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Fermionic Representation of Two-Dimensional Dimer Models

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Abstract Dimer models in two dimensions give rise to well-known statistical lattice problems, which can be exactly solved by the same combinatorial techniques associated with the Ising model and which have been used to account for the phase transitions in a number of physically interesting systems. More recently, dimer models have been regarded as classical limits of the quantum ground state of some antiferromagnetic systems. We then revisit an early transfer-matrix calculation for the dimer model on the simple square lattice. We write a spin representation for the transfer matrix associated with the canonical partition function of two paradigmatic dimers models, on the 4–8 lattice, with an Ising-type transition, and on the brick lattice, with a peculiar commensurate–incommensurate transition. Using standard techniques, the problem is reduced to the calculation of the eigenvalues of a system of free fermions.

Keywords Dimer statistics · Fermionic representation · Lattice models · Dimer model

1 Introduction

Dimer models are associated with nontrivial lattice problems, which can be exactly solved in two dimensions with recourse to the same combinatorial techniques that have been used to analyze two-dimensional Ising models [1–4]. There has been a fruitful interplay between methods and techniques to investigate dimer and Ising problems. Fisher has shown that a two-dimensional Ising model can be mapped onto a certain planar dimer model [5]. The converse, however, is not always true. Dimer models on a honeycomb (brick) lattice, which we call Kasteleyn models [6], display a peculiar transition that has been associated with the commensurate–incommensurate phase transitions of domain wall systems [3, 7]. The dimer model on a 4–8 (bathroom tile) lattice, with a suitable choice of activities, has been used to account for the highly symmetric Ising-type phase transition of some hydrogen-bonded crystals [8]. In contrast to Ising models, the critical behavior of dimer models and the very existence of a phase transition depend on the geometry of the lattice and on the choice of bond activities. More recently, quantum dimer models have been introduced to construct the resonance valence bond wave functions of quantum antiferromagnets in the ground state [9–11]. In these problems, it is possible to regain the standard dimer model in the classical limit and to check a number of properties of the quantum models.

The use of combinatorial methods, in particular of the Pfaffian technique, which is the standard way to obtain the exact solutions of several representative two-dimensional dimer models, is already an indication of the fermionic character of these problems. Although there are some dimer model calculations using Grassmann fields [12, 13] and an early work of Lieb [14] transforming the square lattice dimer model into a problem of free fermions, the combinatorial solutions are still better known and much more used [4, 15, 16]. We then decided to revisit Lieb's work.

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We have used the transfer matrix technique to formulate the statistical problem and to analyze two distinct paradigmatic cases, the 4–8 lattice and the Kasteleyn dimer models.

In Section 1, we review Lieb's work [14] for the dimer model on the square lattice. We establish the notation and the main definitions. The canonical partition function of the dimer problem is formulated as a transfer matrix, which is written in terms of spin-1/2 operators, and a Jordan–Wigner transformation is then used to go from paulions to fermions. Lieb restricts the analysis to a close-packed dimer model on a square lattice, which displays no phase transition. In the following sections, we generalize Lieb's procedures to treat the representative cases of the dimer model on the 4–8 lattice [8], which displays an Ising critical behavior, with a logarithmically diverging and symmetrical specific heat about the critical temperature, and the Kasteleyn model [6], with a strongly diverging specific heat above the critical temperature (with critical exponent $\alpha = 1/2$). In particular, we show that the spectrum of eigenvalues associated with the 4–8 lattice indicates the existence of a disorder–disorder transition, without the characteristic degeneracy of the two-dimensional Ising model at low temperatures.

2 Transfer Matrix for the Dimer Model on the Square Lattice

Consider a simple quadratic lattice with M rows and N columns. A dimer is a rigid rod (diatomic molecule) connecting two nearest-neighbor vertices of a lattice bond. The dimer problem consists in the calculation of the number of dimer configurations on the lattice so that lattice sites (vertices) cannot be occupied by more than one dimer. In the close-packed limit, there are no monomers (empty sites).

In the formulation of the dimer–monomer problem, we assign different activities (statistical weights) to horizontal and vertical dimer bonds and to the monomer vertices. We then write the canonical partition function,

$$Z = \sum_C z_H^h z_V^v z_M^m = z_V^{\frac{1}{2}MN} \sum_C \gamma_H^h \gamma_M^m, \tag{1}$$

where h and v are the numbers of horizontal and vertical dimers, with activities z_H and z_V, respectively; m is the number of monomers, with activity z_M (it is convenient to define $\gamma_H = z_H/z_V$ and $\gamma_M = z_M/z_V^{1/2}$); and the sum is over allowed configurations C of dimers and monomers on the simple square lattice. In the close-packed limit, with no monomers, this partition function can be obtained by standard combinatorial techniques. Also, according to the work of Lieb [14], we can introduce spin-1/2 operators and formulate the problem in terms of a transfer matrix.

