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NUCLEAR PHYSICS

Magnetization of High Density Hadronic Fluid

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Abstract In the present paper the magnetization of a high density relativistic fluid of elementary particles is studied. At very high densities, such as may be found in the interior of a neutron star, when the external magnetic field is gradually increased, the energy of the normal phase of the fluid remains practically constant before extremely high magnetic fields are reached. However, if pion condensation occurs, the energy decreases linearly while the magnetic field strength increases, so that a non vanishing magnetization, independent of the magnetic field, is present. The expression of the magnetization is derived by first considering and solving the Dirac equation of a fermion in interaction with a magnetic field and with a chiral sigma-pion pair. The solution provides the energies of single-particle states. The energy of the system is found by summing up contributions from all particles in the particle fluid. For nuclear densities above 2 to $3\rho_0$, where ρ_0 is the equilibrium nuclear density, the resulting magnetic field turns out to be rather huge, of the order of 10^{17} Gauss.

Keywords Strong magnetic field · Magnetized hadronic matter · Landau levels · Magnetars

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1 Introduction

The properties of a highly compressed fermion fluid, especially with respect to magnetization, are of great interest since they are relevant for stellar objects such as neutron stars and, possibly, quark stars [1]. Recently, magnetars, which are a kind of neutron stars exhibiting extremely powerful magnetic fields, have been discovered [2] and are still being found [3]. We investigate the magnetization of a quark (or nucleon) fluid due to neutral pion condensation under the breaking of chiral symmetry, according to an ansatz which has been proposed in [4] and further investigated in [5, 6]. The main result of this note consists of Eqs. (16) and (19) for the magnetization density. In the Introduction and in Section 2, we follow closely [6].

We consider the Lagrangian density for the fermion field

$$\mathcal{L} = i\bar{\psi}\gamma^{\mu} D_{\mu}\psi - g\bar{\psi}(\sigma + i\gamma_5 \boldsymbol{\tau}\cdot\boldsymbol{\pi})\psi, \tag{1}$$

where $D_{\mu} = \partial_{\mu} - iQA_{\mu}$, with $A^{\mu} = (0, By/2, -Bx/2, 0)$. This Lagrangian describes a Dirac Fermion (quark or nucleon, with electric charge Q) in interaction with a magnetic field and with a chiral $\sigma, \boldsymbol{\pi}$ pair. The corresponding single-particle hamiltonian reads

$$h = -i\boldsymbol{\alpha}\cdot\nabla - Q\boldsymbol{\alpha}\cdot\mathbf{A} + g(\sigma\beta + i\beta\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\pi}).$$

The components of the vector $\boldsymbol{\tau}$ are the familiar isospin Pauli matrices. The charge operator Q is given by $Q = (a + b\tau_3)/2$, with $\tau_3 = \mathrm{diag}(1, -1)$, being $a = b = |e|$ for nuclear matter and $a = |e|/3,\ b = |e|$ for quark matter with flavor $SU(2)$ symmetry. By e we denote the charge of the electron. For simplicity, the coupling with the anomalous magnetic moment of the nucleon is neglected throughout. Its effect is important only for fields stronger than 5×10^{18}G [7]. We will be concerned with weaker fields. In order to simplify the notation, no distinction will be made between the operator Q and the respective eigenvalues.

In the Section 2, we study a quark fluid in interaction with a pion condensate, and, therefore, we consider $\sigma = \bar{\sigma}\cos(\mathbf{q}\cdot\mathbf{r})$, $\pi_1 = \pi_2 = 0$, $\pi_3 = \bar{\sigma}\sin(\mathbf{q}\cdot\mathbf{r})$. The symbols $\mathbf{q}$ and $q = |\mathbf{q}|$ refer to the wave vector associated with pion condensation. Then, h becomes

$$h = -i\boldsymbol{\alpha}\cdot\nabla - \frac{1}{2}QB(x\alpha_y - y\alpha_x) + \beta g\bar{\sigma}\mathrm{e}^{i\gamma_5\tau_3(\mathbf{q}\cdot\mathbf{r})}.$$

Under the transformation $h \to \mathrm{e}^{i\frac{1}{2}\gamma_5\tau_3(\mathbf{q}\cdot\mathbf{r})}h\mathrm{e}^{-i\frac{1}{2}\gamma_5\tau_3(\mathbf{q}\cdot\mathbf{r})}$, and for $\mathbf{q} = (0,0,q)$ directed along the z axis, h reduces to

$$h = -i\boldsymbol{\alpha}\cdot\nabla - \frac{1}{2}q\alpha_z\gamma_5\tau_3 - \frac{1}{2}QB(x\alpha_y - y\alpha_x) + \beta g\bar{\sigma}.$$

