Brazilian Journal of Physics
ISSN: 0103-9733
luizno.bjp@gmail.com
Sociedade Brasileira de Física
Brasil

Braz J Phys (2011) 41:184–194
DOI 10.1007/s13538-011-0032-3

PARTICLES AND FIELDS

A Note on Classification of Cylindrically Symmetric Non-static Space–Times According to Their Teleparallel Killing Vector Fields in the Teleparallel Theory of Gravitation

Ghulam Shabbir · Suhail Khan · M. Jamil Amir

Received: 16 May 2011 / Published online: 29 July 2011

Abstract In this paper, we explored the conservation laws of cylindrically symmetric non-static space–times by using direct integration technique. This classification also covers non-static plane symmetric space–times, static cylindrically symmetric space–times and plane symmetric static space–times. In this paper, we will only present the results of non-static cylindrically symmetric and non-static plane symmetric space–times. The results of static cylindrically symmetric space–times and plane static space–times can be found in Shabbir and Khan (Mod Phys Lett A 25:525, 2010). It turns out that the non-static cylindrically symmetric space–times admit four, five, or seven conservation laws. It is important to note that the above space–times admit at least one or at the most four extra conservation laws.

Keywords Weitzenböck geometry · Torsion · Conservation laws

PACS · 04.20.-q · 04.20.-Jb · 11.30.-j

1 Introduction

Symmetries in general relativity plays a vital role in exploring some crucial aspects of the space–time physics, for instance refer to [1] where the authors discussed the utility of Killing vector fields in determination of cosmological and gravitational red shifts and [2] where the authors presented a formalism which generalizes a simple 2+1 action of fluids with anisotropic stress. Symmetry restrictions are also imposed on the space–time metrics to solve the Einstein's field equations. These symmetry restrictions are well expressed in terms of Killing vector fields, which give rise to conservation laws [3].

General relativity is not the only theory which describes the gravitational interaction between matters. The teleparallel theory of gravity is an alternative theory of gravitation which was proposed by Einstein for the unification of general theory of relativity and quantum gravity. Although he did not succeed in his attempt, we study teleparallel theory of gravity as an alternative theory of gravity without unifying it with quantum gravity. The basic difference between general relativity and teleparallel theory is that gravitation is attributed to curvature in general relativity while in teleparallel theory torsion is responsible for gravitational interaction which plays the role of a force [4–6]. In the next section, a procedure for finding torsion in the space–time will be discussed in detail.

A natural question in one's mind arises about the conservation laws in teleparallel theory of gravitation. Keeping this point in mind, Sharif and Amir [7] introduced the teleparallel version of the Lie derivative for Killing vector fields and used those equations to find the teleparallel Killing vector fields in Einstein universe. Using the teleparallel Lie derivative proposed in [7], we have explored teleparallel Killing vector fields in Bianchi type I space–times [8], Bianchi type II space–times [9], Bianchi type III and Kantowski–Sachs space–times [10], spatially homogenous rotating space–times[11], and cylindrically symmetric static space–times [12]. In [13], the author

G. Shabbir (✉) · S. Khan
Faculty of Engineering Sciences,
GIK Institute of Engineering Sciences and Technology,
Topi, Swabi,
Khyber Pukhtoonkhwa, Pakistan
e-mail: shabbir@giki.edu.pk

M. J. Amir
Department of Mathematics, University of Sargodha,
Sargodha 40100, Pakistan

calculated energy, momentum, angular momentum, and teleparallel Killing vector fields for Brane world black hole in teleparallel theory, and he showed that when the results of energy, momentum, and angular momentum are in agreement with the results of general relativity, teleparallel Killing vector fields are also in agreement with the results of general relativity. Keeping this point in mind, it will be interesting to investigate the Killing symmetry in teleparallel theory, which, in turn, will help in understanding and estimating the various physical properties of the space–time such as energy, momentum, and angular momentum in this alternate description of gravity. Our current study covers the classification of static cylindrically symmetric space–times, plane symmetric static space–times, and non-static plane symmetric space–times. In this paper, we will only present the results of non-static cylindrically symmetric and non-static plane symmetric space–times. The results of static cylindrically symmetric space–times and plane static space–times can be found in [12]. The current study will not only help to understand the geometrical and physical properties of the space–time but the effect of torsion on the gravitational laws can also be deduced.

2 Overview

The teleparallel covariant derivative $\nabla\rho$ of a covariant tensor of rank 2 is defined as [14]

$$\nabla_\rho A_{\mu\nu} = A_{\mu\nu,\rho} - \Gamma^{\theta}{}_{\rho\nu}A_{\mu\theta} - \Gamma^{\theta}{}_{\mu\rho}A_{\nu\theta}, \tag{1}$$

where comma denotes the partial derivative and $\Gamma^{\theta}_{\rho\nu}$ are Weitzenböck connections defined as [14]

$$\Gamma^{\theta}{}_{\mu\nu} = K^{\theta}_{a}\partial_\nu K^{a}_{\mu}, \tag{2}$$

where K^{ν}_{a} is the non-trivial tetrad field. Its inverse field is denoted by K^{a}_{μ} and satisfies the relations

$$K^{a}_{\mu}K^{\nu}_{a} = \delta^{\nu}_{u}, \quad K^{a}_{u}K^{u}_{b} = \delta^{a}_{b} \tag{3}$$

Throughout this paper, $a, b, c, ... = 0, 1, 2, 3$ denote the tangent space indices and $\mu, \nu, \rho, ... = 0, 1, 2, 3$ denote the space–time indices. The Remannian metric can be generated from the tetrad field as

$$g_{\mu\nu} = \eta_{ab}K^{a}_{\mu}K^{b}_{\nu}. \tag{4}$$

where η_{ab} is the Minkowski metric given by $\eta_{ab} = \mathrm{diag}(-1, 1, 1, 1)$. Also the torsion of the space–time in terms of Weitzenböck connections is defined as

$$T^{\theta}_{\mu\nu} = \Gamma^{\theta}{}_{\nu\mu} - \Gamma^{\theta}{}_{\mu\nu}, \tag{5}$$

which is anti-symmetric with respect to its last two indices. It is important to note that the curvature of the Weitzenböck connection vanishes identically [15]. The TP Killing equation is defined as [7]

$$\underset{X}{\overset{T}{L}} g_{\mu\nu} = g_{\mu\nu,\rho}X^\rho + g_{\rho\nu}X^\rho{}_{,\mu} + g_{\mu\rho}X^\rho{}_{,\nu} + X^\rho\left(g_{\theta\nu}T^{\theta}_{\mu\rho} + g_{\mu\theta}T^{\theta}_{\nu\rho}\right) = 0, \tag{6}$$

where $\underset{X}{\overset{T}{L}}$ denotes the teleparallel Lie derivative with respect to the vector field X and $T^{\theta}_{\mu\nu}$ denotes the torsion tensor.

