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PARTICLES AND FIELDS

Discrete Energy Spectrum for a Spin-1/2 Quantum Particle Under the Influence of a Constant Force Field due to the Presence of Topological Defects

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Abstract We study a spin-1/2 quantum particle under a constant force field in the presence of a uniform distribution of parallel screw dislocations. We solve the Schröndiger equation, exactly, and find a discrete energy spectrum.

Keywords Screw dislocation · Torsion field · Bound states

1 Introduction

The description of linear topological defects in solids with the mathematical tools of general relativity proposed by Katanaev and Volovich [1] has attracted a great deal of attention [2]. In this description, disclinations, screw dislocations, and other linear topological defects are represented by the spatial part of the line element of the cosmic string spacetime and the cosmic dislocation spacetime, respectively. Recently, linear topological defects have been studied by the Katanaev–Volovich approach [1] in such systems as circular orbits [3–5], quantum scattering problems [6], bound states [7–9], and in the analog of the Aharonov–Bohm effect for bound states [10–12]. The study of the influence of linear topological defects has been extended to the interaction between the topological defect and the harmonic oscillator [13, 14], the application of the self-adjoint extension method [15–17], geometric phases for neutral particles [18–23], the Landau quantization for a charged particle and a neutral particle [24–30], and the Holonomic quantum computation [31, 32]. An interesting work [33] has established an analogy between a uniform distribution of parallel screw dislocations and a uniform magnetic field, shown that Landau quantization can be achieved and coined the expression "elastic Landau quantization" [33].

In this paper, we study the influence of a uniform distribution of screw dislocations on the quantum dynamics of a spin-1/2 particle under the action of a constant force field. The motion of a spinless quantum particle in a uniform force field was discussed in [34, 35], which found a unique solution to the Schrödinger equation for each fixed value of the energy. Furthermore, bound states for a spinless quantum particles were obtained for the quantum bouncer [36, 37] and in the interaction with a harmonic oscillator [38]. In this work, we solve the Schrödinger equation, exactly, and show that a discrete energy spectrum for a spin-1/2 quantum particle results from the presence of a uniform distribution of parallel screw dislocations. Moreover, we show that the energy levels are degenerate.

The structure of this paper is as follows: in Section 2, we present the line element representing the uniform distribution of parallel screw dislocations and study the quantum dynamics of a spin-1/2 particle under the action of a uniform force field in the presence of this uniform distribution of topological defects. In Section 3, we present our conclusions.

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2 Spin-1/2 Particle in a Elastic Landau System Under the Action of a Constant Force Field

We wish to study the behavior of a spin-1/2 particle under the action of a constant force field in an elastic Landau system. The behavior of a spinless quantum particle in an elastic Landau system, i.e., in an elastic medium with a uniform distribution of parallel screw disclinations, was studied in [33]. This uniform distribution of parallel screw dislocations can be described by the line element [33]

$$ds^2 = d\rho^2 + \rho^2 d\varphi^2 + \left(dz + \Omega\rho^2 d\varphi\right)^2. \tag{1}$$

Here, $\Omega = b_i\, A/2$, where A is the two-dimensional density of dislocations. Given the cylindrical symmetry, we can work with curvilinear coordinates and use the formulation of quantum field theory in curved spacetime [39] to describe this system. In this approach, the spin of the quantum particle is defined locally through a noncoordinate basis $\hat{\theta}^a = e^a_{\ \mu}(x)\, dx^\mu$, where the components $e^a_{\ \mu}(x)$ are called triads [2, 39, 40]. The triads satisfy the relation $g_{\mu\nu} = e^a_{\ \mu}(x)\, e^b_{\ \nu}(x)\, \eta_{ab}$, where $\eta_{ab} = \mathrm{diag}\,(+++)$. The Greek indices indicate the cylindrical coordinates (related to the topological defect) and the Latin indices indicate the local reference frame of the observer (related to the flat space, that is, the absence of defect). The inverse of the triads is given by $dx^\mu = e^\mu_{\ a}(x)\, \hat{\theta}^a$, where they satisfy $e^a_{\ \mu}(x)\, e^\mu_{\ b}(x) = \delta^a_{\ b}$ and $e^\mu_{\ a}(x)\, e^a_{\ \nu}(x) = \delta^\mu_{\ \nu}$. In this work, we choose the following triad field:

$$\hat{\theta}^1 = d\rho;\quad \hat{\theta}^2 = \rho\, d\varphi;\quad \hat{\theta}^3 = dz + \Omega\rho^2 d\varphi. \tag{2}$$

In the formulation of the quantum field theory in curved spacetime and in the presence of a torsion field [39, 41], the partial derivative ∂_μ becomes the covariant derivative $\nabla_\mu = \partial_\mu + \Gamma_\mu(x)$, where $\Gamma_\mu(x) = \frac{i}{4}\left[\omega_{\mu ab}(x) + K_{\mu ab}(x)\right]\Sigma^{ab}$ is called the spinorial connection and Σ^{ab} is defined for two-spinors in the form $\Sigma^{ab} = \frac{i}{2}\left[\sigma^a, \sigma^b\right]$, where $\sigma^0 = I$ is the identity matrix. The one-form connection $K_{\mu ab}(x)$ is related to the contortion tensor via the expression [20, 21, 41]

$$K_{\mu ab}(x) = K_{\beta\nu\mu}(x) \times \left[e^\nu_{\ a}(x)\, e^\beta_{\ b}(x) - e^\nu_{\ b}(x)\, e^\beta_{\ a}(x)\right]. \tag{3}$$

Moreover, the contortion tensor is related to the torsion tensor by $K^\beta_{\ \nu\mu} = \frac{1}{2}\left(T^\beta_{\ \nu\mu} - T_\nu^{\ \beta}{}_\mu - T_\mu^{\ \beta}{}_\nu\right)$. Note that the torsion tensor is antisymmetric in the last two indices, while the contortion tensor is antisymmetric in the first two indices. Following these definitions, it is convenient to represent the torsion tensor in terms of three irreducible components: the trace vector $T_\mu = T^\beta_{\ \mu\beta}$, the axial vector $S^\alpha = \epsilon^{\alpha\beta\nu\mu}\, T_{\beta\nu\mu}$, and the tensor $q_{\beta\nu\mu}$, which satisfies the conditions $q^\beta_{\ \mu\beta} = 0$ and $\epsilon^{\alpha\beta\nu\mu}\, q_{\beta\nu\mu} = 0$. In this way, the torsion tensor becomes $T_{\beta\nu\mu} = \frac{1}{3}\left(T_\nu\, g_{\beta\mu} - T_\mu\, g_{\beta\nu}\right) - \frac{1}{6}\,\epsilon_{\beta\nu\mu\gamma}\, S^\gamma + q_{\beta\nu\mu}$, and we can rewrite the one-form connection (3) in terms of these irreducible components [20, 21, 41]. We also have that $\omega^a_{\ b} = \omega_{\mu\ \ b}^{\ a}(x)\, dx^\mu$ is the one-form connection. Both one-form connections can be obtained by solving the Cartan structure equations $T^a = d\hat{\theta}^a + \omega^a_{\ b} \wedge \hat{\theta}^b$ [40], where the operator d is the exterior derivative, the symbol $\wedge$ denotes the wedge product, and T^a is the torsion two-form.