We have used the relations

$$\beta\mathrm{e}^{i\frac{1}{2}\gamma_5\tau_3(\mathbf{q}\cdot\mathbf{r})} = \mathrm{e}^{-i\frac{1}{2}\gamma_5\tau_3(\mathbf{q}\cdot\mathbf{r})}\beta, \quad \alpha_j\mathrm{e}^{i\frac{1}{2}\gamma_5\tau_3(\mathbf{q}\cdot\mathbf{r})} = \mathrm{e}^{i\frac{1}{2}\gamma_5\tau_3(\mathbf{q}\cdot\mathbf{r})}\alpha_j.$$

We use the chiral representation, throughout,

$$\beta = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \alpha_i = \begin{pmatrix} \sigma_i & 0 \\ 0 & -\sigma_i \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix},$$

so that the Dirac equation reduces to

$$\begin{pmatrix} -\frac{1}{2}q\tau_3\sigma_z - i\sigma_j D_j - E & M \\ M & -\frac{1}{2}q\tau_3\sigma_z + i\sigma_j D_j - E \end{pmatrix} \times \begin{pmatrix} \phi \\ \psi \end{pmatrix} = 0, \tag{2}$$

where $j \in \{x, y, z\}$, $D_x = \partial_x + \frac{1}{2}iQBy$, $D_y = \partial_y - \frac{1}{2}iQBx$, $D_z = \partial_z = ik_z$, E is the energy eigenvalue. The mass $M = g\bar{\sigma}$ is generated dynamically (all of it). The effect under investigation disappears for $M = 0$, that is, after chiral symmetry has been restored. From (2) it follows that

$$\begin{aligned}&\Big(M^2 - D_x^2 - D_y^2 + k_z^2 - QB\sigma_z - (\frac{1}{2}q\tau_3\sigma_z + E)^2 \\ &\quad + q\tau_3(\sigma_y D_x - \sigma_x D_y)\Big)\phi = 0, \\ &\psi = \frac{1}{M}\left(\frac{1}{2}q\tau_3\sigma_z + i\sigma_x D_x + i\sigma_y D_y - \sigma_z k_z + E\right)\phi. \end{aligned} \tag{3}$$

The operator $(\sigma_y D_x - \sigma_x D_y)$, which acts on 2-spinors, may be represented as a 2×2 matrix,

$$(\sigma_y D_x - \sigma_x D_y) = \begin{pmatrix} 0 & -iD_x - D_y \\ iD_x - D_y & 0 \end{pmatrix}.$$

We have $[iD_x - D_y, iD_x + D_y] = 2QB$. Assuming $QB > 0$, we may define the boson operators

$$a^\dagger = \frac{1}{\sqrt{2QB}}(iD_x + D_y), \quad a = \frac{1}{\sqrt{2QB}}(iD_x - D_y) = \frac{1}{\sqrt{2QB}}(iD_x + D_y)^\dagger.$$

Thus,

$$-D_x^2 - D_y^2 = QB(2a^\dagger a + 1), \quad iD_x + D_y = \sqrt{2QB}\,a^\dagger, \quad iD_x - D_y = \sqrt{2QB}\,a.$$

Since ϕ is a 2-spinor, (3) may be written as

$$\begin{pmatrix} \alpha_+ + \beta\, a^\dagger a & -\delta a^\dagger \\ \delta a & \alpha_- + \beta\, a^\dagger a \end{pmatrix}\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = 0, \quad (QB > 0), \tag{4}$$

where,

$$\alpha_+ = M^2 + k_z^2 - (\frac{1}{2}q\tau_3 + E)^2, \quad \beta = 2QB, \quad \delta = q\tau_3\sqrt{2QB},$$
$$\alpha_- = M^2 + 2QB + k_z^2 - (\frac{1}{2}q\tau_3 - E)^2.$$

The system

$$\begin{aligned}&(\alpha_+ + \beta\, a^\dagger a)\phi_1 - \delta\, a^\dagger\phi_2 = 0, \\ &\delta\, a\,\phi_1 + (\alpha_- + \beta\, a^\dagger a)\phi_2 = 0\end{aligned} \tag{5}$$

is equivalent to (4) and leads to

$$(\alpha_+ + \beta\, a^\dagger a)\,\phi_1 + \delta\, a^\dagger\frac{1}{\alpha_- + \beta\, a^\dagger a}\delta\, a\,\phi_1 = 0,$$

which is equivalent to

$$\left(\alpha_+ + \beta\, a^\dagger a + \frac{1}{\alpha_- + \beta\,(a^\dagger a - 1)}\delta^2 a^\dagger a\right)\phi_1 = 0.$$

The upper component ϕ_1 is an eigenstate of $a^\dagger a$. The lower component ϕ_2 is easily expressed in terms of ϕ_1. Denoting by $\nu = 0, 1, 2, \cdots$, the eigenvalues of $a^\dagger a$, the dispersion relation is the solution of

$$(\alpha_+ + \beta\nu)(\alpha_- + \beta\,(\nu - 1)) + \delta^2\nu = 0.$$

The solutions of this equation read

$$E_{\pm}^{(\pm)}(k_z,\nu)=\pm\sqrt{\frac{1}{4}q^2+k_z^2+M^2+2QB\nu\pm 2\sqrt{\frac{1}{4}q^2(k_z^2+M^2)}}.$$