3 Main Results

Consider cylindrically symmetric non-static space–times in usual coordinates (t, r, θ, z) (labeled by (x^0, x^1, x^2, x^3), respectively) with the line element [16]

$$ds^2 = -e^{2A(t,r)}dt^2 + dr^2 + e^{2B(t,r)}\,d\theta^2 + e^{2C(t,r)}\,dz^2, \tag{7}$$

where A, B, and C are functions of t and r only. The case when $B(t,r) = C(t,r)$, the above space–times (7) become plane symmetric non-static space–times. The tetrad components and its inverse can be obtained as follows:

$$K^{a}_{\mu} = \mathrm{diag}\left(e^{A(t,r)},\, 1,\, e^{B(t,r)},\, e^{C(t,r)}\right),$$
$$K^{\mu}_{a} = \mathrm{diag}\left(e^{-A(t,r)},\, 1,\, e^{-B(t,r)},\, e^{-C(t,r)}\right). \tag{8}$$

The corresponding non-vanishing Weitzenböck connections for (8) are obtained as

$$\Gamma^{0}_{00} = \dot{A},\ \Gamma^{0}_{01} = A',\ \Gamma^{2}_{20} = \dot{B},\ \Gamma^{2}_{21} = B',\ \Gamma^{3}_{30} = \dot{C},\ \Gamma^{3}_{31} = C', \tag{9}$$

where "dash" denotes the derivative with respect to r and "dot" denotes the derivative with respect to t. Thus, the non-vanishing torsion components are obtained as

$$T^{0}_{10} = A'(t,r), T^{2}_{02} = \dot{B}(t,r), T^{3}_{03} = \dot{C}(t,r), T^{2}_{12} = B'(t,r),$$
$$T^{3}_{13} = C'(t,r). \tag{10}$$

A vector field X is said to be a teleparallel Killing vector field if it satisfies (6). One can write (6) explicitly using (7) and (10) as

$$\dot{A}X^0 + X^0_{,0} = 0, \tag{11}$$

$$X^1_{,1} = 0,\ \ X^2_{,2} = 0,\ \ X^3_{,3} = 0, \tag{12}$$

$$e^{2C(t,r)}\,X^3_{,2} + e^{2B(t,r)}X^2_{,3} = 0, \tag{13}$$

$$e^{2B(t,r)}X^2_{,0} - e^{2A(t,r)}X^0_{,2} + \dot{B}e^{2B}X^2 = 0, \quad (14)$$

$$e^{2C(t,r)}X^3_{,0} - e^{2A(t,r)}X^0_{,3} + \dot{C}e^{2C}X^3 = 0, \quad (15)$$

$$e^{2B(t,r)}X^2_{,1} + X^1_{,2} + B'e^{2B(t,r)}X^2 = 0, \quad (16)$$

$$e^{2C(t,r)}X^3_{,1} + X^1_{,3} + C'\,e^{2C(t,r)}X^3 = 0, \quad (17)$$

$$X^1_{,0} - e^{2A(t,r)}X^0_{,1} - A'e^{2A(t,r)}X^0 = 0. \quad (18)$$

Equations (11) and (12) give

$$X^0 = e^{-A(t,r)}P^1(r,\theta,z),\ X^1 = P^2(t,\theta,z),\ X^2 = P^3(t,r,z),\ X^3 = P^4(t,r,\theta), \quad (19)$$

where $P^1(r,\theta,z)$, $P^2(t,\theta,z)$, $P^3(t,r,z)$, and $P^4(t,r,\theta)$ are functions of integration which are to be determined. To avoid lengthy details here, we shall write only the results which are:

Case A1 In this case, we have $A(t,r) \neq B(t,r)$, $A(t,r) \neq C(t,r)$, and $B(t,r) \neq C(t,r)$. The space–time is given in (7). Solution of (11) to (18) is given by

$$X^0 = e^{-A(t,r)}c_1,\quad X^1 = c_2,\quad X^2 = e^{-B(t,r)}c_3,\quad X^3 = e^{-C(t,r)}c_4, \quad (20)$$

where c_1, c_2, c_3, c_4, $\in R$. Here the above space–time (7) admits four linearly independent teleparallel Killing vector fields which are $e^{-A(t,r)}\frac{\partial}{\partial t}$, $\frac{\partial}{\partial r}$, $e^{-B(t,r)}\frac{\partial}{\partial\theta}$, and $e^{-C(t,r)}\frac{\partial}{\partial z}$. The Killing vector fields in general relativity are $\frac{\partial}{\partial\theta}$ and $\frac{\partial}{\partial z}$. Here one can see that we get two more conservation laws in teleparallel theory.

Case A2 In this case, there exist the following three possibilities which are

(a) $A = A(t,r)$, $B = B(t,r)$ and $C =$ constant.

(b) $A = A(t,r)$, $C = C(t,r)$ and $B =$ constant.

(c) $B = B(t,r)$, $C = C(t,r)$ and $A =$ constant.

In (a) the space–time (7) can, after a suitable rescaling of z, be written in the form

$$ds^2 = -e^{2A(t,r)}dt^2 + dr^2 + e^{2B(t,r)}d\theta^2 + dz^2. \quad (21)$$

Teleparallel Killing vector fields in this case are

$$X^0 = e^{-A(t,r)}c_1,\quad X^1 = c_2,\quad X^2 = e^{-B(t,r)}c_3,\quad X^3 = c_4, \quad (22)$$

where c_1, c_2, c_3, c_4, $\in R$. Here the above space–time (21) admits four linearly independent teleparallel Killing vector fields which can be written as $e^{-A(t,r)}\frac{\partial}{\partial t}$, $\frac{\partial}{\partial r}$, $e^{-B(t,r)}\frac{\partial}{\partial\theta}$, and $\frac{\partial}{\partial z}$. The Killing vector fields in general relativity are $\frac{\partial}{\partial\theta}$ and $\frac{\partial}{\partial z}$. Here one can see that we get two more conservation laws in teleparallel theory and one teleparallel Killing vector field is same in both the theories. Cases (b) and (c) are exactly the same.

Case A3 In this case, the following three possibilities exist, which are

(d) $A = A(t,r)$, and $B(t,r) = C(t,r)$.

(e) $B = B(t,r)$, and $A(t,r) = C(t,r)$.

(f) $C = C(t,r)$, and $A(t,r) = B(t,r)$.

In (d) the space–time (7) takes the form

$$ds^2 = -e^{2A(t,r)}dt^2 + dr^2 + e^{2B(t,r)}(d\theta^2 + dz^2). \quad (23)$$

The teleparallel Killing vector fields in this case are

$$X^0 = e^{-A(t,r)}c_1,\quad X^1 = c_2,\quad X^2 = e^{-B(t,r)}c_3 + z\,c_5\,e^{-B(t,r)},\quad X^3 = e^{-B(t,r)}c_4 - \theta\,c_5\,e^{-B(t,r)}, \quad (24)$$

where c_1, c_2, c_3, c_4, $c_5 \in R$. Here the above space–time (23) admits five linearly independent teleparallel Killing vector fields which can be written as $e^{-A(t,r)}\frac{\partial}{\partial t}$, $\frac{\partial}{\partial r}$, $e^{-B(t,r)}\frac{\partial}{\partial\theta}$, $e^{-B(t,r)}\frac{\partial}{\partial z}$, and $e^{-B(t,r)}(z\frac{\partial}{\partial\theta} - \theta\frac{\partial}{\partial z})$. Killing vector fields in general relativity are $\frac{\partial}{\partial\theta}$, $\frac{\partial}{\partial z}$, and $(z\frac{\partial}{\partial\theta} - \theta\frac{\partial}{\partial z})$. Here one can see that we get two more conservation laws in teleparallel theory. Cases (e) and (f) are exactly the same.