Let us associate a spin-1/2 variable with each vertical bond, so that spin up, $|\uparrow\rangle$, means the existence of a dimer on that bond and spin down, $|\downarrow\rangle$, means that there is no dimer. Consider a row of vertical bonds. If there is a spin up on a particular bond of this row, there should be a spin down on the vertical bond belonging to the next row along the vertical direction, since we cannot allow the double occupancy of any lattice site. On the other hand, if there is a spin down, there are three possibilities for the vertical bond of the next row: (a) spin up, $|\uparrow\rangle$, if there is a dimer on this bond; (b) spin down, $|\downarrow\rangle$, if there is a monomer on the connecting site; and (c) spin down, $|\downarrow\rangle$, in a cooperative way with a neighboring vertical bond, if there is a horizontal dimer.

The transfer matrix for this problem is formed by three operators. The first operator is given by

$$\mathbf{V}_1 = \prod_{n=1}^{N} \sigma_n^x, \tag{2}$$

where σ_n^x is a spin-1/2 operator and we are using the notation

$$|\cdots\uparrow\cdots\rangle \equiv |\cdot\rangle_1 \otimes |\cdot\rangle_2 \otimes \cdots \otimes |\uparrow\rangle_n \otimes \cdots \otimes |\cdot\rangle_N, \tag{3}$$

with the properties

$$\begin{aligned} \sigma_n^x|\cdots\uparrow\cdots\rangle &= |\cdots\downarrow\cdots\rangle, \\ \sigma_n^x|\cdots\downarrow\cdots\rangle &= |\cdots\uparrow\cdots\rangle. \end{aligned} \tag{4}$$

The application of $\mathbf{V}_1$ on the spin variables of a row of vertical bonds changes all of these spin variables, so that the next row along the vertical direction belongs to an allowed configuration (no vertex is occupied by more than one dimer).

We now apply an operator that creates monomers, that is, an operator that turns a spin up into a spin down, as can be done by σ_n^-, with

$$\sigma_n^-|\cdots\uparrow\cdots\rangle = |\cdots\downarrow\cdots\rangle, \qquad \sigma_n^-|\cdots\downarrow\cdots\rangle = 0. \tag{5}$$

In order to create p monomers in a row, taking into account that $(\sigma_n^-)^2 = 0$, we introduce the operator $\left(\sum_{n=1}^N \sigma_n^-\right)^p / p!$, and write

$$\mathbf{V}_2 = \exp\left(\gamma_M \sum_{n=1}^{N} \sigma_n^-\right), \tag{6}$$

which creates an arbitrary number of monomers.

In order to create dimers on a horizontal row, we have to change spin variables associated with two

neighboring vertices. We then use operators of the form $\sigma_n^- \sigma_{n+1}^-$. With periodic boundary conditions in the horizontal direction, $\sigma_{N+1}^- = \sigma_1^-$, we produce a configuration with an arbitrary number of horizontal dimers by the application of the operator

$$\mathbf{V}_3 = \exp\left(\gamma_H \sum_{n=1}^{N} \sigma_n^- \sigma_{n+1}^-\right). \tag{7}$$

We then define the transfer matrix

$$\mathbf{V} = \mathbf{V}_3 \mathbf{V}_2 \mathbf{V}_1, \tag{8}$$

and write the partition function for the monomer–dimer problem on the square lattice,

$$Z = z_V^{\frac{1}{2}MN} Tr\left(\mathbf{V}^M\right). \tag{9}$$

In the close-packed limit, with no monomers ($\gamma_M = 0$), Lieb has used a standard Jordan–Wigner transformation to show that the calculation of the eigenvalues of this transfer matrix is reduced to a problem of free fermions, from which it is easy to recover the well-known combinatorial results [4]. With this simple choice of activities, there are no singularities in the free energy of the closed-packed dimer model on the simple square lattice.

