger + \frac{1}{\beta}\right)\left(a - \frac{1}{\beta}\right), \left(a - \frac{1}{\beta}\right)^n\right] = -n\left(a - \frac{1}{\beta}\right)^n.$$

It follows that

$$H_\beta|R_n\rangle = \left(\frac{1}{\beta} + \beta\left(n + \frac{1}{2}\right)\right)|R_n\rangle,$$

where

$$\left(a - \frac{1}{\beta}\right)|R_0\rangle = 0, \quad |R_n\rangle = \left(a^\dagger + \frac{1}{\beta}\right)^n |R_o\rangle,$$

which fixes the right eigenvectors. The left eigenvectors are given by

$$\langle L_n|H_\beta = \left(\frac{1}{\beta} + \beta\left(n + \frac{1}{2}\right)\right)\langle L_n|,$$

where

$$\langle L_n|\left(a^\dagger + \frac{1}{\beta}\right) = 0, \quad \langle L_n| = \langle L_0|\left(a - \frac{1}{\beta}\right)^n.$$

The spectrum of H_β is given by

$$\sigma(H_\beta) = \frac{1}{\beta} + \beta\left\{\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \cdots\right\}.$$

We have assumed $\beta \neq 0$, so the spectrum is obviously real. The right eigenvectors and the left eigenvectors are represented, respectively, by the functions

$$R_n(x) = K_n\left(x + \frac{\sqrt{2}}{\beta} - \frac{\mathrm{d}}{\mathrm{d}x}\right)^n \exp\left(-\frac{1}{2}\left(x - \frac{\sqrt{2}}{\beta}\right)^2\right)$$

and

$$L_n(x) = K_n\left(x - \frac{\sqrt{2}}{\beta} - \frac{\mathrm{d}}{\mathrm{d}x}\right)^n \exp\left(-\frac{1}{2}\left(x + \frac{\sqrt{2}}{\beta}\right)^2\right),$$

where K_n denotes a normalization constant. Obviously, $L_n(x) = (-1)^n R_n(-x)$, that is, the left eigenvectors are essentially the involution of the right eigenvectors generated by $\mathcal{P}$. For

$$K_n^2 = \frac{\sqrt{\pi}}{2^n}\exp(-\beta^{-2})$$

the eigenfunctions are orthonormal, $\langle L_n|R_m\rangle = \delta_{nm}$.

It is clear that the operators $(a^\dagger + 1/\beta), (a - 1/\beta)$ are related to the operators $a^\dagger, a$ by a similarity transformation,

$$a^\dagger + \frac{1}{\beta} = \mathrm{e}^{(a^\dagger+a)/\beta}\, a^\dagger\, \mathrm{e}^{-(a^\dagger+a)/\beta},$$

$$a - \frac{1}{\beta} = \mathrm{e}^{(a^\dagger+a)/\beta}\, a\, \mathrm{e}^{-(a^\dagger+a)/\beta}.$$

Thus, the operator

$$\widetilde{H}_\beta = \mathrm{e}^{-(a^\dagger+a)/\beta}\, H_\beta\, \mathrm{e}^{(a^\dagger+a)/\beta}$$

is Hermitian. Moreover, $|L_n\rangle = \mathrm{e}^{-2(a^\dagger+a)/\beta}|R_n\rangle$ and $\langle R_n|\mathrm{e}^{-2(a^\dagger+a)/\beta}|R_m\rangle = \delta_{nm}$.