Case A4 In this case, there exist the following three possibilities, which are

(g) $A =$ constant and $B(t,r) = C(t,r)$.

(h) $B =$ constant and $A(t,r) = C(t,r)$.

(i) $C =$ constant and $A(t,r) = B(t,r)$.

In (g) the space–time (7) can, after a suitable rescaling of t, be written in the form

$$ds^2 = -dt^2 + dr^2 + e^{2B(t,r)}\left(d\theta^2 + dz^2\right). \quad (25)$$

Teleparallel Killing vector fields in this case are

$$X^0 = c_1, \qquad X^1 = c_2,$$
$$X^2 = e^{-B(t,r)}c_3 + z\,c_5\,e^{-B(t,r)}, \quad (26)$$
$$X^3 = e^{-B(t,r)}c_4 - \theta\,c_5\,e^{-B(t,r)},$$

where $c_1, c_2, c_3, c_4, c_5 \in R$. Here the above space–time (25) admits five linearly independent teleparallel Killing vector fields which can be written as $\frac{\partial}{\partial t}, \frac{\partial}{\partial r}$, $e^{-B(t,r)}\frac{\partial}{\partial \theta}$, $e^{-B(t,r)}\frac{\partial}{\partial z}$, and $e^{-B(t,r)}(z\frac{\partial}{\partial\theta} - \theta\frac{\partial}{\partial z})$. Killing vector fields in general relativity are $\frac{\partial}{\partial\theta}$, $\frac{\partial}{\partial z}$, and $\left(z\frac{\partial}{\partial\theta} - \theta\frac{\partial}{\partial z}\right)$. Here one can see that we get two more conservation laws in teleparallel theory. Cases (h) and (i) are exactly the same.

Case A5 In this case, there exist the following three possibilities, which are

(j) $A = A(t,r)$ and $B(t,r) = C(t,r) =$ constant.
(k) $B = B(t,r)$ and $A(t,r) = C(t,r) =$ constant.
(l) $C = C(t,r)$ and $A(t,r) = B(t,r) =$ constant.

In (j) the space–time (7) can, after a suitable rescaling of θ and z, be written in the form

$$ds^2 = -e^{2A(t,r)}dt^2 + dr^2 + \left(d\theta^2 + dz^2\right). \quad (27)$$

Teleparallel Killing vector fields in this case are

$$X^0 = e^{-A(t,r)}c_1,\ X^1 = c_2, \quad (28)$$
$$X^2 = c_3 + zc_5,\ X^3 = c_4 - \theta c_5,$$

where $c_1, c_2, c_3, c_4, c_5 \in R$. Here the above space–time (27) admits five linearly independent teleparallel Killing vector fields which can be written as $e^{-A(t,r)}\frac{\partial}{\partial t}, \frac{\partial}{\partial r}$, $\frac{\partial}{\partial\theta}$, $\frac{\partial}{\partial z}$, and $\left(z\frac{\partial}{\partial\theta} - \theta\frac{\partial}{\partial z}\right)$. Killing vector fields in general relativity are $\frac{\partial}{\partial\theta}$, $\frac{\partial}{\partial z}$, and $\left(z\frac{\partial}{\partial\theta} - \theta\frac{\partial}{\partial z}\right)$. Here one can see that we get two more conservation laws in teleparallel theory, and three teleparallel Killing vector fields are exactly the same in both the theories. Cases (k) and (l) are exactly the same.

Case A6 In this case, we have $A(t,r) = B(t,r) = C(t,r)$. The line element takes the form

$$ds^2 = e^{2A(t,r)}\left(-dt^2 + d\theta^2 + dz^2\right) + dr^2. \quad (29)$$

Teleparallel Killing vector fields in this case are

$$X^0 = e^{-A(t,r)}c_1 + zc_4e^{-A(t,r)} + \theta c_5e^{-A(t,r)}, \quad X^1 = c_7,$$
$$X^2 = e^{-A(t,r)}c_2 + tc_5e^{-A(t,r)} + zc_6e^{-A(t,r)},$$
$$X^3 = e^{-A(t,r)}c_3 + tc_4e^{-A(t,r)} - \theta c_6e^{-A(t,r)}, \quad (30)$$

where $c_1, c_2, c_3, c_4, c_5\ c_6, c_7 \in R$. Here the above space–time (29) admits seven linearly independent teleparallel Killing vector fields which can be written as $e^{-A(t,r)}\frac{\partial}{\partial t}, \frac{\partial}{\partial r}, e^{-A(t,r)}\frac{\partial}{\partial\theta}, e^{-A(t,r)}\frac{\partial}{\partial z}$, $e^{-A(t,r)}(z\frac{\partial}{\partial\theta} - \theta\frac{\partial}{\partial z})$, $e^{-A(t,r)}(z\frac{\partial}{\partial t} + t\frac{\partial}{\partial z})$, and $e^{-A(t,r)}(\theta\frac{\partial}{\partial t} + t\frac{\partial}{\partial\theta})$. Killing vector fields in general relativity are $\frac{\partial}{\partial\theta}$, $\frac{\partial}{\partial z}$, and $(z\frac{\partial}{\partial\theta} - \theta\frac{\partial}{\partial z})$. Here one can see that we get four more conservation laws in teleparallel theory.

Case B1 In this case, we have $A(t) \neq B(t)$, $A(t) \neq C(t)$, and $B(t) \neq C(t)$. The space–time is given as

$$ds^2 = -e^{2A(t)}dt^2 + dr^2 + e^{2B(t)}d\theta^2 + e^{2C(t)}dz^2, \quad (31)$$

where A, B, and C are functions of t only. Solution of (11) to (18) is

$$X^0 = e^{-A(t)}c_1, \quad X^1 = c_2, \quad X^2 = e^{-B(t)}c_3,$$
$$X^3 = e^{-C(t)}c_4, \quad (32)$$

where $c_1, c_2, c_3, c_4 \in R$. Here the above space–time (31) admits four linearly independent teleparallel Killing vector fields which are $e^{-A(t)}\frac{\partial}{\partial t}, \frac{\partial}{\partial r}$, $e^{-B(t)}\frac{\partial}{\partial\theta}$, and $e^{-C(t)}\frac{\partial}{\partial z}$. Killing vector fields in general relativity are $\frac{\partial}{\partial r}$, $\frac{\partial}{\partial\theta}$, and $\frac{\partial}{\partial z}$. Here one can see that we get one more conservation law in teleparallel theory and one teleparallel Killing vector field is exactly the same in both the theories.

Case B2 In this case, there exist the following three possibilities:

(a1) $A = A(t)$, $B = B(t)$ and $C =$ constant.
(b1) $A = A(t)$, $C = C(t)$ and $B =$ constant.
(c1) $B = B(t)$, $C = C(t)$ and $A =$ constant.