As discussed in [42, 43], we can either follow the treatment of the Dirac equation [43] and use the expression for the covariant derivative by using the spinorial connection to write the Schrödinger equation in the Minkowsky spacetime background in cylindrical coordinates or transform the coordinates directly from Cartesian to cylindrical coordinates (to treat the Dirac equation in this fashion, we would have to write the σ^μ matrices in cylindrical coordinates). By substituting $\hat{p}_i = -i\partial_i$ for $\hat{\pi}_i = -i\partial_i - i\Gamma_i$, we can rewrite the Schrödinger equation in the form [20, 21]

$$i\frac{\partial\psi}{\partial t} = \frac{1}{2m}\left[\vec{p} + \vec{\Xi}\right]^2\psi + \frac{1}{8}\,\vec{\sigma}\cdot\vec{S}\,\psi + V(\rho)\,\psi, \tag{4}$$

where the vector $\vec{\Xi}$ is defined in such a way that its components are given in the local reference frame by $\Xi_k = +\frac{1}{2}\,\sigma^3\, e^\varphi_{\ k}(x) - \frac{1}{8}\, S^0\,\sigma_k$. By solving the Cartan structure equations $T^a = d\hat{\theta}^a + \omega^a_{\ b}\,\hat{\theta}^b$ [40], we obtain $\omega_{\varphi\ 2}^{\ 1}(x) = -\omega_{\varphi\ 1}^{\ 2}(x) = -1$ and $T^3 = 2\Omega\rho\, d\rho \wedge d\varphi$. Hence, we obtain one non-null component of the axial four-vector S^μ, which is $S^0 = -4\Omega$. With these information in our hands, the Schrödinger equation (4) becomes

$$\begin{aligned} i\frac{\partial\psi}{\partial t} = &-\frac{1}{2m}\left[\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2} - 2\Omega\,\frac{\partial^2}{\partial z\partial\varphi}\right.\\ &\left. + \left(1 + \Omega^2\rho^2\right)\frac{\partial^2}{\partial z^2}\right]\psi + \frac{1}{2m}\frac{i\sigma^3}{\rho^2}\frac{\partial\psi}{\partial\varphi}\\ &-\frac{i\sigma^3}{2m}\,\Omega\frac{\partial\psi}{\partial z} + \frac{1}{8m\rho^2}\psi + \frac{\Omega^2}{8m}\psi + V(\rho)\,\psi. \end{aligned} \tag{5}$$

From (5) we see that ψ is an eigenfunction of σ^3, whose eigenvalues are $s = \pm 1$ and that the Hamiltonian on the right-hand side commutes with the operators

$\hat{J}_z = -i\partial_\varphi + (1/2)\sigma^3$ and $\hat{p}_z = -i\partial_z$; thus, we can write the solution of the Schrödinger equation (5) on the basis of the eigenfunctions of the operators $\hat{J}_z$ and $\hat{p}_z$, that is, $\psi_s = e^{-i\mathcal{E}t}\, e^{i(l+\frac{1}{2})\varphi}\, e^{ikz}\, R_s(\rho)$, where $l = 0, \pm 1, \pm 2, \ldots$ and k is a constant. Thus, substituting this general solution into the second-order differential equation (5), we have that

$$\mathcal{E} R_s = -\frac{1}{2m}\left[\frac{d^2 R_s}{d\rho^2} + \frac{1}{\rho}\frac{dR_s}{d\rho}\right] + \frac{1}{2m}\frac{\gamma_s^2}{\rho^2} R_s - \frac{\Omega k}{m}\gamma_s\, R_s + \frac{1}{2m}\left(k + s\frac{\Omega}{2}\right)^2 R_s + \Omega^2 k^2 \rho^2\, R_s + V(\rho)\; R_s, \tag{6}$$

where we have defined $\gamma_s = l + \frac{1}{2}(1-s)$. Now, let us consider a spin-1/2 quantum particle under the action of a constant radial force whose modulus is F. Hence, (6) becomes

$$\mathcal{E} R_s = -\frac{1}{2m}\left[\frac{d^2 R_s}{d\rho^2} + \frac{1}{\rho}\frac{dR_s}{d\rho}\right] + \frac{1}{2m}\frac{\gamma_s^2}{\rho^2} R_s - \frac{\Omega k}{m}\gamma_s\, R_s + \frac{1}{2m}\left(k + s\frac{\Omega}{2}\right)^2 R_s + \Omega^2 k^2 \rho^2\, R_s + F\,\rho\, R_s, \tag{7}$$