The case $\nu=0$ deserves special attention. We investigate when $\phi_1=|\nu\rangle$, for $\nu=0$, leads to a solution of (4). Consider the levels

$$E_{-}^{(+)}(k_z,\nu)=\sqrt{\frac{1}{4}q^2+k_z^2+M^2+2QB\nu-2\sqrt{\frac{1}{4}q^2(k_z^2+M^2)}},$$

with the $-$ sign. For $\nu=0$, we have $E_{-}^{(+)}(k_z,0)=\sqrt{k_z^2+M^2}-\frac{1}{2}|q|$. Now, for $\nu=0$, $\phi_1=|\nu\rangle$ leads to a solution of (4) if $\phi_2=0$ and $\alpha_+=0$. The last condition implies $E=\sqrt{k_z^2+M^2}-\frac{1}{2}q\tau_3$, so that $q,\ \tau_3$ should be such that $q\tau_3=|q|$. Obviously, this condition determines the spin polarization of the lowest Landau level (LLL) (see Appendix A). The symbol τ_3 is throughout used to denote either the Pauli matrix or the relevant eigenvalue. Assuming $Q>0$, $B>0$, we must have $\tau_3=1$ so that $q>0$. Since $\phi_2=0,\ \ \phi_1\neq 0$, the spin polarization is also positive. The case $QB<0$ is treated similarly. The only difference is that iD_x+D_y behaves now as the annihilation operator a and iD_x-D_y as the creation operator $a^\dagger$, except for normalization, so that the expressions for $a, a^\dagger$ are essentially interchanged. Moreover, in the dispersion relation, QB should be replaced by $-QB=|QB|$ and (4) should be replaced by

$$\begin{pmatrix}\alpha_+ +\beta\, a^\dagger a & -\delta a\\ \delta a^\dagger & \alpha_- +\beta\, a^\dagger a\end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}=0,\quad (QB<0),\qquad (6)$$

where

$$\alpha_+=M^2-2QB+k_z^2-\left(\frac{1}{2}q\tau_3+E\right)^2,\quad \beta=-2QB,$$
$$\delta=q\tau_3\sqrt{-2QB},$$
$$\alpha_-=M^2+k_z^2-\left(\frac{1}{2}q\tau_3-E\right)^2.$$

It follows that, for the LLL, instead of $q\tau_3=|q|$, now we have $q\tau_3=-|q|$, so that the roles played by the components $\phi_1,\ \phi_2$, are interchanged and spin polarization is inverted with respect to the previous case. We have shown that the spin polarization of the LLL has the direction of the polarizing magnetic field B if the charge of the fermion is positive and has the opposite direction if that charge is negative. The flavors $u,\ d$ have opposite charges and in a spin polarized phase they have also opposite spin polarizations, but they have concordant magnetization.

Pion condensation only occurs in the phase with broken chiral symmetry. It has been argued that the presence of a strong external magnetic field tends to increase the dynamically generated vacuum mass of quarks [8, 9] so that such field might hinder chiral symmetry restoration and stabilize the hadronic fluid [10]. The interplay between an external strong magnetic field and pion condensation is a challenging topic.

2 Magnetized Phase

In this section we study the magnetized phase and we compute the magnetization density, which is the negative slope of the energy density as function of B for constant baryon density. Denoting by $\mathcal{M}$ the magnetization density, we have, by definition

$$\mathcal{M}L^3=-\left(\frac{\partial\mathcal{E}}{\partial B}\right)_{\mathcal{N}}.\qquad (7)$$

We assume that the Fermi energy μ is such that only levels of the type $E_{-}^{(+)}(k_z,\nu)$ are filled up. The LLL with $\nu=0$ belongs to this set of levels. The pion-condensation wave vector $\mathbf{q}$ insures that the levels of the type $E_{+}^{(+)}(k_z,\nu)$ satisfy the condition $E_{+}^{(+)}(k_z,\nu)>\mu$ so that they remain empty. Explicitly, $M+|q|/2>\mu>M-|q|/2\geq 0$. The energy of quark matter reads

$$\begin{aligned}\mathcal{E}&=\frac{L^2|QB|}{2\pi}\sum_{k_z=-\sqrt{\mu^2-M^2}}^{\sqrt{\mu^2-M^2}}\\&\quad\times\left(\frac{1}{2}\left(M_{k_z}-\frac{1}{2}|q|\right)+\sum_{\nu=1}^{\nu_{max}}\sqrt{\left(M_{k_z}-\frac{1}{2}|q|\right)^2+2|QB|\nu}\right),\\&=\Delta\mathcal{E}_0+\mathcal{E}_1,\end{aligned}\qquad (8)$$

where $M_{k_z}=\sqrt{M^2+k_z^2}$, and we have separated the contribution from the LLL, $\Delta\mathcal{E}_0$, from the other terms. The upper limit of the sum on the Landau levels (LL) in (8) is