In (a1) the space–time (7) can, after a suitable rescaling of z, be written in the form

$$ds^2 = -e^{2A(t)}dt^2 + dr^2 + e^{2B(t)}d\theta^2 + dz^2. \quad (33)$$

Teleparallel Killing vector fields in this case are

$$X^0 = e^{-A(t)}c_1, \quad X^1 = c_2, \quad X^2 = e^{-B(t)}c_3, \quad X^3 = c_4, \tag{34}$$

where c_1, c_2, c_3, $c_4 \in R$. Here the above space–time (33) admits four linearly independent teleparallel Killing vector fields which can be written as $e^{-A(t)}\frac{\partial}{\partial t}$, $\frac{\partial}{\partial r}$, $e^{-B(t)}\frac{\partial}{\partial \theta}$, and $\frac{\partial}{\partial z}$. Killing vector fields in general relativity are $\frac{\partial}{\partial r}$, $\frac{\partial}{\partial \theta}$, and $\frac{\partial}{\partial z}$. Here one can see that we get one more conservation law in teleparallel theory and two teleparallel Killing vector field are exactly the same in both the theories. Cases (b1) and (c1) are exactly the same.

Case B3 In this case, the following three possibilities exist, which are

(d1) $A = A(t)$, and $B(t) = C(t)$.

(e1) $B = B(t)$, and $A(t) = C(t)$.

(f1) $C = C(t)$, and $A(t) = B(t)$.

In (d1) the space–time (7) takes the form

$$ds^2 = -e^{2A(t)}\,dt^2 + dr^2 + e^{2B(t)}\left(d\theta^2 + dz^2\right). \tag{35}$$

Teleparallel Killing vector fields in this case are

$$X^0 = e^{-A(t)}c_1, \quad X^1 = c_2, \quad X^2 = e^{-B(t)}c_3 + z\,c_5\,e^{-B(t)}, \quad X^3 = e^{-B(t)}c_4 - \theta\,c_5\,e^{-B(t)}, \tag{36}$$

where $c_1, c_2, c_3, c_4, c_5 \in R$. Here the above space–time (35) admits five linearly independent teleparallel Killing vector fields which can be written as $e^{-A(t)}\frac{\partial}{\partial t}$, $\frac{\partial}{\partial r}$, $e^{-B(t)}\frac{\partial}{\partial \theta}$, $e^{-B(t)}\frac{\partial}{\partial z}$, and $e^{-B(t)}(z\frac{\partial}{\partial\theta} - \theta\frac{\partial}{\partial z})$. Killing vector fields in general relativity are $\frac{\partial}{\partial r}$, $\frac{\partial}{\partial \theta}$, $\frac{\partial}{\partial z}$, and $(z\frac{\partial}{\partial\theta} - \theta\frac{\partial}{\partial z})$. Here one can see that we get one more conservation law in teleparallel theory. Cases (e1) and (f1) are exactly the same.

Case B4 In this case, there exist the following three possibilities, which are

(g1) $A =$ constant and $B(t) = C(t)$.

(h1) $B =$ constant and $A(t) = C(t)$.

(i1) $C =$ constant and $A(t) = B(t)$.

In (g1) the space–time (7) can, after a suitable rescaling of t, be written in the form

$$ds^2 = -dt^2 + dr^2 + e^{2B(t)}\left(d\theta^2 + dz^2\right). \tag{37}$$

Teleparallel Killing vector fields in this case are

$$X^0 = c_1, \quad X^1 = c_2, \quad X^2 = e^{-B(t)}c_3 + zc_5e^{-B(t)}, \quad X^3 = e^{-B(t)}c_4 - \theta\,c_5\,e^{-B(t)}, \tag{38}$$

where c_1, c_2, c_3, c_4, c_5, $\in R$. Here the above space–time (37) admits five linearly independent teleparallel Killing vector fields which can be written as $\frac{\partial}{\partial t}$, $\frac{\partial}{\partial r}$, $e^{-B(t)}\frac{\partial}{\partial \theta}$, $e^{-B(t)}\frac{\partial}{\partial z}$, and $e^{-B(t)}(z\frac{\partial}{\partial\theta} - \theta\frac{\partial}{\partial z})$. Killing vector fields in general relativity are $\frac{\partial}{\partial r}$, $\frac{\partial}{\partial \theta}$, $\frac{\partial}{\partial z}$, and $(z\frac{\partial}{\partial\theta} - \theta\frac{\partial}{\partial z})$. Here one can see that we get one more conservation law in teleparallel theory and one teleparallel Killing vector field is same in both the theories. Cases (h1) and (i1) are exactly the same.

Case B5 In this case, there exist the following three possibilities, which are

(j1) $B = B(t)$ and $A(t) = C(t) =$ constant.

(k1) $C = C(t)$ and $A(t) = B(t) =$ constant.

(m1) $A = A(t)$ and $B(t) = C(t) =$ constant.

In (j1) the space–time (7) can, after a suitable rescaling of t and z, be written in the form

$$ds^2 = -dt^2 + dr^2 + e^{2B(t)}d\theta^2 + dz^2. \tag{39}$$

Teleparallel Killing vector fields in this case are

$$X^0 = c_1, \quad X^1 = c_2 + zc_5, \quad X^2 = e^{-B(t)}c_3, \quad X^3 = c_4 - r\,c_5, \tag{40}$$

where $c_1, c_2, c_3, c_4, c_5 \in R$. Here the above space–time (39) admits five linearly independent teleparallel Killing vector fields which can be written as $\frac{\partial}{\partial t}$, $\frac{\partial}{\partial r}$, $e^{-B(t)}\frac{\partial}{\partial \theta}$, $\frac{\partial}{\partial z}$, and $(z\frac{\partial}{\partial r} - r\frac{\partial}{\partial z})$. Killing vector fields in general relativity are $\frac{\partial}{\partial r}$, $\frac{\partial}{\partial \theta}$, $\frac{\partial}{\partial z}$, and $(z\frac{\partial}{\partial r} - r\frac{\partial}{\partial z})$. Here one can see that we get one more conservation law in teleparallel theory and three teleparallel Killing vector fields are same in both the theories. Other cases (k1) and (m1) are exactly the same.

Case B6 In this case, we have $A(t) = B(t) = C(t)$. The line element takes the form

$$ds^2 = e^{2A(t)}\left(-dt^2 + d\theta^2 + dz^2\right) + dr^2. \quad (41)$$

Teleparallel Killing vector fields in this case are

$$X^0 = e^{-A(t)}c_1 + zc_4e^{-A(t)} + \theta c_5e^{-A(t)}, \quad X^1 = c_7,$$
$$X^2 = e^{-A(t)}c_2 + tc_5e^{-A(t)} + zc_6e^{-A(t)},$$
$$X^3 = e^{-A(t)}c_3 + tc_4e^{-A(t)} - \theta c_6e^{-A(t)}, \quad (42)$$

where c_1, c_2, c_3, c_4, c_5, c_6, $c_7 \in R$. Here the above space–time (41) admits seven linearly independent teleparallel Killing vector fields which can be written as $e^{-A(t)}\frac{\partial}{\partial t}$, $\frac{\partial}{\partial r}$, $e^{-A(t)}\frac{\partial}{\partial \theta}$, $e^{-A(t)}\frac{\partial}{\partial z}$, $e^{-A(t)}(z\frac{\partial}{\partial \theta} - \theta\frac{\partial}{\partial z})$, $e^{-A(t)}(z\frac{\partial}{\partial t} + t\frac{\partial}{\partial z})$, and $e^{-A(t)}(\theta\frac{\partial}{\partial t} + t\frac{\partial}{\partial \theta})$. Killing vector fields in general relativity are $\frac{\partial}{\partial r}$, $\frac{\partial}{\partial \theta}$, $\frac{\partial}{\partial z}$, and $(z\frac{\partial}{\partial \theta} - \theta\frac{\partial}{\partial z})$. In this case, we get three more conservation laws in teleparallel theory and one teleparallel Killing vector field is same in both the theories.