Other operators share with Q the property of rendering H_β Hermitian. With respect to the orthonormal basis constituted by the eigenvectors $|\phi_n\rangle$ of the harmonic oscillator (3), the operator H_β is represented by the tridiagonal matrix

$$M_\beta = \begin{pmatrix} \beta/2 & \sqrt{1} & 0 & 0 & \cdots \\ -\sqrt{1} & 3\beta/2 & \sqrt{2} & 0 & \cdots \\ 0 & -\sqrt{2} & 5\beta/2 & \sqrt{3} & \cdots \\ 0 & 0 & -\sqrt{3} & 7\beta/2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The matrix M_β is J-Hermitian for $J = \mathrm{diag}(1, -1, 1, -1, \cdots)$. Let $\mathcal{J}$ denote the operator which is represented by the matrix J with respect to the basis constituted by the eigenvectors $|\phi_n\rangle$. In the present case, $\mathcal{J}$ is precisely the parity operator $\mathcal{P}$. If $|R_n\rangle$ is a right eigenvector of H_β, then $\langle R_n|\mathcal{J}$ is a left eigenvector. For a vector $|\Xi\rangle$ normalized according to $\langle\Xi|Q|\Xi\rangle = 1$, where $Q = \mathrm{e}^{-2(a^\dagger+a)/\beta}$, and for the eigenvectors of H_β normalized according to $\langle R_i|\mathcal{J}|R_j\rangle = (-1)^i\delta_{ij}$, the quantity $|\langle R_i|\mathcal{J}|\Xi\rangle|^2$ has the meaning of a probability.

We observe that $\hat{J}(\log Q)\mathcal{J} = -\log Q$, as may be easily checked, keeping in mind that $\log Q = -2(a^\dagger + a)/\beta$. This is in agreement with Bender's [10] criterium for the complete specification of the norm operator Q.

3.2 The Swanson Hamiltonian

Next, we consider an operator of a class which has been proposed by Swanson et al. [11–13], namely the non-Hermitian Hamiltonian with a real spectrum

$$\begin{aligned} H_\theta &= \frac{1}{2}(p^2(1 - i\tan 2\theta) + x^2(1 + i\tan 2\theta)) \\ &= \frac{p^2\mathrm{e}^{-2i\theta} + x^2\mathrm{e}^{2i\theta}}{2\cos 2\theta}, \end{aligned} \quad (5)$$

where θ is a real parameter, $-\pi/4 < \theta < \pi/4$. Let us consider the replacements $x \to x\mathrm{e}^{-2i\theta}$, $p \to p\mathrm{e}^{2i\theta}$ and let $\mathcal{P}$ be the Hermitian operator which produces them, that is, $\mathcal{P}$ is such that

$$\mathcal{P}x\mathcal{P}^{-1} = x\mathrm{e}^{-2i\theta}, \quad \mathcal{P}p\mathcal{P}^{-1} = p\mathrm{e}^{2i\theta}.$$

Then, we have

$$\mathcal{P}H_\theta = H_\theta^\dagger\mathcal{P},$$

which means that the non-Hermitian operator H_θ is actually $\mathcal{P}$-Hermitian and, so, may have real eigenvalues. In terms of the differential operators

$$c = \frac{1}{\sqrt{2}}(x e^{i\theta} + ip e^{-i\theta}), \quad c^\ddagger = \frac{1}{\sqrt{2}}(x e^{i\theta} - ip e^{-i\theta}) \qquad (6)$$

or, equivalently,

$$c = \frac{e^{i\theta}}{\sqrt{2}}\, x + \frac{e^{-i\theta}}{\sqrt{2}} \frac{d}{dx}, \quad c^\ddagger = \frac{e^{i\theta}}{\sqrt{2}}\, x - \frac{e^{-i\theta}}{\sqrt{2}} \frac{d}{dx},$$

a diagonal, oscillator-like form, is obtained,

$$H_\theta = \omega\left(c^\ddagger c + \frac{1}{2}\right), \quad \omega = \frac{1}{\cos 2\theta}.$$

The operators $c, c^\ddagger$ satisfy $[c, c^\ddagger] = 1$. For completeness, we give their expression in terms of the operators $a^\dagger, a$ of (2),

$$c = \cos\theta\, a + i \sin\theta\, a^\dagger = \mathcal{P}^{-1/2} a \mathcal{P}^{1/2},$$
$$c^\ddagger = \cos\theta\, a^\dagger + i\ \sin\theta\, a = \mathcal{P}^{-1/2} a^\dagger \mathcal{P}^{1/2}.$$