$$\nu_{max}=\left[\frac{\mu^2-(M_{kz}-\frac{1}{2}|q|)^2}{2|QB|}\right].$$

To evaluate $\mathcal{E}_1$ (which excludes the LLL) we consider magnetic fields which give rise to a large number of LL. In other words, the ratio $(k_F^Q)^2/(2|QB|)$, where k_F^Q denotes the Fermi momentum of quark Q, must be larger than 1 by at least one order of magnitude. From this point of view, a field is considered to be strong if the number of involved Landau levels is small. Under these conditions, we may replace the sum over ν by an integral over the variable $x = 2|QB|\nu$ plus a term linear in $|QB|$, $\Delta\mathcal{E}$, which may be found using the formula for the sum

$$a\sum_{\nu=1}^{N} f(a\nu) \approx \int_0^{Na} \mathrm{d}x f(x) + \frac{1}{2}a(f(Na) - f(0)) + \frac{1}{12}a^2(f'(Na) - f'(0)) + \cdots . \tag{9}$$

The contribution of the LL $\nu = 1, \cdots, \nu_{max}$ to the energy of quark matter reads

$$\mathcal{E}_1 = \frac{L^2|QB|}{2\pi} \sum_{k_z=-\sqrt{\mu^2-M^2}}^{\sqrt{\mu^2-M^2}} \sum_{\nu=1}^{\nu_{max}} \sqrt{(M_{k_z} - \frac{1}{2}|q|)^2 + 2|QB|\nu}, \\ = \frac{L^3}{2(2\pi)^2}\cdot\frac{1}{12}\left(-6M^2(|q|^2+M^2)\log\frac{\frac{1}{2}|q| + \mu + \sqrt{(\frac{1}{2}|q|+\mu)^2 - M^2}}{M} + 2\left(\frac{1}{4}|q|^3 + \frac{13}{4}|q|^2 M - \frac{1}{2}|q|^2\mu - 3M^2\mu + |q|\mu^2 + 6\mu^3\right)\sqrt{(\frac{1}{2}|q|+\mu)^2 - M^2}\right) + \Delta\mathcal{E}, \tag{10}$$

where the correction linear in $|QB|$, associated with the levels $\nu = 1, \cdots, \nu_{max}$, reads

$$\Delta\mathcal{E} = -\frac{L^3}{2(2\pi)^2}|QB| \times \left(M^2 \log\frac{\frac{1}{2}|q| + \mu + \sqrt{(\frac{1}{2}|q|+\mu)^2 - M^2}}{M} + \left(\mu - \frac{1}{2}|q|\right)\sqrt{\left(\frac{1}{2}|q| + \mu\right)^2 - M^2}\right). \tag{11}$$

The calculation has been performed as follows: first, for a fixed ν, we have performed the sum over k_z by replacing it by an integral, as usual; second, we have approximated the sum over ν, from 1 to $[(\mu^2 - (M - \frac{1}{2}|q|)^2)/(2|QB|)]$, by an integral over ν from 0 to $(\mu^2 - (M - \frac{1}{2}|q|)^2)/(2|QB|)$, plus the correction $\Delta\mathcal{E}$.

The contribution of the LLL $\nu = 0$ to the energy reads

$$\Delta\mathcal{E}_0 = \frac{L^3}{(2\pi)^2}|QB| \times \left(M^2 \log\frac{\frac{1}{2}|q| + \mu + \sqrt{(\frac{1}{2}|q|+\mu)^2 - M^2}}{M} + \left(\mu - \frac{1}{2}|q|\right)\sqrt{\left(\frac{1}{2}|q| + \mu\right)^2 - M^2}\right).$$

Next we determine the quark number, in the same limit, e.g. when a large number of LL is involved. We have

$$\mathcal{N} = \mathcal{N}_1 + \Delta\mathcal{N}_0, \tag{12}$$

where the first term contains the contribution of the LL $\nu = 1, \cdots, \nu_{max}$

$$\mathcal{N}_1 = \frac{L^2|QB|}{2\pi}\sum_{k_z}\sum_{\nu=1}^{\nu_{max}} 1 \\ = \frac{L^3}{3(2\pi)^2}\left(\left(-\frac{1}{4}|q|^2 + \frac{1}{2}|q|\mu - 2M^2 + 2\mu^2\right) \times \sqrt{\left(\frac{1}{2}|q| + \mu\right)^2 - M^2} + \frac{3}{2}|q|M^2\log\frac{\frac{1}{2}|q| + \mu - \sqrt{\left(\frac{1}{2}|q|+\mu\right)^2 - M^2}}{M}\right) + \Delta\mathcal{N}, \tag{13}$$

the ranges of the summations and integrations being as in (10). The correction linear in $|QB|$, $\Delta\mathcal{N}$ analogous to the one obtained for the energy, reads

$$\Delta\mathcal{N} = -\frac{L^3}{(2\pi)^2}|QB|\sqrt{\left(\frac{1}{2}|q| + \mu\right)^2 - M^2}. \tag{14}$$

and the contribution of the Landau level $\nu = 0$ to the quark number reads

$$\Delta\mathcal{N}_0 = 2\frac{L^3}{(2\pi)^2}|QB|\sqrt{\left(\frac{1}{2}|q|+\mu\right)^2 - M^2}.$$