Case C1 In this case, there exist the following possibilities

(a2) $A = A(t)$, $B = B(t,r)$, and $C = C(t,r)$.
(b2) $B = B(t)$, $A = A(t,r)$, and $C = C(t,r)$.
(c2) $C = C(t)$, $A = A(t,r)$, and $B = B(t,r)$.

In (a2) the space–time (7) takes the form

$$ds^2 = -e^{2A(t)}\,dt^2 + dr^2 + e^{2B(t,r)}\,d\theta^2 + e^{2C(t,r)}dz^2. \quad (43)$$

The solution of (11) to (18) is given by

$$X^0 = e^{-A(t)}c_1, \quad X^1 = c_2, \quad X^2 = e^{-B(t,r)}c_3,$$
$$X^3 = e^{-C(t,r)}c_4, \quad (44)$$

where $c_1, c_2, c_3, c_4 \in R$. Here the above space–time (43) admits four linearly independent teleparallel Killing vector fields which are $e^{-A(t)}\frac{\partial}{\partial t}$, $\frac{\partial}{\partial r}$, $e^{-B(t,r)}\frac{\partial}{\partial \theta}$, and $e^{-C(t,r)}\frac{\partial}{\partial z}$. Killing vector fields in general relativity are $\frac{\partial}{\partial \theta}$ and $\frac{\partial}{\partial z}$. In this case, one can see that we get two more conservation laws in teleparallel theory. Cases (b2) and (c2) are exactly the same.

Case C2 In this case, there exist the following three possibilities which are

(d2) $A = A(t)$ and $B(t,r) = C(t,r)$.
(e2) $B = B(t)$ and $A(t,r) = C(t,r)$.
(f2) $C = C(t)$ and $A(t,r) = B(t,r)$.

In (d2) the space–time (7) takes the form

$$ds^2 = -e^{2A(t)}\,dt^2 + dr^2 + e^{2B(t,r)}\,(d\theta^2 + dz^2). \quad (45)$$

Teleparallel Killing vector fields in this case are

$$X^0 = e^{-A(t)}c_1, \quad X^1 = c_2,$$
$$X^2 = e^{-B(t,r)}c_3 + z\,c_5\,e^{-B(t,r)}, \quad (46)$$
$$X^3 = e^{-B(t,r)}c_4 - \theta\,c_5\,e^{-B(t,r)},$$

where $c_1, c_2, c_3, c_4, c_5 \in R$. Here the above space–time (45) admits five linearly independent teleparallel Killing vector fields which can be written as $e^{-A(t)}\frac{\partial}{\partial t}$, $\frac{\partial}{\partial r}$, $e^{-B(t,r)}\frac{\partial}{\partial \theta}$, $e^{-B(t,r)}\frac{\partial}{\partial z}$, and $e^{-B(t,r)}(z\frac{\partial}{\partial \theta} - \theta\frac{\partial}{\partial z})$. Killing vector fields in general relativity are $\frac{\partial}{\partial \theta}$, $\frac{\partial}{\partial z}$, and $(z\frac{\partial}{\partial \theta} - \theta\frac{\partial}{\partial z})$. Here one can see that we get two more conservation laws in teleparallel theory. Cases (e2) and (f2) are exactly the same.

Case D1 In this case, there exist the following three possibilities, which are

(a3) $A = A(t)$, $B = B(t)$, and $C = C(t,r)$.
(b3) $A = A(t)$, $B = B(t,r)$, and $C = C(t)$.
(c3) $A = A(t,r)$, $B = B(t)$, and $C = C(t)$.

In (a3) the space–time (7) takes the form

$$ds^2 = -e^{2A(t)}\,dt^2 + dr^2 + e^{2B(t)}\,d\theta^2 + e^{2C(t,r)}dz^2. \quad (47)$$

Solution of (11) to (18) is given by

$$X^0 = e^{-A(t)}c_1, \quad X^1 = c_2, \quad X^2 = e^{-B(t)}c_3,$$
$$X^3 = e^{-C(t,r)}c_4, \quad (48)$$

where $c_1, c_2, c_3, c_4 \in R$. Here the above space–time (47) admits four linearly independent teleparallel Killing vector fields which are $e^{-A(t)}\frac{\partial}{\partial t}$, $\frac{\partial}{\partial r}$, $e^{-B(t)}\frac{\partial}{\partial \theta}$, and $e^{-C(t,r)}\frac{\partial}{\partial z}$. Killing vector fields in general relativity are $\frac{\partial}{\partial \theta}$ and $\frac{\partial}{\partial z}$.

Here one can see that we get two more conservation laws in teleparallel theory. Cases (b3) and (c3) are exactly the same.

Case D2 In this case, there exist the following three possibilities which are

(d3) $A(t) = B(t)$ and $C = C(t,r)$.
(e3) $A(t) = C(t)$ and $B = B(t,r)$.
(f3) $A = A(t,r)$ and $B(t) = C(t)$.

In (d3) the space–time (7) takes the form

$$ds^2 = -e^{2A(t)}(dt^2 + d\theta^2) + dr^2 + e^{2C(t,r)}dz^2. \tag{49}$$

Teleparallel Killing vector fields in this case are

$$X^0 = e^{-A(t)}c_1 + \theta\, c_5\, e^{-A(t)}, \quad X^1 = c_2,$$
$$X^2 = e^{-A(t)}c_3 + t\, c_5\, e^{-A(t)}, \quad X^3 = e^{-C(t,r)}c_4, \tag{50}$$

where $c_1, c_2, c_3, c_4, c_5 \in R$. Here the above space–time (49) admits five linearly independent teleparallel Killing vector fields which can be written as $e^{-A(t)}\frac{\partial}{\partial t}$, $\frac{\partial}{\partial r}$, $e^{-A(t)}\frac{\partial}{\partial \theta}$, $e^{-C(t,r)}\frac{\partial}{\partial z}$, and $e^{-A(t)}(t\frac{\partial}{\partial\theta} + \theta\frac{\partial}{\partial t})$. Killing vector fields in general relativity are $\frac{\partial}{\partial\theta}$ and $\frac{\partial}{\partial z}$. Here one can see that we get three more conservation laws in teleparallel theory. Cases (e3) and (f3) are exactly the same.

