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Geometrothermodynamics of a Charged Black Hole of String Theory

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Abstract The thermodynamics of the Gibbons–Maeda–Garfinkle–Horowitz–Strominger charged black hole from string theory is reformulated within the context of the recently developed formalism of geometrothermodynamics. The geometry of the space of equilibrium states is curved, but we show that the thermodynamic curvature does not diverge at the phase transition point expected when the black hole solution becomes a naked singularity. This provides a counterexample to the conventional notion that such a divergence signals the occurrence of a second-order phase transition.

Keywords Quantum aspects of black holes · Thermodynamics · Strings and branes

1 Introduction

The thermodynamics of black holes has been studied extensively since the work of Hawking [1]. The notion of critical behaviour has arisen in several contexts for black holes, ranging from the Hawking–Page [2] phase transition in hot anti-de-Sitter space and the pioneering work by Davies [3] on the thermodynamics of Kerr–Newman black holes, to the idea that the extremal limit of various black hole families might themselves

be regarded as genuine critical points [4–7]. The use of geometry in statistical mechanics was pioneered by Ruppeiner [8–10] and Weinhold [11, 12], who suggested that the curvature of a metric defined on the space of parameters of a statistical mechanical theory could provide information about the phase structure. However, some puzzling anomalies become apparent when these methods are applied to the study of black hole thermodynamics. A possible resolution was suggested by Quevedo’s geometrothermodynamics (GTD) whose starting point [13] was the observation that standard thermodynamics was invariant with respect to Legendre transformations since one expects consistent results whatever starting potential one takes.

The formalism of GTD indicates that phase transitions occur at those points where the thermodynamic curvature is singular. In particular, these singularities represent critical points where the geometric description of GTD does not hold anymore, for example, at a point where a naked singularity of spacetime appears. In this paper, we apply the GTD formalism to the Gibbons–Maeda–Garfinkle–Horowitz–Strominger (GMGHS) charged black hole from string theory to investigate the behaviour of the thermodynamical curvature.

1.1 Geometrothermodynamics in Brief

The formulation of GTD is based on the use of contact geometry as a framework for thermodynamics. Consider the $(2n + 1)$ -dimensional thermodynamic phase space \mathcal{T} coordinatized by the thermodynamic potential Φ , extensive variables E^a , and intensive variables I^a , with $a = 1, \dots, n$. Consider on \mathcal{T} a non-degenerate metric $G = G(Z^A)$, with $Z^A = \{\Phi, E^a, I^a\}$, and the Gibbs

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1-form $\Theta = d\Phi - \delta_{ab} I^a dE^b$, with $\delta_{ab} = \text{diag}(1, 1, \dots, 1)$. If the condition $\Theta \wedge (d\Theta)^n \neq 0$ is satisfied, the set (\mathcal{T}, Θ, G) defines a contact Riemannian manifold. The Gibbs 1-form is invariant with respect to Legendre transformations, while the metric G is Legendre invariant if its functional dependence on Z^A does not change under a Legendre transformation. Legendre invariance guarantees that the geometric properties of G do not depend on the thermodynamic potential used in its construction.

The n -dimensional subspace $\mathcal{E} \subset \mathcal{T}$ is called the space of equilibrium thermodynamic states if it is determined by the smooth mapping

$$\begin{aligned} \varphi: \mathcal{E} &\longrightarrow \mathcal{T} \\ (E^a) &\longmapsto (\Phi, E^a, I^a) \end{aligned} \quad (1)$$

with $\Phi = \Phi(E^a)$, and the condition $\varphi^*(\Theta) = 0$ is satisfied, i.e.

$$d\Phi = \delta_{ab} I^a dE^b \quad (2)$$

$$\frac{\partial \Phi}{\partial E^a} = \delta_{ab} I^b. \quad (3)$$

The first of these equations corresponds to the first law of thermodynamics, whereas the second one is usually known as the condition for thermodynamic equilibrium (the intensive thermodynamic variables are dual to the extensive ones). The mapping φ defined above implies that the equation $\Phi = \Phi(E^a)$ must be explicitly given. This relation is known as the fundamental equation from which all the equations of state can be derived. Finally, the second law of thermodynamics is equivalent to the convexity condition on the thermodynamic potential,

$$\partial^2 \Phi / \partial E^a \partial E^b \geq 0. \quad (4)$$

The thermodynamic potential satisfies the homogeneity condition $\Phi(\lambda E^a) = \lambda^\beta \Phi(E^a)$ for constant parameters λ and β . Therefore, it satisfies the Euler's identity,