In order to compute the magnetization density (7), while B increases, μ must be allowed to change in a well defined way, since $\mathcal{N}$ must remain fixed. The variations $\Delta\mathcal{E}$ (11) and $\Delta\mathcal{N}$ (14), associated to a non-vanishing $|QB|$, have been computed keeping μ fixed. Let $\Delta\mu$, be the variation of μ which insures that the number of fermions does not change when B is introduced. Clearly, $\Delta\mu$ must satisfy

$$\Delta\mathcal{N} + \Delta\mathcal{N}_0 + \partial_\mu\mathcal{N}\Delta\mu = 0.$$

Since

$$\frac{\partial\mathcal{N}}{\partial\mu} = \frac{L^3}{(2\pi)^2}2\mu\sqrt{\left(\frac{1}{2}|q|+\mu\right)^2 - M^2},$$

we find

$$2\mu\Delta\mu = -|QB|.$$

To compute the magnetization for a fixed quark number we must take into account the change in the energy of the system (8) due to the required change in the chemical potential, namely, $\partial_\mu\mathcal{E}\Delta\mu$, where

$$\partial_\mu\mathcal{E} = \frac{\partial\mathcal{E}}{\partial\mu} = \frac{L^3}{2(2\pi)^2}4\mu^2\sqrt{\left(\frac{1}{2}|q|+\mu\right)^2 - M^2}.$$

The terms of the total energy (8) linear in the magnetic field, which define the magnetization density, give

$$\begin{aligned}\Delta\mathcal{E}+\Delta\mathcal{E}_0+\frac{\partial\mathcal{E}}{\partial\mu}\Delta\mu &= -L^3\mathcal{M}|B| \\ &= \frac{L^3}{2(2\pi)^2}|QB|\left(M^2\log\frac{\frac{1}{2}|q|+\mu+\sqrt{(\frac{1}{2}|q|+\mu)^2-M^2}}{M} - \left(\mu+\frac{1}{2}|q|\right)\sqrt{\left(\frac{1}{2}|q|+\mu\right)^2-M^2}\right).\end{aligned} \quad (15)$$

The contribution of the fermions with charge Q to the magnetization density, absolute value of the slope of the energy density for constant baryon density, reads, therefore

$$\begin{aligned}\mathcal{M} &= \frac{|Q|}{2(2\pi)^2} \\ &\times\left(-M^2\log\frac{\frac{1}{2}|q|+\mu+\sqrt{(\frac{1}{2}|q|+\mu)^2-M^2}}{M}\right. \\ &\left.+\left(\mu+\frac{1}{2}|q|\right)\sqrt{\left(\frac{1}{2}|q|+\mu\right)^2-M^2}\right).\end{aligned} \quad (16)$$

Clearly, $\mathcal{M} \geq 0$, as is appropriate for a ferromagnetic phase. A negative slope, indicating a positive magnetization, is typical of a ferromagnetic phase. We recall that the potential energy of a magnetic moment $\mathbf{m}$ subject to an external magnetic field $\mathbf{B}$ is $-\mathbf{m}\cdot\mathbf{B}$. If the phase was paramagnetic, the slope would have been proportional to the applied magnetic field. The argument in favor of a ferromagnetic phase is in line with the findings in [6] where the spontaneous generation of the magnetic field was obtained. We have assumed that $M \neq 0$, so that μ should not be so high that chiral symmetry has been restored. The spin polarization of the LLL $\nu = 0$ is such that the spin polarizations of quarks u and d are opposite, so that each flavor is in agreement with the other one with respect to the orientation of the corresponding magnetization.

Under the approximation, which has been adopted, of neglecting the coupling to the anomalous magnetic moment, the expressions in (10) and (13) also apply to the proton component of a nucleon fluid, as may be easily seen, using the standard treatment.

For completeness, we present the thermodynamical potential $\Omega = \mathcal{E} - \mu\mathcal{N} = \mathcal{E}_1 + \Delta\mathcal{E}_0 - \mu(\mathcal{N}_1 + \Delta\mathcal{N}_0)$, which reads

$$\Omega = \frac{L^3}{2(2\pi)^2}\cdot\frac{1}{48}\left(-24M^2(|q|^2+2|q|\mu^2+M^2)\log\frac{\frac{1}{2}|q|+\mu+\sqrt{(\frac{1}{2}|q|+\mu)^2-M^2}}{M}\right.$$
$$\left.+\left(|q|^3+13|q|^2M+4|q|^2\mu+20M^2\mu-4|q|\mu^2-8\mu^3\right)\sqrt{(\frac{1}{2}|q|+\mu)^2-M^2}\right)-\frac{L^3|QB|}{2(2\pi)^2}$$
$$\times\left((\mu+\frac{|q|}{2})\sqrt{(\frac{1}{2}|q|+\mu)^2-M^2}-M^2\log\frac{\frac{1}{2}|q|+\mu+\sqrt{(\frac{1}{2}|q|+\mu)^2-M^2}}{M}\right).$$

It is well known that the magnetization may equivalently be obtained either from the thermodynamical potential or from the energy,

$$L^3\mathcal{M} = -\left(\frac{\partial\Omega}{\partial B}\right)_\mu = -\left(\frac{\partial\mathcal{E}}{\partial B}\right)_{\mathcal{N}},$$

confirming (16).