Case E1 In this case, there exist the following six possibilities which are

(a4) $A = A(t,r)$, $B = B(t)$, and $C = C(r)$.
(b4) $A = A(t,r)$, $B = B(r)$, and $C = C(t)$.
(c4) $A = A(t)$, $B = B(t,r)$, and $C = C(r)$.
(d4) $A = A(r)$, $B = B(t,r)$, and $C = C(t)$.
(e4) $A = A(t)$, $B = B(r)$, and $C = C(t,r)$.
(f4) $A = A(r)$, $B = B(t)$, and $C = C(t,r)$.

The space–time (7) in case (a4) takes the form

$$ds^2 = -e^{2A(t,r)}\,dt^2 + dr^2 + e^{2B(t)}\,d\theta^2 + e^{2C(r)}dz^2. \tag{51}$$

Solution of (11) to (18) is given by

$$X^0 = e^{-A(t,r)}c_1, \quad X^1 = c_2, \quad X^2 = e^{-B(t)}c_3,$$
$$X^3 = e^{-C(r)}c_4, \tag{52}$$

where $c_1, c_2, c_3, c_4 \in R$. Here the above space–time (51) admits four linearly independent teleparallel Killing vector fields which are $e^{-A(t,r)}\frac{\partial}{\partial t}$, $\frac{\partial}{\partial r}$, $e^{-B(t)}\frac{\partial}{\partial\theta}$, and $e^{-C(r)}\frac{\partial}{\partial z}$. Killing vector fields in general relativity are $\frac{\partial}{\partial\theta}$ and $\frac{\partial}{\partial z}$. Here one can see that we get two more conservation laws in teleparallel theory. The remaining cases (b4)–(f4) are exactly the same.

Case E2 In this case, there exist the following six possibilities which are

(g4) $A = A(t,r)$, $B = B(t)$, and $C =$ constant.
(h4) $A = A(t,r)$, $B =$ constant, and $C = C(t)$.
(i4) $B = B(t,r)$, $A = A(t)$, and $C =$ constant.
(j4) $B = B(t,r)$, $A =$ constant, and $C = C(t)$.
(k4) $C = C(t,r)$, $A =$ constant, and $B = B(t)$.
(l4) $C = C(t,r)$, $A = A(t)$, and $B =$ constant.

In case (g4) the space–time (7) can, after a suitable rescaling of z, takes the form

$$ds^2 = -e^{2A(t,r)}dt^2 + dr^2 + e^{2B(t)}d\theta^2 + dz^2. \tag{53}$$

The solution of (11) to (18) is given by

$$X^0 = e^{-A(t,r)}c_1, \quad X^1 = c_2, \quad X^2 = e^{-B(t)}c_3,$$
$$X^3 = c_4, \tag{54}$$

where $c_1, c_2, c_3, c_4 \in R$. Here the above space–time (53) admits four linearly independent teleparallel Killing vector fields which are $e^{-A(t,r)}\frac{\partial}{\partial t}$, $\frac{\partial}{\partial r}$, $e^{-B(t)}\frac{\partial}{\partial\theta}$, and $\frac{\partial}{\partial z}$. Killing vector fields in general relativity are $\frac{\partial}{\partial\theta}$ and $\frac{\partial}{\partial z}$. Here one can see that we get two more conservation laws in teleparallel theory. The remaining cases (h4)–(l4) are exactly the same.

Case E3 In this case, there exist the following six possibilities which are

(m4) $A = A(t,r)$, $B = B(r)$, and $C =$ constant.
(n4) $A = A(t,r)$, $B =$ constant, and $C = C(r)$.
(o4) $B = B(t,r)$, $A = A(r)$, and $C =$ constant.
(p4) $B = B(t,r)$, $A =$ constant, and $C = C(r)$.
(q4) $C = C(t,r)$, $A = A(r)$, and $B =$ constant.
(r4) $C = C(t,r)$, $A =$ constant, and $B = B(r)$.

The space–time in (m4), after a suitable rescaling of z, takes the form

$$ds^2 = -e^{2A(t,r)}dt^2 + dr^2 + e^{2B(r)}d\theta^2 + dz^2. \tag{55}$$

Solution of (11) to (18) is given by

$$X^0 = e^{-A(t,r)}c_1, \quad X^1 = c_2, \quad X^2 = e^{-B(r)}c_3, \quad X^3 = c_4, \tag{56}$$

where $c_1, c_2, c_3, c_4, c_5 \in R$. Here the above space–time (55) admits four linearly independent teleparallel Killing vector fields which are $e^{-A(t,r)}\frac{\partial}{\partial t}$, $\frac{\partial}{\partial r}$, $e^{-B(r)}\frac{\partial}{\partial\theta}$, and $\frac{\partial}{\partial z}$. Killing vector fields in general relativity are $\frac{\partial}{\partial\theta}$ and $\frac{\partial}{\partial z}$. Here one can see that we get two more conservation laws in teleparallel theory and one teleparallel Killing vector field is same in both the theories. The remaining cases (n4)–(r4) are exactly the same.

Case F1 In this case, there exist the following three possibilities which are

(a5) $A = A(t,r)$, $B = B(r)$, and $C = C(t,r)$.
(b5) $A = A(t,r)$, $B = B(t,r)$, and $C = C(r)$.
(c5) $B = B(t,r)$, $A = A(r)$, and $C = C(t,r)$.

The space–time (7) in case (a5) takes the form

$$ds^2 = -e^{2A(t,r)}\,dt^2 + dr^2 + e^{2B(r)}\,d\theta^2 + e^{2C(t,r)}dz^2. \tag{57}$$

Solution of (11) to (18) is given by

$$X^0 = e^{-A(t,r)}c_1, \quad X^1 = c_2, \quad X^2 = e^{-B(r)}c_3, \quad X^3 = e^{-C(t,r)}c_4, \tag{58}$$

where $c_1, c_2, c_3, c_4 \in R$. Here the above space–time (57) admits four linearly independent teleparallel Killing vector fields which are $e^{-A(t,r)}\frac{\partial}{\partial t}$, $\frac{\partial}{\partial r}$, $e^{-B(r)}\frac{\partial}{\partial\theta}$, and $e^{-C(t,r)}\frac{\partial}{\partial z}$. Killing vector fields in general relativity are $\frac{\partial}{\partial\theta}$ and $\frac{\partial}{\partial z}$. Here one can see that we get two more conservation laws in teleparallel theory. Cases (b5) and (c5) are exactly the same.

Case F2 In this case, there exist the following three possibilities which are

(a5) $A(t,r) = B(t,r)$, and $C = C(r)$.
(b5) $A(t,r) = C(t,r)$, and $B = B(r)$.
(c5) $A = A(r)$ and $B(t,r) = C(t,r)$.