3 Dimer Model on the 4–8 Lattice

3.1 Transfer Matrix

The 4–8 lattice may be schematically drawn as a square lattice with the removal of a certain set of bonds (see Fig. 1). The dimer model on the 4–8 lattice, with the choice of activities indicated in the figure, where $z = \exp(-\beta\varepsilon)$, β is the inverse of the temperature, and $\varepsilon > 0$ is a positive energy parameter, leads to a phase transition associated with the hydrogen-bonded crystal SCD [8].

It is convenient to introduce a unit cell, as indicated by the dashed lines in Fig. 1, and consider a transfer matrix along the vertical direction. Considering the vertical bonds at the bottom of the lattice in Fig. 1, we apply the operator

$$\mathbf{V}_4 = z^{\mathbf{N}_+} = \exp\left[(\ln z)\mathbf{N}_+\right], \tag{10}$$

where $\mathbf{N}_+$, given by

$$\mathbf{N}_+ = \sum_{n=1}^{N} \sigma_n^+ \sigma_n^-, \tag{11}$$

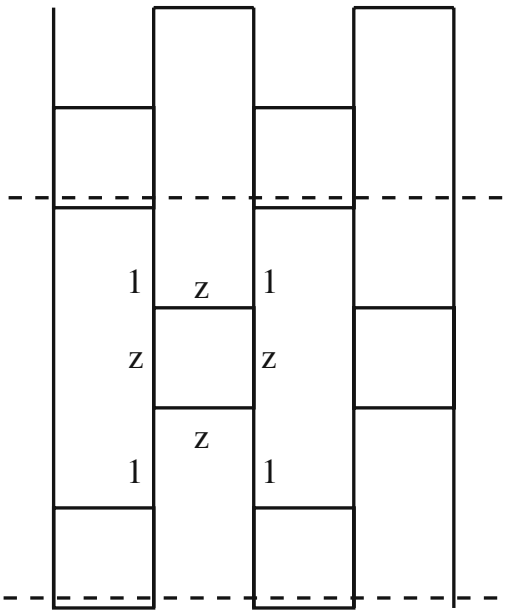

Fig. 1 Schematic representation of the 4–8 lattice with a choice of dimer activities. The *dashed lines* indicate a unit cell along the vertical direction

calculates the number of spins up along the row. This operator is used every time that we have activities different from 1 in a row of vertical bonds.

The next operator changes all spin variables. We use the same operator as in the case of the square lattice, given by (2).

We now apply an operator to create dimers on the vertical bonds. Taking into account that dimers are created on a special set of positions, since we do not have all bonds as in the square lattice, we introduce two distinct operators,

$$\mathbf{V}_2 = \exp\left[z \sum_{n=1}^{N/2} \sigma_{2n-1}^- \sigma_{2n}^-\right],$$

$$\mathbf{V}_2' = \exp\left[z \sum_{n=1}^{N/2} \sigma_{2n}^- \sigma_{2n+1}^-\right], \tag{12}$$

for N even, and using periodic boundary conditions, $\sigma_{N+1}^- = \sigma_1^-$, along the horizontal direction. The first operator creates dimers on bonds (1,2), (3,4), (5,6),..., and the second operator creates dimers on bonds (2,3), (4,5), (6,7),...

In the close-packed limit, we then write the transfer matrix

$$\begin{aligned}\mathbf{V} &= \mathbf{V}_2 \mathbf{V}_1 \mathbf{V}_2' \mathbf{V}_1 \mathbf{V}_4 \mathbf{V}_2' \mathbf{V}_1 \mathbf{V}_2 \mathbf{V}_1 \mathbf{V}_4 \\ &= \mathbf{V}_2 \overline{\mathbf{V}}_2' \mathbf{V}_4 \mathbf{V}_2' \overline{\mathbf{V}}_2 \mathbf{V}_4,\end{aligned} \tag{13}$$

where it is convenient to use the definitions

$$\mathbf{V}_1\mathbf{V}_2\mathbf{V}_1 \equiv \overline{\mathbf{V}}_2 = \exp\left[z\sum_{n=1}^{N/2}\sigma_{2n-1}^{+}\sigma_{2n}^{+}\right], \tag{14}$$

and

$$\mathbf{V}_1\mathbf{V}_2'\mathbf{V}_1 \equiv \overline{\mathbf{V}}_2' = \exp\left[z\sum_{n=1}^{N/2}\sigma_{2n}^{+}\sigma_{2n+1}^{+}\right]. \tag{15}$$

The partition function is given by

$$Z = Tr\left[\mathbf{V}^{M/4}\right], \tag{16}$$

in which we are assuming that N is even and that M is a multiple of 4.