3 Normal Phase

In the absence of pion condensation, the contribution to the energy coming from all LL with a specific spin orientation (either ↑ or ↓) will have to be taken into account. If we consider anly the LL with $\nu \geq 1$ we have

$$\mathcal{E}'_{1,\downarrow} = \mathcal{E}'_{1,\uparrow} = \frac{L^2|QB|}{2\pi}\sum_{k_z}\sum_{\nu=1}^{\nu_{max}}\sqrt{k_z^2+M^2+2|QB|\nu},$$
$$= \frac{L^3}{4(2\pi)^2}\left(\mu(2\mu^2-M^2)\sqrt{\mu^2-M^2}-M^4\right.$$
$$\left.\log\frac{\mu+\sqrt{\mu^2-M^2}}{M}\right)-\frac{L^3|QB|}{2(2\pi)^2}\left(\mu\sqrt{\mu^2-M^2}\right.$$
$$\left.+M^2\log\frac{\mu+\sqrt{\mu^2-M^2}}{M}\right),$$

the ranges of the summations and integrations being as in (10). The total energy is

$$\mathcal{E}' = \mathcal{E}'_{1,\uparrow} + \mathcal{E}'_{1,\downarrow} + \mathcal{E}'_0, \tag{17}$$

where the contribution from the LLL is

$$\mathcal{E}'_0 = \frac{L^3|QB|}{(2\pi)^2}\left(\mu\sqrt{\mu^2-M^2}+M^2\log\frac{\mu+\sqrt{\mu^2-M^2}}{M}\right).$$

The term linear in $|QB|$, coming from the $\mathcal{E}'_{1,\uparrow}$ and $\mathcal{E}'_{1,\downarrow}$ terms, cancels the contribution of the LLL to the energy of the quark gas. Also for the quark number a similar effect occurs. The contribution to the quark number coming from all LL with a specific spin orientation (either ↑ or ↓) with $\nu \geq 1$ reads

$$\mathcal{N}'_\downarrow = \mathcal{N}'_\uparrow = \frac{L^2|QB|}{2\pi}\sum_{k_z}\sum_{\nu=1}^{\nu_{max}}1$$
$$= \frac{L^3}{(2\pi)^2}\left(\frac{2}{3}(\mu^2-M^2)^{\frac{3}{2}}-|QB|\sqrt{\mu^2-M^2}\right).$$

The term linear in $|QB|$ cancels the contribution of the LLL to the quark number, which reads

$$\mathcal{N}_0 = \frac{2L^3}{(2\pi)^2}|QB|\sqrt{\mu^2-M^2}\,.$$

We conclude that, in the normal phase, when the magnetic field intensity is such that very many LL contribute to the total energy, the energy of the quark gas is practically independent of $|QB|$, which means that practically no magnetization arises, even for quite large magnetic fields. The same conclusion applies to leptons. In that case, the mass M is not dynamically generated, but is the actual lepton mass. However, as has been observed in [10], if the external field is so large that for the density under consideration the number of relevant Landau levels is too small, then the magnetization becomes actually noticeable. Strictly speaking, formula (9) does not fully apply to the present situation because the coefficient of a^k diverges, for $k \geq 2$. One easily verifies that the derivatives $f^{(k)}(x)$ behave as $\epsilon^{3/2-k}$ for $x = Na - \epsilon$. Numerical experiments indicate that to obtain the total energy of unpolarized quark matter when $2|QB|/(\mu^2 - M^2)$ is very small, the correction term

$$-0.4\mu\left(\frac{2|QB|}{\mu^2 - M^2}\right)^{\frac{3}{2}}$$

must be added to $\mathcal{E}'_{1,\uparrow} + \mathcal{E}'_{1,\downarrow}$. This implies a very weak magnetization behaving as $\sqrt{|B|}$.