Although $c^\dagger \neq c^\ddagger$, these operators realize a certain representation of the Weil–Heisenberg algebra. It follows that the spectrum is

$$\sigma(H_\theta) = \omega\left\{\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \cdots\right\}, \quad \omega = \frac{1}{\cos 2\theta}.$$

The operator $\mathcal{P}$ allows for the definition of a $\mathcal{P}$-inner product, according to which the eigenvectors associated with different eigenvalues are $\mathcal{P}$-orthogonal. However, if the operator $\mathcal{P}$ is not positive definite, the $\mathcal{P}$-inner product will be indefinite and the corresponding $\mathcal{P}$-norm will not be appropriate for the interpretation of quantum mechanics. We will see that, actually, $\mathcal{P}$ is positive definite, by considering the left and right eigenvectors. They are easily obtained, keeping in mind the explicit expressions of the differential operators (6). The right eigenvectors satisfy

$$H_\theta |R_n\rangle = \left(\frac{1}{2} + n\right) \omega |R_n\rangle$$

and are given by

$$c|R_0\rangle = 0, \quad |R_n\rangle = K_n (c^\ddagger)^n |R_0\rangle,$$

where K_n is a real normalization constant. The left eigenvectors satisfy

$$\langle L_n| H_\theta = \left(\frac{1}{2} + n\right) \omega \langle L_n|$$

and are given by

$$\langle L_0| c^\ddagger = 0, \quad \langle L_n| = K_n \langle L_0| c^n.$$

The last equation is equivalent to

$$(c^\ddagger)^\dagger |L_0\rangle = 0, \quad |L_n\rangle = K_n (c^\dagger)^n |L_0\rangle.$$

As an example, we present the explicit expressions of the lowest right and left eigenvectors,

$$R_0(x) = K_0 \exp\left(-\frac{x^2}{2}\, e^{2i\theta}\right),$$

$$L_0(x) = K_0 \exp\left(-\frac{x^2}{2}\, e^{-2i\theta}\right).$$

For an appropriate value of the normalization constant K_n, the eigenfunctions are orthonormal, $\langle L_n | R_m\rangle = \delta_{nm}$.

It is clear that the operators $c^\ddagger, c$ are related to the operators $a^\dagger, a$ by a similarity transformation,

$$c^\ddagger = e^{i\frac{\theta}{2}(a^2 - a^{\dagger 2})}\, a^\dagger\, e^{-i\frac{\theta}{2}(a^2 - a^{\dagger 2})}, \quad c = e^{i\frac{\theta}{2}(a^2 - a^{\dagger 2})}\, a\, e^{-i\frac{\theta}{2}(a^2 - a^{\dagger 2})}.$$

Thus,

$$\mathcal{P} = e^{-i\theta(a^2 - a^{\dagger 2})}.$$

Being the exponential of an Hermitian operator, it is positive definite. Besides, the Hermitian operator

$$\widetilde{H}_\theta = e^{-i\frac{\theta}{2}(a^2 - a^{\dagger 2})}\, H_\theta\, e^{i\frac{\theta}{2}(a^2 - a^{\dagger 2})} = \omega\left(\frac{1}{2} + a^\dagger a\right)$$

is the harmonic oscillator operator with frequency ω, as observed by Jones [12]. Moreover,

$$|L_n\rangle = e^{-i\theta(a^2 - a^{\dagger 2})}\, |R_n\rangle$$

and

$$\left\langle R_n | e^{-i\theta(a^2 - a^{\dagger 2})} | R_m \right\rangle = \delta_{nm}.$$

For a vector $|\Xi\rangle$ normalized according to $\langle \Xi | \mathcal{P} | \Xi\rangle = 1$, the quantity $|\langle R_i | \mathcal{P} | \Xi\rangle|^2 = |\langle L_i | \Xi\rangle|^2$ has the meaning of a probability. We observe that the knowledge of the relative probability does not require the determination of the full set of left- and right-eigenfunctions, but only of the eigenfunctions related to the relevant transitions. Our conclusions are similar to those of [12]. However, our approach is different, since it is based on the explicit determination of the left- and right-eigenfunctions.