3.2 Free-Fermion Solution

We now introduce a Jordan–Wigner transformation to change from spin to fermion operators, as used by Schultz et al. [17] to solve the two-dimensional Ising model. Let us define the fermion creation and annihilation operators,

$$\mathbf{C}_n = \left[\exp\left(\pi i\sum_{j=1}^{n-1}\sigma_j^{+}\sigma_j^{-}\right)\right]\sigma_n^{-}, \tag{17}$$

and

$$\mathbf{C}_n^{\dagger} = \left[\exp\left(\pi i\sum_{j=1}^{n-1}\sigma_j^{+}\sigma_j^{-}\right)\right]\sigma_n^{+}, \tag{18}$$

which obey the anticommutation rules

$$\left\{\mathbf{C}_m, \mathbf{C}_n^{\dagger}\right\} = \delta_{mn}, \qquad \left\{\mathbf{C}_m^{\dagger}, \mathbf{C}_n^{\dagger}\right\} = \{\mathbf{C}_m, \mathbf{C}_n\} = 0, \tag{19}$$

leading to a number of useful relations, $\sigma_n^{+}\sigma_{n+1}^{+} = \mathbf{C}_n^{\dagger}\mathbf{C}_{n+1}^{\dagger}$ and $\sigma_n^{-}\sigma_{n+1}^{-} = -\mathbf{C}_n\mathbf{C}_{n+1}$, for $n \neq N$, and $\sigma_N^{+}\sigma_1^{+} = -(-1)^{\mathbf{N}_+}\mathbf{C}_N^{\dagger}\mathbf{C}_1^{\dagger}$ and $\sigma_N^{-}\sigma_1^{-} = (-1)^{\mathbf{N}_+}\mathbf{C}_N\mathbf{C}_1$.

It is now convenient to use a unit cell with two distinct sites. We then introduce two subscripts and redefine the spin-1/2 operators,

$$\sigma_{n,1}^{\star} = \sigma_{2n-1}^{\star}, \tag{20}$$

and

$$\sigma_{n,2}^{\star} = \sigma_{2n}^{\star}, \tag{21}$$

where the superscript $\star$ stands for $+$, $-$, x, y, and z. Using this notation and assuming even values of N, the boundary condition becomes

$$\sigma_{\frac{N}{2}+1,1}^{\star} = \sigma_{1,1}^{\star}. \tag{22}$$

We then have

$$\mathbf{V}_2 = \exp\left[-z\sum_{n=1}^{N/2}\mathbf{C}_{n,1}\mathbf{C}_{n,2}\right],$$

$$\overline{\mathbf{V}}_2 = \exp\left[z\sum_{n=1}^{N/2}\mathbf{C}_{n,1}^{\dagger}\mathbf{C}_{n,2}^{\dagger}\right], \tag{23}$$

$$\mathbf{V}_2' = \exp\left[-z\left(\sum_{n=1}^{N/2-1}\mathbf{C}_{n,2}\mathbf{C}_{n+1,1} - (-1)^{\mathbf{N}_+}\mathbf{C}_{\frac{N}{2},2}\mathbf{C}_{1,1}\right)\right], \tag{24}$$

$$\overline{\mathbf{V}}_2' = \exp\left[z\left(\sum_{n=1}^{N/2-1}\mathbf{C}_{n,2}^{\dagger}\mathbf{C}_{n+1,1}^{\dagger} - (-1)^{\mathbf{N}_+}\mathbf{C}_{\frac{N}{2},2}^{\dagger}\mathbf{C}_{1,1}^{\dagger}\right)\right], \tag{25}$$

and

$$\mathbf{V}_4 = \exp\left[(\ln z)\sum_{n=1}^{N/2}\left(\mathbf{C}_{n,1}^{\dagger}\mathbf{C}_{n,1} + \mathbf{C}_{n,2}^{\dagger}\mathbf{C}_{n,2}\right)\right]. \tag{26}$$