In (d5) the space–time (7) takes the form

$$ds^2 = -e^{2A(t,r)}\,(dt^2 + d\theta^2) + dr^2 + e^{2C(r)}\,dz^2. \tag{59}$$

Teleparallel Killing vector fields in this case are

$$X^0 = e^{-A(t,r)}c_1 + \theta\, c_5\, e^{-A(t,r)}, \quad X^1 = c_2, \quad X^2 = e^{-A(t,r)}c_3 + t\, c_5\, e^{-A(t,r)}, \quad X^3 = e^{-C(r)}c_4, \tag{60}$$

where $c_1, c_2, c_3, c_4, c_5 \in R$. Here the above space–time (59) admits five linearly independent teleparallel Killing vector fields which can be written as $e^{-A(t,r)}\frac{\partial}{\partial t}$, $\frac{\partial}{\partial r}$, $e^{-A(t,r)}\frac{\partial}{\partial\theta}$, $e^{-C(r)}\frac{\partial}{\partial z}$, and $e^{-A(t,r)}(\theta\frac{\partial}{\partial t} + t\frac{\partial}{\partial\theta})$. Killing vector fields in general relativity are $\frac{\partial}{\partial\theta}$ and $\frac{\partial}{\partial z}$. Here one can see that we get three more conservation laws in teleparallel theory. Cases (e5) and (f5) are exactly the same.

Case G1 In this case, there exist the following three possibilities which are

(a6) $A = A(t,r)$, $B = B(r)$, and $C = C(r)$
(b6) $A = A(r)$, $B = B(t,r)$, and $C = C(r)$
(c6) $A = A(r)$, $B = (r)$, and $C = C(t,r)$

In (a6), the space–time (7) takes the form

$$ds^2 = -e^{2A(t,r)}\,dt^2 + dr^2 + e^{2B(r)}\,d\theta^2 + e^{2C(r)}\,dz^2. \tag{61}$$

Teleparallel Killing vector fields in this case are

$$X^0 = e^{-A(t,r)}c_1, \; X^1 = c_2, \; X^2 = e^{-B(r)}c_3, \; X^3 = e^{-C(r)}c_4, \tag{62}$$

where $c_1, c_2, c_3, c_4 \in R$. Here the above space–time (61) admits four linearly independent teleparallel Killing vector fields which can be written as $e^{-A(t,r)}\frac{\partial}{\partial t}$, $\frac{\partial}{\partial r}$, $e^{-B(r)}\frac{\partial}{\partial\theta}$, and $e^{-C(r)}\frac{\partial}{\partial z}$. Killing vector fields in general relativity are $\frac{\partial}{\partial\theta}$ and $\frac{\partial}{\partial z}$. Here one can see that we get two more conservation laws in teleparallel theory. Cases (b6) and (c6) are exactly the same.

Case G2 In this case, there exist the following three possibilities which are

(d6) $A = A(t,r)$ and $B(r) = C(r.$

(e6) $A(r) = C(r)$ and $B = B(t,r.$

(f6) $A(r) = B(r)$ and $C = C(t,r)$

In (d6) the space–time (7) takes the form

$$ds^2 = -e^{2A(t,r)}\,dt^2 + dr^2 + e^{2B(r)}\left(d\theta^2 + dz^2\right). \tag{63}$$

Teleparallel Killing vector fields in this case are

$$X^0 = e^{-A(t,r)}c_1, \qquad X^1 = c_2,$$
$$X^2 = e^{-B(r)}c_3 + z\,c_5\,e^{-B(r)}, \quad X^3 = e^{-B(r)}c_4 - \theta\,c_5\,e^{-B(r)}, \tag{64}$$

where c_1, c_2, c_3, c_4, $c_5 \in R$. Here the above space–time (63) admits five linearly independent teleparallel Killing vector fields which can be written as $e^{-A(t,r)}\frac{\partial}{\partial t}$, $\frac{\partial}{\partial r}$, $e^{-B(r)}\frac{\partial}{\partial\theta}$, $e^{-B(r)}\frac{\partial}{\partial z}$, and $e^{-B(r)}\left(z\frac{\partial}{\partial\theta} - \theta\frac{\partial}{\partial z}\right)$. Killing vector fields in general relativity are $\frac{\partial}{\partial\theta}$, $\frac{\partial}{\partial z}$, and $\left(z\frac{\partial}{\partial\theta} - \theta\frac{\partial}{\partial z}\right)$. Here one can see that we get two more conservation laws in teleparallel theory. Cases (e6) and (f6) are exactly the same.

Case H1 In this case, there exist the following three possibilities which are

(a7) $A = A(t)$, $B = B(r)$, and $C = C(r)$.

(b7) $A = A(r)$, $B = B(t)$, and $C = C(r)$.

(c7) $A = A(r)$, $B = B(r)$, and $C = C(t)$.

In (a7) the space–time (7) takes the form

$$ds^2 = -e^{2A(t)}\,dt^2 + dr^2 + e^{2B(r)}\,d\theta^2 + e^{2C(r)}\,dz^2. \tag{65}$$

Teleparallel Killing vector fields in this case are

$$X^0 = e^{-A(t)}c_1, \quad X^1 = c_2,$$
$$X^2 = e^{-B(r)}c_3, X^3 = e^{-C(r)}c_4, \tag{66}$$

where $c1$, $c2$, $c3$, $c4$, $\in R$. Here the above space–time (65) admits four linearly independent teleparallel Killing vector fields which can be written as $e^{-A(t)}\frac{\partial}{\partial t}$, $\frac{\partial}{\partial r}$, $e^{-B(r)}\frac{\partial}{\partial\theta}$, and $e^{-C(r)}\frac{\partial}{\partial z}$. Killing vector fields in general relativity are $\frac{\partial}{\partial\theta}$ and $\frac{\partial}{\partial z}$. Here one can see that we get two more conservation laws in teleparallel theory. Cases (b7) and (c7) are exactly the same.

Case H2 In this case, there exist the following three possibilities which are

(d7) $A = A(t)$ and $B(r) = C(r)$.

(e7) $A(r) = C(r)$ and $B = B(t)$.

(f7) $A(r) = B(r)$ and $C = C(t)$.

In (d7) the space–time (7) can, after a suitable rescaling of t, takes the form

$$ds^2 = -dt^2 + dr^2 + e^{2B(r)}(\,d\theta^2 + dz^2). \tag{67}$$

Teleparallel Killing vector fields in this case are

$$X^0 = c_1, \qquad X^1 = c_2,$$
$$X^2 = e^{-B(r)}c_3 + z\,c_5\,e^{-B(r)}, \quad X^3 = e^{-B(r)}c_4 - \theta\,c_5\,e^{-B(r)}, \tag{68}$$

where $c_1, c_2, c_3, c_4, c_5 \in R$. Here the above space–time (67) admits five linearly independent teleparallel Killing vector fields which can be written as $\frac{\partial}{\partial t}$, $\frac{\partial}{\partial r}$, $e^{-B(r)}\frac{\partial}{\partial\theta}$, $e^{-B(r)}\frac{\partial}{\partial z}$, and $e^{-B(r)}\,(z\frac{\partial}{\partial\theta} - \theta\frac{\partial}{\partial z})$. Killing vector fields in general relativity are $\frac{\partial}{\partial t}$, $\frac{\partial}{\partial\theta}$, $\frac{\partial}{\partial z}$, and $(z\frac{\partial}{\partial\theta} - \theta\frac{\partial}{\partial z})$. Here one can see that we get one more conservation law in teleparallel theory. Cases (e7) and (f7) are exactly the same.