The wave function of the spin-1/2 quantum particle resulting from the solution of (7) must be regular at the origin. To discuss a general solution including the origin, [15–17] applied the self-adjoint extension approach. Here, we focus the functions R_s that are regular at the origin, which allows us to write the solution of (7) in the form:

$$R_s(\rho) = e^{-\frac{1}{2}\Omega k \rho^2}\, e^{-\frac{F}{\Omega k}\rho}\, \rho^{|\zeta_s|}\, H_s\left(\sqrt{\Omega k}\rho\right), \tag{8}$$

where we have considered only the positive values for $k > 0$. The function H_s is the Heun biconfluent function: $H_s = H\left[2|\gamma_s|, \frac{F}{(\Omega k)^{3/2}}, \frac{\beta_s}{\Omega, k} + \frac{F^2}{4\Omega^3 k^3}, 0, \sqrt{\Omega k}\rho\right]$ [44], where in the third parameter,

$$\beta_s \equiv 2m\mathcal{E} + 2\Omega k\gamma_s - \left(k + s\frac{\Omega}{2}\right)^2. \tag{9}$$

To make the radial solution finite everywhere, we then require the Heun biconfluent series to become a polynomial, a condition that will be satisfied if [44–46]

$$\frac{\beta_s}{\Omega k} + \frac{F^2}{4\Omega^3 k^3} = 2(n+1) + 2|\gamma_s| \qquad (n = 0, 1, 2, \ldots). \tag{10}$$

Thus, to ensure that the wave function be finite everywhere, the energy levels for the spin-1/2 quantum particle under a constant force field must be of the form

$$\mathcal{E}_{n,l} = \frac{\Omega k}{m}\left[n + |\gamma_s| + \gamma_s + 1\right] - \frac{1}{2m}\frac{F^2}{4\Omega^2 k^2} - \frac{1}{2m}\left[k - s\frac{\Omega}{2}\right]^2 \qquad (n = 0, 1, 2, \ldots). \tag{11}$$

Note that each energy level n is degenerate. To obtain the energy levels of the bound states, previous studies of the quantum bouncer [36, 37] have imposed upon the quantum particle a new potential that restricts its motion and hence leads to nondegenerate states, for as shown in [34] and [35], the normalization of the wave function for a fixed value of the energy yields a unique, i. e., nondegenerate solution of the Schrödinger equation. Here, by contrast, we have shown that when the spin-1/2 quantum particle is subject to a constant radial force, the presence of a uniform distribution of screw dislocations parallel to the z-axis yields normalized solutions of the Schrödinger equation in correspondence with a discrete spectrum comprising only degenerate energy levels.

3 Conclusions

In this paper, we have studied the quantum dynamics of a spin-1/2 particle under a constant force field in the presence of a uniform distribution of screw dislocations. We have seen that the Schröndiger equation can be solved exactly and that a discrete spectrum constituted of degenerate energy levels results.

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References

1. M.O. Katanaev, I.V. Volovich, Ann. Phys. (NY) **216**, 1 (1992)
2. H. Kleinert, in *Gauge Fields in Condensed Matter*, vol. 2 (World Scientific, Singapore, 1989)
3. F. Moraes, Phys. Lett. A **214**, 189 (1996)
4. A. de Padua et al., Phys. Lett. A **238**, 153 (1998)
5. K. Bakke, A.M. de M. Carvalho, C. Furtado, Eur. Phys. J. C **63**, 149 (2009)
6. C. Furtado, V.B. Bezerra, F. Moraes, Phys. Lett. A **289**, 160 (2001)
7. C. Furtado, F. Moraes, Phys. Lett. A **188**, 394 (1994)
8. E. Aurell, J. Phys. A: Math. Gen. **32**, 571 (1999)
9. K. Bakke, J. Math. Phys. **51**, 093516 (2010)
10. M. Peshkin, A. Tonomura, in *The Aharonov–Bohm Effect*. Lecture Notes in Physics, vol. 340 (Springer, Berlin, 1989)
11. S. Azevedo, F. Moraes, Phys. Lett. A **246**, 374 (1998)