4 Polarized Hadronic Matter

We are primarily concerned with magnetization, but not with the mechanism responsible for it and for spin-polarization of hadronic matter. Pion condensation, which has been considered in the previous sections, is only one possibility. Other possibilities have been described in the literature, as, for instance, those proposed in refs. [11, 12]. The occurrence, in these works, of permanent magnetization at quite low densities appears to be related to the use of non-relativistic Skyrme forces. However, a relativistic counterpart of these forces, involving a 4-fermion tensor-tensor interaction, may easily be found, and it would probably lead to a similar behavior. The relativistic, chiral invariant analog of the Skyrme interaction, which is appropriate for an effective QCD approach, reads

$$\begin{aligned}\mathcal{L}_{TT} = G_{TT}[&(\bar{\psi}\gamma^\mu\gamma^\nu\tau_k\psi)(\bar{\psi}\gamma_\mu\gamma_\nu\tau_k\psi)\\ &-(\bar{\psi}\gamma^\mu\gamma^\nu\gamma_5\psi)(\bar{\psi}\gamma_\mu\gamma_\nu\gamma_5\psi)].\end{aligned}$$

If we postulate this interaction, the mean field approximation leads to the Lagrangian density

$$\mathcal{L}_{MFA} = i(\overline{\psi}\gamma^\mu\partial_\mu\psi) - F_k(\overline{\psi}\Sigma_3\tau_k\psi) - \frac{F_k^2}{2G},$$

$$F_k = G\langle\overline{\psi}\Sigma_3\tau_k\psi\rangle,$$

which in turn leads to the Dirac equation

$$(-i\boldsymbol{\alpha}\cdot\nabla + \epsilon_\tau F\beta\Sigma_3)\psi = \varepsilon\psi,$$

where $\epsilon_\tau = 1$ for quarks u and $\epsilon_\tau = -1$ for quarks d denote the eigenvalues of τ_3. At very high density we may assume that chiral symmetry has been restored, so that the current quark mass may be neglected. The eigenvalues of the Dirac equation read

$$\varepsilon_p = \pm\sqrt{\left(|F| \pm \sqrt{p_1^2 + p_2^2}\right)^2 + p_3^2}\,.$$

Let us assume, for simplicity, that $|F| > \mu$. Then the system is fully polarized and only the levels $\sqrt{(|F| - \sqrt{p_1^2 + p_2^2})^2 + p_3^2}$ are filled up so that the Fermi surface is not spherical but doughnut shaped and, taking into account that 6 is the color-flavor degeneracy, the particle number and the energy read

$$N = \frac{6V}{(2\pi)^3}\int \mathrm{d}^3\mathbf{p}\,\theta(\varepsilon_p - \mu) = \frac{12V}{(2\pi)^3}\pi^2|F|\mu^2\,,$$

$$\begin{aligned}E = \frac{6V}{(2\pi)^3}\int \mathrm{d}^3\mathbf{p}\,\theta(\varepsilon_p - \mu)\epsilon_p + \frac{VF^2}{2G} = \frac{8V}{(2\pi)^3}\pi^2|F|\mu^3\\ +\frac{VF^2}{2G}\,,\end{aligned}$$

where the term $VF^2/(2G)$ comes from the standard mean field approximation. Notice that $-\mu \le p_3 \le \mu$ and $F - \mu \le \sqrt{p_1^2 + p_2^2} \le F + \mu$. The thermodynamical potential becomes

$$\Phi = E - N\mu = -\frac{4V}{(2\pi)^3}\pi^2|F|\mu^3 + \frac{VF^2}{2G}\,,$$

implying that

$$F = 4G\frac{\pi^2\mu^3}{(2\pi)^3}\,.$$

The condition $|F| > \mu$ implies $\frac{4\pi^2G\mu^2}{(2\pi)^3} > 1$ ($G\mu^2 > 6.28$) which may occur, depending on the G value, if μ is high enough. Obviously, quarks u and d are oppositely polarized.

4.1 Magnetization of Polarized Matter

In the absence of pion condensation, we find for the partial contribution of the fermions of charge Q to the energy of totally polarized quark (or proton) matter,

$$\begin{aligned}\mathcal{E}(Q) = &\frac{L^3}{2(2\pi)^2}\cdot\frac{1}{2}\\ &\times\left(-M^4\log\frac{\mu + \sqrt{\mu^2 - M^2}}{M}\right.\\ &\left.+ (2\mu^3 - M^2\mu)\sqrt{\mu^2 - M^2}\right) + \frac{L^3}{2(2\pi)^2}|QB|\\ &\times\left(M^2\log\frac{\mu + \sqrt{\mu^2 - M^2}}{m} - \mu\sqrt{\mu^2 - M^2}\right) + \cdots\end{aligned} \tag{18}$$