The extra terms in the expressions of operators $\mathbf{V}_2'$ and $\overline{\mathbf{V}}_2'$, depending on $(-1)^{\mathbf{N}_+}$, can be dealt with by choosing appropriate boundary conditions for the fermion operators. We first note that $\left[\mathbf{V}, (-1)^{\mathbf{N}_+}\right] = 0$, since the $\mathbf{V}$ matrix is formed by quadratic fermion operators, which always commute with the number operator. Therefore, the parity of the number of particles is a conserved quantity, so that we are allowed to separately consider eigenstates of $\mathbf{V}$ with even and odd number of fermions. The transfer matrix is finally written as

$$\begin{aligned}\mathbf{V}_{\pm} = {} & \exp\left[-z\sum_{n=1}^{\frac{N}{2}}\mathbf{C}_{n,1}\mathbf{C}_{n,2}\right]\exp\left[z\sum_{n=1}^{\frac{N}{2}}\mathbf{C}_{n,2}^{\dagger}\mathbf{C}_{n+1,1}^{\dagger}\right] \\ & \times\exp\left[(\ln z)\mathbf{N}_+\right]\exp\left[-z\sum_{n=1}^{\frac{N}{2}}\mathbf{C}_{n,2}\mathbf{C}_{n+1,1}\right] \\ & \times\exp\left[z\sum_{n=1}^{\frac{N}{2}}\mathbf{C}_{n,1}^{\dagger}\mathbf{C}_{n,2}^{\dagger}\right]\exp\left[(\ln z)\mathbf{N}_+\right],\end{aligned} \tag{27}$$

where

$$\mathbf{N}_+ = \sum_{n=1}^{\frac{N}{2}}\left(\mathbf{C}_{n,1}^{\dagger}\mathbf{C}_{n,1} + \mathbf{C}_{n,2}^{\dagger}\mathbf{C}_{n,2}\right). \tag{28}$$

The + subscript means that $\mathbf{N}_+$ is even and that we are assuming anticyclic boundary conditions,

$$\mathbf{C}^\dagger_{\frac{N}{2}+1,1} = -\mathbf{C}^\dagger_{1,1}; \qquad \mathbf{C}_{\frac{N}{2}+1,1} = -\mathbf{C}_{1,1}. \tag{29}$$

The − subscript means that $\mathbf{N}_+$ is odd and that we assume cyclic boundary conditions,

$$\mathbf{C}^\dagger_{\frac{N}{2}+1,1} = \mathbf{C}^\dagger_{1,1}; \qquad \mathbf{C}_{\frac{N}{2}+1,1} = \mathbf{C}_{1,1}. \tag{30}$$

It is important to emphasize that the set of eigenstates and eigenvalues of $\mathbf{V}$ is formed by the eigenstates and eigenvalues of $\mathbf{V}_+$ with an even number of fermions and the eigenstates and eigenvalues of $\mathbf{V}_-$ with an odd number of fermions. Other eigenstates and eigenvalues should be discarded.

We now have a quadratic form in terms of fermion operators and should introduce a discrete Fourier transformation,

$$\mathbf{C}_{n,\alpha} = \left(\frac{2}{N}\right)^{\frac{1}{2}} e^{-i\frac{\pi}{4}} \sum_q e^{iqn}\eta_{q,\alpha}, \tag{31}$$

$$\mathbf{C}^\dagger_{n,\alpha} = \left(\frac{2}{N}\right)^{\frac{1}{2}} e^{i\frac{\pi}{4}} \sum_q e^{-iqn}\eta^\dagger_{q,\alpha}, \tag{32}$$

where the index q is chosen according to the boundary conditions. For anticyclic boundary conditions, we choose

$$q \equiv l = \pm\frac{2\pi}{N}, \pm\frac{6\pi}{N}, \pm\frac{10\pi}{N}, \cdots, \pm\frac{(N-2)\pi}{N}. \tag{33}$$

For cyclic boundary conditions, we choose

$$q \equiv k = 0, \pm\frac{4\pi}{N}, \pm\frac{8\pi}{N}, \pm\frac{12\pi}{N}, \cdots, \pm\frac{(N-4)\pi}{N}, \pi. \tag{34}$$

The transfer matrix can now be written in terms of fermions in the Fourier space,

$$\mathbf{V}_\pm = \prod_q \mathbf{V}_q; \quad \mathbf{V}_q = \mathbf{V}_{2,q}\overline{\mathbf{V}}_{2,q}\mathbf{V}_{4,q}\mathbf{V}'_{2,q}\overline{\mathbf{V}}_{2,q}\mathbf{V}_{4,q}, \tag{35}$$

with

$$\begin{aligned}\mathbf{V}_{2,q} &= \exp\left(iz\eta_{q,1}\eta_{-q,2}\right),\\ \overline{\mathbf{V}}_{2,q} &= \exp\left(iz\eta^\dagger_{q,1}\eta^\dagger_{-q,2}\right),\end{aligned} \tag{36}$$