$$\beta \Phi(E^a) = \delta_{ab} I^b E^a, \quad (5)$$

and using the first law of thermodynamics, we obtain the Gibbs–Duhem relation,

$$(1 - \beta) \delta_{ab} I^a dE^b + \delta_{ab} E^a dI^b = 0. \quad (6)$$

Now, we will formulate the main statement of geometrothermodynamics. A thermodynamic system is described by a metric G on \mathcal{T} which is called a thermodynamic metric [13] if it is invariant with respect to

transformations which do not modify the contact structure of \mathcal{T} . In particular, G must be invariant with respect to Legendre transformations in order for geometrothermodynamics to be able to describe thermodynamic properties in terms of geometric concepts in a manner which must be invariant with respect to changes of the thermodynamic potential.

As is shown in [13], a Legendre invariant metric G induces a Legendre invariant metric g on \mathcal{E} defined by the pullback φ^* as $g = \varphi^*(G)$. There is a vast number of metrics on \mathcal{T} that satisfy the Legendre invariance condition. For instance, the metric structure

$$G = \Theta^2 + (\delta_{ab} E^a I^b)(\delta_{cd} dE^c dI^d) \quad (7)$$

is Legendre invariant (because of the invariance of the Gibbs 1-form) and induces on \mathcal{E} the Quevedo's metric

$$g = \Phi(E^c) \frac{\partial^2 \Phi}{\partial E^a \partial E^b} dE^a dE^b. \quad (8)$$

The geometry described by the metric $g = \varphi^*(G)$ is invariant with respect to arbitrary diffeomorphisms performed on \mathcal{E} .

Now, Weinhold's metric g^W is defined as the Hessian in the mass representation [11, 12], whereas Ruppeiner's metric g^R is given as minus the Hessian in the entropy representation [8–10],

$$g^W = \frac{\partial^2 M}{\partial E^a \partial E^b} dE^a dE^b \quad (9)$$

$$g^R = -\frac{\partial^2 S}{\partial F^a \partial F^b} dF^a dF^b \quad (10)$$

where $E^a = \{S, Q\}$ and $F^a = \{M, Q\}$. As is well known, [13], Weinhold's and Ruppeiner's metrics are not Legendre invariant and from the analysis given above, it is clear that these metrics must be related by $g^W = \left(\frac{\partial S}{\partial M}\right)^{-1} g^R = T g^R$. In GTD, the simplest way to reach the Legendre invariance for g^W is to apply a conformal transformation with the thermodynamic potential as the conformal factor as given in (8),

$$g = M \frac{\partial^2 M}{\partial E^a \partial E^b} dE^a dE^b = M g^W, \quad (11)$$

or, using (10), it can be written in terms of the Ruppeiner's metric as

$$g = -M \left(\frac{\partial S}{\partial M}\right)^{-1} \frac{\partial^2 S}{\partial F^a \partial F^b} dF^a dF^b = M T g^R. \quad (12)$$

2 The GMGHS Black Hole

The low-energy effective action of the heterotic string theory in four dimensions is given by

$$\mathcal{A} = \int d^4x \sqrt{-\tilde{g}} e^{-\psi} \left(-R + \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} - \tilde{G}^{\mu\nu} \partial_\mu \psi \partial_\nu \psi + \frac{1}{8} F_{\mu\nu} F^{\mu\nu} \right), \quad (13)$$

where R is the Ricci scalar, $\tilde{G}_{\mu\nu}$ is the metric that arises naturally in the σ model,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad (14)$$

is the Maxwell field associated with a $U(1)$ subgroup of $E_8 \times E_8$, ψ is the dilaton field and

$$H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu} - [\Omega_3(A)]_{\mu\nu\rho}, \quad (15)$$

where $B_{\mu\nu}$ is the antisymmetric tensor gauge field and

$$[\Omega_3(A)]_{\mu\nu\rho} = \frac{1}{4} (A_\mu F_{\nu\rho} + A_\nu F_{\rho\mu} + A_\rho F_{\mu\nu}) \quad (16)$$

is the gauge Chern–Simons term. Considering $H_{\mu\nu\rho} = 0$ and working in the conformal Einstein frame, the action becomes

$$\mathcal{A} = \int d^4x \sqrt{-\tilde{g}} (-R + 2(\nabla\psi)^2 + e^{-2\psi} F^2), \quad (17)$$

where the Einstein frame metric $\tilde{g}_{\mu\nu}$ is related to $\tilde{G}_{\mu\nu}$ through the dilaton,

$$\tilde{g}_{\mu\nu} = e^{-\psi} \tilde{G}_{\mu\nu}. \quad (18)$$

The charged black hole solution, known as the GMGHS solution, is given by [14–18]