Next we obtain a Hermitian involutive operator $\mathcal{J}$ (such that $\mathcal{J}^2 = Id$) which also renders H_θ Hermitian, that is, $\mathcal{J} H_\theta = H_\theta^\dagger \mathcal{J}$. With respect to the orthonormal basis constituted by the eigenvectors $|\phi_n\rangle$ of the harmonic oscillator (3), H_θ is represented by the matrix

$$M_\theta = \begin{pmatrix} 1/2 & 0 & i\beta\sqrt{1.2} & 0 & 0 & \cdots \\ 0 & 3/2 & 0 & i\beta\sqrt{2.3} & 0 & \cdots \\ i\beta\sqrt{1.2} & 0 & 5/2 & 0 & i\beta\sqrt{3.4} & \cdots \\ 0 & i\beta\sqrt{2.3} & 0 & 7/2 & 0 & \cdots \\ 0 & 0 & i\beta\sqrt{3.4} & 0 & 9/2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

$$\beta = \frac{\tan 2\theta}{2}.$$

It is clear that $JM_\theta = M_\theta^\dagger J$ for $J = I_2 \oplus -I_2 \oplus I_2 \oplus -I_2 \oplus \cdots$. Let $\mathcal{J}$ denote the operator which is represented by the matrix J with respect to the system of the eigenvectors $|\phi_n\rangle$ (3). It follows that $\mathcal{J}H_\theta = H_\theta^\dagger \mathcal{J}$.

If $|R_n\rangle$ is a right eigenvector of H_θ, then $\langle R_n|\mathcal{J}$ is a left eigenvector, so that $\mathcal{J}$ is the required involutive operator. For a vector $|\Xi\rangle$ normalized according to $\langle\Xi|Q|\Xi\rangle = 1$ and for eigenvectors of H_θ normalized according to $\langle R_i|\mathcal{J}|R_j\rangle = \eta_i\delta_{ij}$, where $\eta_i = 1$ for $i = 4n$, $(4n+1)$, $n = 0, 1, 2, \cdots$ and $\eta_i = -1$ for $i = (4n-2)$, $(4n-1)$, $n = 1, 2, \cdots$, the quantity $|\langle R_i|\mathcal{J}|\Xi\rangle|^2$ has the meaning of a probability. We observe that $\mathcal{J}(\log Q)\mathcal{J} = -\log Q$, for $\log Q = -i\theta(a^2 - a^{\dagger 2})$, as may be easily checked. This is in agreement with Bender's [10] criterium for the complete specification of the norm operator Q.

3.3 Non-Hermitian Extensions of the Poeschl–Teller Hamiltonian

Let us now consider the Poeschl–Teller Hamiltonian [14, 15]

$$H = p^2 - \frac{\gamma(\gamma-1)}{\cosh^2 x}, \quad \gamma > 1. \tag{7}$$

In terms of the operators

$$A^\dagger = -ip + (\gamma - 1)\tanh x, \quad A = ip + (\gamma - 1)\tanh x,$$

we obtain

$$H = A^\dagger A - (\gamma - 1)^2.$$

For a detailed algebraic treatment of the Poeschl–Teller Hamiltonian, see [16]. The discrete spectrum of this Hamiltonian is easily determined. It is the set of eigenvalues:

$$E_n = -(\gamma - 1 - n)^2, \quad n = 0, 1, \cdots \leq \gamma - 1.$$

Next we investigate non-Hermitian extensions of the Poeschl–Teller Hamiltonian [17]. To start with, we consider the $\mathcal{PT}$-symmetric extension, obtained from (7) through the replacement $x \to x - i\alpha$,

$$\begin{aligned} H_\alpha &= p^2 - \frac{\gamma(\gamma-1)}{\cosh^2(x - i\alpha)}, \\ &= p^2 - \frac{2\gamma(\gamma-1)(1+\cosh 2x\cos 2\alpha + i\sinh 2x \sin 2\alpha)}{(\cosh 2x - \cos 2\alpha)^2}, \\ \gamma &> 1, \end{aligned} \tag{8}$$