$$\begin{aligned}\mathbf{V}'_{2,q} &= \exp\left(-ize^{-iq}\eta_{-q,1}\eta_{q,2}\right),\\ \overline{\mathbf{V}}'_{2,q} &= \exp\left(-ize^{iq}\eta^\dagger_{-q,1}\eta^\dagger_{q,2}\right),\end{aligned} \tag{37}$$

and

$$\mathbf{V}_{4,q} = \exp\left[(\ln z)\left(\eta^\dagger_{q,1}\eta_{q,1} + \eta^\dagger_{q,2}\eta_{q,2}\right)\right], \tag{38}$$

where the set of values of q (either k or l) depends on the parity.

As $\mathbf{V}_q$ and $\mathbf{V}_{q'}$ commute, these operators can be diagonalized simultaneously, and the largest eigenvalue will be the product of the largest eigenvalues of $\mathbf{V}_q$. Also, we note that $\mathbf{V}_q$ includes couplings between operators with indices $(q, 1)$ and $(-q, 2)$, so that it becomes interesting to rewrite the equations in terms of a single kind of couplings,

$$\mathbf{V}_\pm = \prod_{q>0}\left(\mathbf{V}_q\mathbf{V}_{-q}\right) = \prod_{q>0}\left(\mathbf{A}_q\mathbf{B}_q\right). \tag{39}$$

Operator $\mathbf{A}_q$ is given by

$$\mathbf{A}_q = \mathbf{A}_{2,q}\overline{\mathbf{A}}'_{2,q}\mathbf{A}_{4,q}\mathbf{A}'_{2,q}\overline{\mathbf{A}}_{2,q}\mathbf{A}_{4,q}, \tag{40}$$

where

$$\begin{aligned}\mathbf{A}_{2,q} &= \exp\left(iz\eta_{q,1}\eta_{-q,2}\right),\\ \overline{\mathbf{A}}_{2,q} &= \exp\left(iz\eta^\dagger_{q,1}\eta^\dagger_{-q,2}\right),\end{aligned} \tag{41}$$

$$\begin{aligned}\mathbf{A}'_{2,q} &= \exp\left(-ize^{iq}\eta_{q,1}\eta_{-q,2}\right),\\ \overline{\mathbf{A}}'_{2,q} &= \exp\left(-ize^{-iq}\eta^\dagger_{q,1}\eta^\dagger_{-q,2}\right),\end{aligned} \tag{42}$$

$$\mathbf{A}_{4,q} = \exp\left[(\ln z)\left(\eta^\dagger_{q,1}\eta_{q,1} + \eta^\dagger_{-q,2}\eta_{-q,2}\right)\right], \tag{43}$$

which account for couplings between $(q, 1)$ and $(-q, 2)$. Operator $\mathbf{B}_q$ is defined by the same expressions with the replacement of q by $-q$.

We now introduce four states,

$$\Phi_0 = |0\rangle, \qquad \Phi_q = \eta^\dagger_{q,1}|0\rangle, \tag{44}$$

$$\Phi_{-q} = \eta^\dagger_{-q,2}|0\rangle, \qquad \Phi_{-qq} = \eta^\dagger_{-q,2}\eta^\dagger_{q,1}|0\rangle. \tag{45}$$

Matrix $\mathbf{A}_q$ is already diagonal in terms of Φ_q and Φ_{-q}. We then look at the space defined by Φ_0 and Φ_{-qq},

$$\begin{aligned}\mathbf{A}_q\Phi_0 = &\left(1 - 2z^2\cos q + 2z^4\right)\Phi_0\\ &+ iz\left(2z^2 - e^{-iq}\right)\Phi_{-qq},\end{aligned} \tag{46}$$

and

$$\mathbf{A}_q\Phi_{-qq} = -iz^3\left(2z^2 - e^{iq}\right)\Phi_0 + 2z^4\Phi_{-qq}. \tag{47}$$

In this two-dimensional subspace, it is easy to see that the eigenvalues are real and positive. The largest eigenvalue, given by

$$\begin{aligned}\lambda_q = \frac{1}{2}\Big[&\left(1 - 2z^2\cos q + 4z^4\right)\\ &+ \sqrt{\left(1 - 2z^2\cos q + 4z^4\right)^2 - 4z^4}\Big],\end{aligned} \tag{48}$$

is symmetric under the change $q \to -q$. Therefore, it also an eigenvalue of $\mathbf{B}_q$, and the largest eigenvalue of the transfer matrix is given by

$$\Lambda_{\pm}^{(\max)} = \prod_{q>0} \lambda_q^2. \tag{49}$$

Note that

$$\lambda_q \geq \lambda_{q=0} = \frac{1}{2}\Big[\left(1 - 2z^2 + 4z^4\right) + \sqrt{\left(1 - 2z^2 + 4z^4\right)^2 - 4z^4}\Big] \geq 0. \tag{50}$$

At the critical point, given by $z^2 = 1/2$, we have $\lambda_{q=0} = 1/2$.