Case I1 In this case, there exist the following three possibilities which are

(a8) $A = A(r)$, $B = B(t)$, and $C = C(t)$.

(b8) $A = A(t)$, $B = B(r)$, and $C = C(t)$.

(c8) $A = A(t)$, $B = B(t)$, and $C = C(r)$.

In (a8) the space–time (7) takes the form

$$ds^2 = -e^{2A(r)}\,dt^2 + dr^2 + e^{2B(t)}\,d\theta^2 + e^{2C(t)}\,dz^2. \tag{69}$$

Teleparallel Killing vector fields in this case are

$$X^0 = e^{-A(r)}c_1,\ X^1 = c_2,$$
$$X^2 = e^{-B(t)}c_3,\ X^3 = e^{-C(t)}c_4, \tag{70}$$

where $c_1, c_2, c_3, c_4 \in R$. Here the above space–time (69) admits four linearly independent teleparallel Killing vector fields which can be written as $e^{-A(r)}\frac{\partial}{\partial t}$, $\frac{\partial}{\partial r}$, $e^{-B(t)}\frac{\partial}{\partial \theta}$, and $e^{-C(t)}\frac{\partial}{\partial z}$. Killing vector fields in general relativity are $\frac{\partial}{\partial \theta}$ and $\frac{\partial}{\partial z}$. Here one can see that we get two more conservation laws in teleparallel theory. Cases (b8) and (c8) are exactly the same.

Case I2 In this case, there exist the following three possibilities which are

(d8) $A = A(r)$ and $B(t) = C(t)$.
(e8) $A(t) = C(t)$ and $B = B(r)$.
(f8) $A(t) = B(t)$ and $C = C(r)$.

In (d8) the space–time (7) takes the form

$$ds^2 = -e^{2A(r)}dt^2 + dr^2 + e^{2B(t)}\left(d\theta^2 + dz^2\right). \tag{71}$$

Teleparallel Killing vector fields in this case are

$$X^0 = e^{-A(r)}c_1, \qquad X^1 = c_2,$$
$$X^2 = e^{-B(t)}c_3 + z\,c_5\,e^{-B(t)}, \quad X^3 = e^{-B(t)}c_4 - \theta\,c_5\,e^{-B(t)}, \tag{72}$$

where c_1, c_2, c_3, c_4, c_5, $\in R$. Here the above space–time (71) admits five linearly independent teleparallel Killing vector fields which can be written as $e^{-A(r)}\frac{\partial}{\partial t}$, $\frac{\partial}{\partial r}$, $e^{-B(t)}\frac{\partial}{\partial \theta}$, $e^{-B(t)}\frac{\partial}{\partial z}$, and $e^{-B(t)}\left(z\frac{\partial}{\partial \theta} - \theta\frac{\partial}{\partial z}\right)$. Killing vector fields in general relativity are $\frac{\partial}{\partial \theta}$, $\frac{\partial}{\partial z}$, and $\left(z\frac{\partial}{\partial \theta} - \theta\frac{\partial}{\partial z}\right)$. Here one can see that we get two more conservation laws in teleparallel theory. Cases (e8) and (f8) are exactly the same.

4 Conclusion

In this paper, we classified cylindrically symmetric non-static space–times according to their teleparallel Killing vector fields. From the above study, it is shown that the above space–times admit four, five, or seven conservation laws. In all the cases of four, five, or seven conservation laws in teleparallel theory of gravitation, the above space–times admit more conservation laws which are different from conservation laws in general relativity. It also turns out that the teleparallel Killing vector fields are multiple of some specific functions (these functions are basically the components of the inverse tetrad field). These functions appear in the teleparallel Killing vector fields because of the non-vanishing torsion components. From the above discussion, it is clear that the presence of the torsion in the non-static cylindrically symmetric space–times increased the number of conservation laws. It will be interesting to see if these extra conservation laws affect the results of the components of energy, momentum, and angular momentum in the context of the teleparallel theory of gravitation as discussed in [13]. From the present work, it can be concluded that torsion has a stronger effect on the space–time than curvature, as torsion produces at least one or at the most four more conservation laws. It is important to mention here that the significance of Killing symmetry in the teleparallel theory is quite clear because the teleparallel Killing vector fields can be studied to compare the results of energy, momentum, and angular momentum in both teleparallel and general relativity theories [13, 17].

Now it is obvious that in this paper we obtained a different set of teleparallel Killing vector fields as compared to Killing vector fields in general relativity. One of the reasons which seems to us is the teleparallel Lie derivative. If one compares the Lie derivative of metric tensor in both the theories, one can easily see that teleparallel Lie derivative contains some extra terms which depends upon torsion of the space–time. These torsion components are dependent upon the metric functions of the space–time. It is clear from the teleparallel Killing equation that both the theories will produce the same Killing vector fields only if the torsion components become zero. This is only possible when the space–time becomes Minkowski [8, 12]. One can easily see that in Minkowski space–times [8, 12] Killing vector fields in both the theories are same. The reasons for compatibility of Killing vector fields other than for Minkowski space–time in both the theories are unknown in literature. A detailed study is needed to find some physical reasons that why Killing vector fields are different in the teleparallel theory as compared to general relativity. May be this will help to study other symmetries in teleparallel theory.

References

1. A. Harvey, E. Schucking, E.J. Surowitz, Am. J. Phys. **74**, 1017 (2006)
2. J.P. Krisch, E.N. Glass, Phy. Rev. D **80**, 044001 (2009)

3. A.Z. Petrov, *Physics Einstein Spaces* (Oxford University Press, Pergamon, 1969)
4. R. Aldrovandi, J.G. Pereira, *An Introduction to Geometrical Physics* (World Scientific, Singapore, 1995)
5. J.G. Pereira, T. Vargas, C.M. Zhang, Class. Quantum Gravity **18**, 833 (2001)
6. V.C. de Andrade, J.G. Pereira, Phy. Rev. D **56**, 4689 (1997)
7. M. Sharif, M.J. Amir, Mod. Physics Lett. A **23**, 963 (2008)
8. G. Shabbir, S. Khan, Mod. Physics Lett. A **25**, 55 (2010)
9. G. Shabbir, S. Khan, Mod. Physics Lett. A **25**, 1733 (2010)
10. G. Shabbir, S. Khan, Commun. Theor. Physics **54**, 469 (2010)
11. G. Shabbir, S. Khan, Commun. Theor. Phys. **55**, 268 (2011)
12. G. Shabbir, S. Khan, Mod. Physics Lett. A **25**, 525 (2010)
13. G.G.L. Nashed, Chin. Phys. B **19**, 020401 (2010)
14. R. Aldrovandi, J.G. Pereira, An Introduction to Gravitation Theory (lecture notes) (2001).
15. V.C. de Andrade, L.C.T. Guillen, J.G. Pereira, In Proceedings of the Ninth Marcel Grossmann Meeting (World Scientific, Singapore, 2000), p. 1022.
16. H. Stephani, D. Kramer, M.A.H. MacCallum, C. Hoenselears, E. Herlt, *Exact Solutions of Einstein's Field Equations* (Cambridge University Press, Cambridge, 2003)
17. G.G.L. Nashed, Astrophys Space Sci. **333**, 317 (2011)