where α denotes a real parameter. The Hamiltonian (8) is $\mathcal{P}$-Hermitian: $\mathcal{P}H_\alpha = H_\alpha^\dagger\mathcal{P}$. In terms of the operators

$$A_\alpha^\ddagger = -ip + (\gamma - 1)\tanh(x - i\alpha),$$

$$A_\alpha = ip + (\gamma - 1)\tanh(x - i\alpha),$$

we obtain

$$H_\alpha = A_\alpha^\ddagger A_\alpha - (\gamma - 1)^2.$$

The spectrum of this Hamiltonian is real and coincides with the spectrum of the Hamiltonian (7), as may be easily seen following the technique described in [16]. Mathematically, it may be interesting to observe that $A_\alpha^\ddagger$ is related to the $\mathcal{P}$-adjoint to A_α, that is, $A_\alpha^\ddagger = -\mathcal{P}A_\alpha^\dagger\mathcal{P}$. The ground state wave function of H_α is annihilated by A_α and reads

$$\phi_0(x) = \frac{1}{(\cosh(x - i\alpha))^{\gamma-1}}.$$

The groundstate energy is, therefore, $E_0 = -(\gamma - 1)^2$.

In order to determine the first excited state of H_α (when it exists), we consider the partner Hamiltonian

$$H_{1;\alpha} = A_\alpha A_\alpha^\ddagger - (\gamma - 1)^2 = p^2 - \frac{(\gamma-1)(\gamma-2)}{\cosh^2(x - i\alpha)},$$

which is obtained from H_α if the order of the operators A_α, $A_\alpha^\ddagger$ is interchanged. In terms of the operators

$$A_{1;\alpha}^\ddagger = -ip + (\gamma - 2)\tanh(x - i\alpha),$$

$$A_{1;\alpha} = ip + (\gamma - 2)\tanh(x - i\alpha),$$

we may write

$$H_{1;\alpha} = A_{1;\alpha}^\ddagger A_{1;\alpha} - (\gamma - 2)^2,$$

so that its ground state wave function, which is annihilated by $A_{1;\alpha}$, reads

$$\phi_{1;0}(x) = \frac{1}{(\cosh(x - i\alpha))^{\gamma-2}}$$

and the groundstate energy is $E_{1;0} = -(\gamma - 2)^2$. It is well-known that the required wave function of the first exited state of H_α is given by

$$\phi_1(x) = A_\alpha\phi_{1;0}(x),$$

the corresponding eigenvalue being precisely $E_1 = E_{1;0} = -(\gamma-2)^2$.

In order to determine the second excited state of H_α (when it exists), which is connected with the first excited states of $H_{1;\alpha}$, we construct the partner Hamiltonian of $H_{1;\alpha}$, namely,

$$H_{2;\alpha} = A_{1;\alpha} A^{\ddagger}_{1;\alpha} - (\gamma-2)^2 = p^2 - \frac{(\gamma-2)(\gamma-3)}{\cosh 2(x-i\alpha)}.$$

In terms of the operators

$$A^{\ddagger}_{2;\alpha} = -ip + (\gamma-3)\tanh(x-i\alpha),$$
$$A_{2;\alpha} = ip + (\gamma-3)\tanh(x-i\alpha),$$

we may write

$$H_{2;\alpha} = A^{\ddagger}_{2;\alpha} A_{2;\alpha} - (\gamma-3)^2,$$
$$A^{\ddagger}_{2;\alpha} = -ip + (\gamma-3)\tanh(x-i\alpha),$$
$$A_{2;\alpha} = ip + (\gamma-3)\tanh(x-i\alpha),$$

so that the ground state wave function of $H_{2;\alpha}$, which is annihilated by $A_{2;\alpha}$, reads