12. C. Furtado, C.A. de Lima Ribeiro, S. Azevedo, Phys. Lett. A **296**, 171 (2002)
13. C. Furtado, F. Moraes, J. Phys. A: Math. Gen **33**, 5513 (2000)
14. S. Azevedo, Phys. Lett. A **288**, 33 (2001)
15. C. Filgueiras, F. Moraes, Phys. Lett. A **361**, 13 (2007)
16. C. Filgueiras, F. Moraes, Ann. Phys. (NY) **323**, 3150 (2008)
17. C. Filgueiras et al., Ann. Phys. (NY) **325**, 2529 (2010)
18. K. Bakke, J.R. Nascimento, C. Furtado, Phys. Rev. D **78**, 064012 (2008)
19. K. Bakke, C. Furtado, Ann. Phys. (Berlin) **522**, 477 (2010)
20. K. Bakke, et al., Eur. Phys. J. C **60**, 501 (2009)
21. K. Bakke, et al., Eur. Phys. J. C **64**, 169 (2009)
22. K. Bakke, C. Furtado, Phys. Rev. D **80**, 024033 (2009)
23. K. Bakke, C. Furtado, Eur. Phys. J. C **69**, 531 (2010)
24. C. Furtado, et al., Phys. Lett. A **195**, 90 (1994)
25. C. Furtado, F. Moraes, EPL **45**, 279 (1999)
26. G.A. Marques, et al., J. Phys. A: Math. Gen. **34**, 5945 (2001)
27. K. Bakke, et al., Phys. Rev. D **79**, 024008 (2009)
28. K. Bakke, et al., Cent. Eur. J. Phys. **8**, 893 (2010)
29. K. Bakke, C. Furtado, Phys. Rev. D **82**, 084025 (2010)
30. K. Bakke, C. Furtado, Int. J. Mod. Phys. D **19**, 1 (2010)
31. K. Bakke, C. Furtado, S. Sergeenkov, EPL **87**, 30002 (2009)
32. K. Bakke, C. Furtado, Quantum Inf. Comp. **11**, 0444 (2011)
33. A.L. Silva Netto, C. Furtado, J. Phys., Condens. Matter **20**, 125209 (2008)
34. L.D. Landau, E.M. Lifshitz, *Quantum Mechanics, the Nonrelativist Theory* 3rd edn. (Pergamon, Oxford, 1977)
35. L.E. Ballentine, *Quantum Mechanics, a Modern Development* (World Scientific, Singapore, 1998)
36. R.L. Gibbs, Am. J. Phys. **43**, 25 (1975)
37. R.D. Desko, D.J. Bord, Am. J. Phys. **51**, 82 (1983)
38. S. Azevedo, Int. J. Quant. Chem. **101**, 127 (2005)
39. N.D. Birrell, P.C.W. Davies, *Quantum Fields in Curved Space* (Cambridge University Press, Cambridge, 1982)
40. M. Nakahara, *Geometry, Topology and Physics* (Institute of Physics, Bristol, 1998)
41. I.L. Shapiro, Phys. Rep. **357**, 113 (2002)
42. P. Schlüter, K.-H. Wietschorke, W. Greiner, J. Phys. A **16**, 1999 (1983)
43. K. Bakke, C. Furtado, Phys. Rev. A **80**, 032106 (2009)
44. A. Ronveaux, *Heun's Differential Equations* (Oxford University Press, Oxford, 1995)
45. B. Leaute, G. Marcilhacy, J. Phys. A: Math. Gen. **19**, 3527 (1986)
46. E.R. Arriola, A. Zarzo, J.S. Dehesa, J. Comput. Appl. Math. **37**, 161 (1991)