We now write the free energy density in the thermodynamic limit,

$$\begin{aligned} -\beta f(z) &= \lim_{M,N\to\infty} \frac{2}{MN} \ln Z = \frac{1}{\pi} \int_0^{\pi} dq \ln \lambda_q \\ &= \frac{1}{\pi} \int_0^{\pi} dq \ln \left\{ \frac{1}{2}\left[1 - 2z^2 \cos q + 4z^4\right] \right. \\ &\quad + \frac{1}{2}\Big[\left(1 - 2z^2 \cos q + 4z^4\right)^2 \\ &\quad \left. - 4z^4 \Big]^{1/2} \right\}. \end{aligned} \tag{51}$$

Using the identity [18]

$$\begin{aligned} &\int_0^{\pi} d\phi \ln \left(a - b\cos\phi - c \sin\phi\right) \\ &\quad = \pi \ln \left[\frac{a + \sqrt{a^2 - b^2 - c^2}}{2} \right], \end{aligned} \tag{52}$$

with $a = 1 - 2z^2 \cos q + 4z^4$, $b = 2z^2$, and $c = 0$, we obtain

$$\begin{aligned} -\beta f(z) &= \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} dq_1 dq_2 \\ &\quad \times \ln \left[1 - 2z^2(\cos q_1 + \cos q_2) + 4z^4\right], \end{aligned} \tag{53}$$

which is the well-known result for the free energy of the dimer model on the 4–8 lattice, in agreement with the combinatorial calculations of Salinas and Nagle [8]. It is easy to see that the critical point is given by $z^2 = 1/2$.

We now compare the energy spectra of the transfer matrices associated with this dimer model and with an Ising model on the square lattice. Equations (41)–(43) can be diagonalized by a Bogoliubov–Valatin transformation [17],

$$\begin{aligned} \xi_{q,1} &= \cos\phi_q \, \eta_{q,1} + \sin\phi_q \, \eta_{-q,2}^{\dagger}, \\ \xi_{-q,2} &= \cos\phi_q \, \eta_{-q,2} - \sin\phi_q \, \eta_{q,1}^{\dagger}, \end{aligned} \tag{54}$$

$$\begin{aligned} \xi_{q,1}^{\dagger} &= \cos\phi_q \, \eta_{q,1}^{\dagger} + \sin\phi_q \, \eta_{-q,2}, \\ \xi_{-q,2}^{\dagger} &= \cos\phi_q \, \eta_{-q,2}^{\dagger} - \sin\phi_q \, \eta_{q,1}, \end{aligned} \tag{55}$$

which keeps the fermionic character of the new operators. It is straightforward to show that

$$\mathbf{V}_{\pm} = \exp\left[-\sum_{q>0} \sum_{\alpha=1,2} \varepsilon_q \left(\xi_{q,\alpha}^{\dagger} \xi_{q,\alpha} - \frac{1}{2} \right) \right], \tag{56}$$

where

$$\varepsilon_q = \ln \lambda_q, \tag{57}$$

with λ_q given by (48). We then see that the largest eigenvalue is associated with the zero particle state, or vacuum, Ψ_0, which has an even number of particles. Of course, Ψ_0 cannot be an acceptable eigenstate of V_-, which requires an odd number of particles. Therefore, in contrast to the Ising model, the largest eigenvalue is non-degenerate both above and below the critical temperature, which explains the highly symmetric specific heat and the disorder–disorder character of the transition in the dimer model on the 4–8 lattice. In the Ising case, there is a two-fold degeneracy below the critical temperature, which is an indication of symmetry breaking. There is no obvious symmetry breaking in the dimer model on the 4–8 lattices, which does correspond to the description of the phase transition in the hydrogen-bonded crystal SCD [8].