$$\phi_{2;0}(x) = \frac{1}{(\cosh(x-i\alpha))^{\gamma-3}}$$

and the groundstate energy is $E_{2;0} = -(\gamma-3)^2$. It is well-known that the wave function of the first exited state of $H_{1;\alpha}$ is given by

$$\phi_{1;1}(x) = A_{1;\alpha}\phi_{2;0}(x),$$

and the wave function of the second exited state of H_α is given by

$$\phi_2(x) = A_\alpha \phi_{1;1}(x) = A_\alpha A_{1;\alpha}\phi_{2;0}(x),$$

the corresponding eigenvalue being precisely $E_2 = E_{1;1} = E_{2;0} = -(\gamma-3)^2$. And so on.

We have seen that the eigenvalues of H_α and H coincide. This is not surprising because these operators are similar,

$$H_\alpha = \mathrm{e}^{-\alpha p} H \mathrm{e}^{\alpha p}.$$

Moreover, H_α is Q-Hermitian, for $Q = \mathrm{e}^{2\alpha p}$, i.e., $QH_\alpha = H_\alpha^\dagger Q$. The norm defined with the help of the positive definite operator Q is appropriate for the usual statistical interpretation of quantum mechanics.

Finally, we consider the non-$\mathcal{PT}$-symmetric extension of the Poeschl–Teller Hamiltonian, obtained from (7) through the replacement $p \to p\,\mathrm{e}^{i\theta},\ x \to x\,\mathrm{e}^{-i\theta}$,

$$\begin{aligned} H_\theta &= (\eta + i\zeta)^2\, p^2 - \frac{\gamma(\gamma-1)}{\cosh^2((\eta - i\zeta)x)}, \\ &= (\eta^2 - \zeta^2 + 2i\eta\zeta)\, p^2 - \frac{2\gamma(\gamma-1)(1+\cosh(2\eta x)\ \cos(2\zeta x) + i\sinh(2\eta x)\ \sin(2\zeta x))}{(\cosh(2\eta x) - \cos(2\zeta x))^2}, \\ &\gamma > 1,\ \eta = \cos\theta,\ \zeta = \sin\theta,\ -\frac{\pi}{4} < \theta < \frac{\pi}{4}. \end{aligned} \tag{9}$$

In terms of the operators

$$A^{\ddagger}_{\theta} = -ip\mathrm{e}^{i\theta} + (\gamma-1)\tanh(x\mathrm{e}^{-i\theta}),$$
$$A_{\theta} = ip\mathrm{e}^{i\theta} + (\gamma-1)\tanh(x\mathrm{e}^{-i\theta}),$$

we obtain

$$H_\theta = A^{\ddagger}_\theta A_\theta - (\gamma-1)^2.$$

The spectrum of the Hamiltonian (9) is real and coincides with the spectrum of the Poeschl–Teller Hamiltonian, as may be easily seen following the technique described in [16]. This is not surprising because H_θ is similar to H,

$$H_\theta = \mathrm{e}^{-\frac{\theta}{2}(px+xp)} H \mathrm{e}^{\frac{\theta}{2}(px+xp)}.$$

Moreover, H_θ is Q-Hermitian, for $Q = \mathrm{e}^{\theta(px+xp)}$, i.e., $QH_\theta = H_\theta^\dagger Q$, so that the norm defined with the help of the positive definite operator Q is appropriate for the conventional quantum-mechanical interpretation.

4 Conclusions

We have presented simple examples of non-Hermitian models in quantum mechanics for which the Hamiltonian H has a real spectrum. An involutive $\mathcal{J}$ operator such that $H^\dagger = \mathcal{J}H\mathcal{J}$ is identified and explicitly constructed in each case. The operator $\mathcal{J}$ allows for the definition of an indefinite norm, which obviously is not suitable for the standard quantum mechanical interpretation. We discuss the important role played by this operator in allowing for the unambiguous definition of a positive definite norm operator Q, suitable for the usual quantum mechanical interpretation. In indefinite inner product spaces, $\mathcal{J}$-Hermitian operators are known to have a spectrum which is symmetric relatively to the real axis or a real spectrum.

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