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r \left(r - \frac{Q^2 e^{-2\psi_0}}{M} \right) d\Omega^2 \quad (19)$$

$$e^{-2\psi} = e^{-2\psi_0} \left(1 - \frac{Q^2}{Mr} \right) \quad (20)$$

$$F = Q \sin\theta d\theta \wedge d\vartheta \quad (21)$$

where M is the mass of the black hole, Q the electric charge and ψ_0 is the asymptotic value of the dilaton. In addition to their mass M and the charge Q , the GMGHS solution is also characterized by the dilaton charge

$$D = -\frac{Q^2 e^{-2\psi_0}}{2M}. \quad (22)$$

Note that this metric become Schwarzschild's solution if $Q = 0$ and has a spherical event horizon at

$$r_H = 2M \quad (23)$$

with an area given by

$$A = 4\pi r_H \left(r_H - \frac{Q^2 e^{-2\psi_0}}{M} \right) \quad (24)$$

$$A = 4\pi r_H^2 - 8\pi Q^2 e^{-2\psi_0}. \quad (25)$$

Equation (25) tell us that the area of the horizon goes to zero if

$$r_H^2 = 2Q^2 e^{-2\psi_0}, \quad (26)$$

i.e. the GMGHS solution becomes a naked singularity when

$$M^2 \leq \frac{1}{2} Q^2 e^{-2\psi_0}. \quad (27)$$

The Hawking temperature is

$$T = \frac{\kappa}{2\pi} = \frac{1}{8\pi M} \quad (28)$$

which is independent of charge. Finally, the electric potential computed on the horizon of the black hole is

$$\phi = \frac{Q}{r_H} e^{-2\psi_0}, \quad (29)$$

while the entropy is

$$S = \frac{A}{4} = \pi r_H^2 - 2\pi Q^2 e^{-2\psi_0}. \quad (30)$$

All these parameters are related by means of the first law of black hole thermodynamics $dM = TdS + \phi dQ$ and for a given fundamental equation $M = M(S, Q)$ we have the conditions for thermodynamic equilibrium

$$T = \frac{\partial M}{\partial S}, \quad \phi = \frac{\partial M}{\partial Q}. \quad (31)$$

Therefore, the phase space \mathcal{T} for black hole's geometrothermodynamics is five-dimensional with coordinates $Z^A = \{M, S, Q, T, \phi\}$ and the fundamental Gibbs 1-form is given by $\Theta = dM - TdS - \phi dQ$. On the other hand, the space of thermodynamic equilibrium states \mathcal{E} is two-dimensional with coordinates $E^a = \{S, Q\}$, and is defined by means of the mapping

$$\varphi : \{S, Q\} \mapsto \left\{ M(S, Q), S, Q, \frac{\partial M}{\partial S}, \frac{\partial M}{\partial Q} \right\} \quad (32)$$

with $\varphi^*(\Theta) = 0$. Here, the mass M plays the role of thermodynamic potential that depends on the extensive variables S and Q , and under this representation, the

metric structure in the phase space \mathcal{T} can be written from (7) as

$$G = (dM - TdS - \phi dQ)^2 + (TS + \phi Q)(dTdS + d\phi dQ). \quad (33)$$

However, Legendre transformations allow us to introduce a set of additional thermodynamic potentials which depend on different combinations of extensive and intensive variables. In particular, it is possible to consider the entropy representation, $S = S(M, Q)$, where the Gibbs 1-form of the phase space can be chosen as

$$\Theta_S = dS - \frac{1}{T}dM + \frac{\phi}{T}dQ. \quad (34)$$

Then, the space of equilibrium states is defined by the smooth mapping

$$\varphi_S : \{M, Q\} \mapsto \{M, S(M, Q), Q, T(M, Q), \phi(M, Q)\}, \quad (35)$$

with

$$\frac{1}{T} = \frac{\partial S}{\partial M}, \quad \frac{\phi}{T} = -\frac{\partial S}{\partial Q}, \quad (36)$$

and such that $\varphi_S^*(\Theta_S) = 0$. In this representation, the second law of thermodynamics corresponds to the concavity condition of the entropy function. Additional representations can easily be analysed within GTD, and the only object that is needed in each case is the smooth mapping φ which guarantees the existence of a well-defined space of equilibrium states. However, the thermodynamic properties of black holes must be independent of the representation.