4 Dimer Model on the Brick Lattice

The brick (honeycomb) lattice is shown in Fig. 2, with a choice of dimer activities leading to a Kasteleyn phase transition [4]. The transfer matrix is obtained with the same type of operators that were used in the previous treatment of the dimer model on the 4–8 lattice. Taking into account that $h + v = MN/2$, we write the partition function

$$Z = \sum_{\{\mathbf{C}\}} z^v = z^{\frac{1}{2}MN} \sum_{\{\mathbf{C}\}} x^h, \tag{58}$$

where $\mathbf{C}$ is a configuration of v vertical dimers, with activity z, and h horizontal dimers with activity 1, and we have chosen $x = 1/z$.

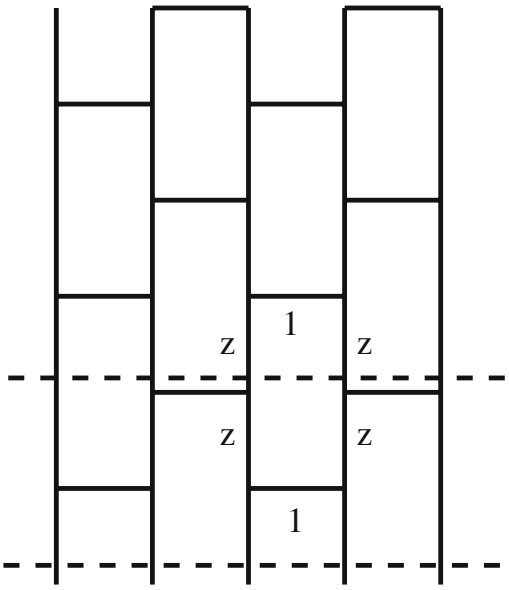

Fig. 2 Schematic representation of the brick lattice

According to the same steps already used to treat the 4–8 lattice, we write

$$Z = z^{\frac{MN}{2}} Tr\left(\mathbf{V}^{\frac{M}{2}}\right), \tag{59}$$

with the transfer matrix

$$\mathbf{V} = \mathbf{V}_2' \mathbf{V}_1 \mathbf{V}_2 \mathbf{V}_1 = \mathbf{V}_2' \overline{\mathbf{V}}_2, \tag{60}$$

where $\mathbf{V}_1$ is given by (2), and

$$\mathbf{V}_2' = \exp\left[x \sum_{n=1}^{\frac{N}{2}} \sigma_{2n}^- \sigma_{2n+1}^- \right],$$

$$\overline{\mathbf{V}}_2 = \exp\left[x \sum_{n=1}^{\frac{N}{2}} \sigma_{2n-1}^+ \sigma_{2n}^+ \right]. \tag{61}$$

It is straightforward to show that the largest eigenvalue can be written as

$$\Lambda^{\max} = \prod_{q>0} \lambda_q \lambda_q^*, \tag{62}$$

with

$$\lambda_q = \frac{1}{2}\left[2 - x^2 e^{iq} - x\left(x^2 e^{2iq} - 4e^{iq} \right)^{\frac{1}{2}} \right]. \tag{63}$$

In the thermodynamic limit, we obtain the free energy density,

$$-\beta f(z) = \frac{2}{MN} \ln Z(z)$$
$$= \frac{1}{4\pi} \int_0^{\pi} dq \ln \left\{ \left[\frac{1}{4} \left(1 + \sqrt{\Delta} + \sqrt{\left(1+\sqrt{\Delta}\right)^2 - 16z^4} \right) \right]^2 \right\}, \tag{64}$$

where

$$\Delta = 1 - 8z^2 \cos q + 16z^4 \geq 0. \tag{65}$$

Note that $\Delta = (1 - 4z^2)^2$ for $q = 0$, so that $\Delta = 0$ at the critical condition, $z = 1/2$. It is not difficult to check that (64) leads to the same results obtained from the Pfaffian technique [4].

5 Conclusions

We use the transfer matrix technique to write the canonical partition function associated with two paradigmatic two-dimensional dimer models. Introducing spin operators and using a standard Jordan–Wigner transformation, the problem is reduced to the calculation of the energy eigenvalues of a system of free fermions. The dimer model on a 4–8 lattice displays a symmetric transition, with a logarithmically diverging specific heat, which is reflected by the fermionic spectrum of energy. The Kasteleyn model displays a peculiar behavior, which is known to mimic the commensurate-incommensurate transition of domain wall systems. In both cases, we recover the well-known combinatorial results for the free energy of these closed-packed dimer models.

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