Using the entropy representation, (30) let us write the entropy as the potential

$$S(M, Q) = 4\pi M^2 - 2\pi Q^2 e^{-2\psi_0}. \quad (37)$$

Therefore, the Ruppenier's metric is given by

$$g^R = -8\pi dM dM + 4\pi e^{-2\psi_0} dQ dQ \quad (38)$$

and the invariant metric under Legendre transformation induced in \mathcal{E} is calculated using (12) as

$$g = -dM dM + \frac{e^{-2\psi_0}}{2} dQ dQ. \quad (39)$$

Here, the curvature vanishes, meaning that the GMGHS black holes do not show any statistical thermodynamic interaction and no phase transition structure. On the other hand, considering the mass representation, we have the potential

$$M(S, Q) = \sqrt{\frac{S}{4\pi} + \frac{Q^2 e^{-2\psi_0}}{2}}, \quad (40)$$

and the Weinhold's metric is

$$g^W = -\frac{1}{64\pi^2 M^3} dS dS + \frac{S e^{-2\psi_0}}{8\pi M^3} dQ dQ - \frac{Q e^{-2\psi_0}}{16\pi M^3} dQ dS. \quad (41)$$

Using (11), we obtain Quevedo's invariant metric,

$$g = -\frac{1}{64\pi^2 M^2} dS dS + \frac{S e^{-2\psi_0}}{8\pi M^2} dQ dQ - \frac{Q e^{-2\psi_0}}{16\pi M^2} dQ dS, \quad (42)$$

but this time, the curvature escalar gives

$$R = \frac{8\pi (2\pi S Q^2 e^{-2\psi_0} + 4\pi^2 Q^4 e^{-4\psi_0} - S^2)}{(S + \pi Q^2 e^{-2\psi_0})^2 (S + 2\pi Q^2 e^{-2\psi_0})}. \quad (43)$$

Note that R diverge when $S = -\pi Q^2 e^{-2\psi_0}$ or $S = -2\pi Q^2 e^{-2\psi_0}$, but although a negative value in S have been related with the indication of a new type of instability (where a transition between a black hole with negative entropy and a black hole with positive entropy occurs [19–21]), in this case there is no indication of such a process. Therefore, we conclude that a negative value of entropy has no physical interest for us while for positive values of entropy there are no curvature singularities, showing that GMGHS metric has no extremal black hole configurations or phase transitions [22]. However, it is not in accordance with the intuitive expectation that naked singularities show the limit of applicability of black hole thermodynamics [22–24]. Remember that the GMGHS solution becomes a naked singularity for $M^2 \leq \frac{1}{2} Q^2 e^{-2\psi_0}$ (i.e. $S \leq 0$), but R has no singular behaviour at this point.

On the other hand, when $S = (1 + \sqrt{5})\pi Q^2 e^{-2\psi_0}$ or $M^2 = \left(\frac{3+\sqrt{5}}{4}\right) Q^2 e^{-2\psi_0}$ the scalar curvature vanishes identically, leading to a flat geometry. At this point the scalar curvature changes its sign, and it is the only point with a positive entropy where this happens. A similar situation appears in [23] for Reissner–Nordstrom solution. There, the author argues that in GTD a phase transition can also be described by a change of sign of the scalar curvature, passing through a state of flat geometry. However, for the GMGHS black hole there is no indication of a phase transition at this point.

Since the thermodynamic interaction is always present and there are no phase transitions, we conclude that the GMGHS solution corresponds to a stable thermodynamic configuration and that the geometrical

description presented here does not break down even for the naked singularity case.

3 Conclusion

The formalism of Quevedo's GTD indicates that phase transitions would occur at those points where the thermodynamic curvature R is singular. As in ordinary thermodynamics, near the points of phase transitions, equilibrium thermodynamics is not valid and therefore, one expects that singularities represent critical points where the geometric description of GTD does not hold anymore and must give place to a more general approach.

However, the relation between the singularities of the specific heat and the thermodynamic curvature calculated with the Quevedo's metric [22] is not consistent for the GMGHS black hole. Our results show that the metric structure of the thermodynamical manifold for the GMGHS solution does not have curvature singularities, which is not in accordance with the intuitive expectation that naked singularities, as the one that appears for this metric when $M^2 \leq \frac{1}{2} Q^2 e^{-2\psi_0}$, show the limit of applicability of black hole thermodynamics.

It is clear that the phase manifold contains information about thermodynamic systems; however, it is necessary a further exploration of the geometric properties in order to understand where the thermodynamic information is encoded. A more detailed investigation along these lines will be reported in the future.

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