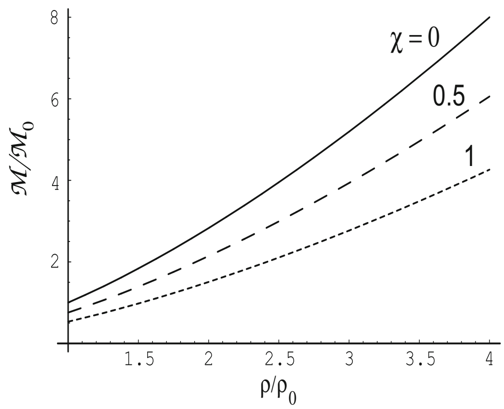

Fig. 1 $\mathcal{M}/\mathcal{M}_0$ versus ρ/ρ_0, for spin-polarized symmetric hadronic matter. Here, $\mathcal{M}_0 = \overline{|Q|}k_{F_0}^2/(8\pi^2) = 8.683 \times 10^{15} G$, ρ_0 is the equilibrium nuclear matter density, k_{F_0} is the corresponding Fermi momentum and $\overline{|Q|} = |Q_u| + |Q_d|$ is the sum of the absolute charges, symmetric hadronic matter being assumed, that is, k_F is the same for both flavors. The curves are for several values of the parameter $\chi = M/k_F$. The upper curve is for $\chi = 0$, the middle curve for $\chi = 1/2$ and lower curve for $\chi = 1$.

where, as previously, $\mu = \sqrt{M^2 + k_F^2}$ is the Fermi energy. For $B \to 0$ the partial contribution of the Fermions of charge Q to the magnetization becomes

$$\mathcal{M}(Q) = \frac{|Q|}{2(2\pi)^2}k_F^2 \left[-(M/k_F)^2 \log\frac{1+\sqrt{1+(M/k_F)^2}}{(M/k_F)} + \sqrt{1+(M/k_F)^2}\right]. \tag{19}$$

For reasonable values of k_F and M, a magnetization of the order of 10^{17} G is found. By reasonable values of k_F and M we mean values of the Fermi energy and of the effective mass for nuclear densities above 2 to $3\rho_0$, where ρ_0 is the equilibrium nuclear density. Fig.1 shows the magnetization for a spin-polarized flavor symmetric fluid of quarks (or nucleons).

5 Conclusions

The magnetization of a high density hadronic fluid of nucleons or quarks has been studied. Pion condensation, which has been proposed as a mechanism of spin-polarization of hadronic matter, is discussed. For nuclear densities above 2 to $3\rho_0$, where ρ_0 is the equilibrium nuclear density, the magnetic field due to spin-polarization of hadronic matter turns out to be rather high, close to 10^{17} Gauss. Such a field is regarded to be "small" if, for the relevant density, the number of involved Landau levels is large enough, that is, if $k_F^2/(2|QB|)$ is larger than 1 by at least an order of magnitude. Our conclusions are in line with the findings in [6], except that we present an explicit expression for the magnetization of hadronic matter. We also suggest that other mechanism, beyond the pion condensation considered in [6], such as the relativistic analogues of those described in [11, 12], may lead to the occurrence of a ferromagnetic phase in hadronic matter, at high densities.

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Appendix A

The eigenfunctions associated with (2) read

$$\phi_1 = |\nu\rangle, \quad \phi_2 = -\frac{\delta\sqrt{\nu}}{\alpha_- + \beta(\nu-1)}|\nu-1\rangle, \quad \nu \neq 0,$$

$$\psi_1 = \frac{1}{M}\left(\frac{1}{2}|q| - k_z + E - \frac{\delta\sqrt{2QB\nu}}{\alpha_- + \beta(\nu-1)}\right)|\nu\rangle,$$

$$\psi_2 = \frac{1}{M}\left(\sqrt{2QB} - \frac{(\frac{1}{2}|q| + k_z + E)\delta}{\alpha_- + \beta(\nu-1)}\right)\sqrt{\nu}|\nu-1\rangle.$$

We wish to investigate the spin of the lowest branch of positive energy states, $E_-^{(+)} = \sqrt{\left(\frac{1}{2}|q| - \sqrt{k_z^2 + M^2}\right)^2 + 2QB\nu}$. We may write

$$\phi_2 = \frac{\sqrt{2|q|^2 QB\nu}}{-(\frac{1}{2}|q| - E)^2 + k_z^2 + M^2 + 2QB\nu}|\nu-1\rangle.$$

The contribution of ψ_1, ψ_2 to the average spin is the same as the contribution of ϕ_1, ϕ_2,

$$\overline{\Sigma}(k_z, \nu) = \frac{\langle\phi_1|\phi_1\rangle - \langle\phi_2|\phi_2\rangle}{\langle\phi_1|\phi_1\rangle + \langle\phi_2|\phi_2\rangle} = \frac{\langle\psi_1|\psi_1\rangle - \langle\psi_2|\psi_2\rangle}{\langle\psi_1|\psi_1\rangle + \langle\psi_2|\psi_2\rangle}.$$

We find that the average spin reads,

$$\overline{\Sigma}(k_z,\nu)=\frac{\sqrt{k_z^2+M^2}-\frac{1}{2}|q|}{\sqrt{(\sqrt{k_z^2+M^2}-\frac{1}{2}|q|)^2+2QB}},$$

so that

$$\frac{L^2|QB|}{(2\pi)}\sum_{k_z\nu}\overline{\Sigma}(k_z,\nu)$$

$$\approx\frac{L^3}{(2\pi)^2}\frac{1}{6}\Big((2\mu^2-|q|^2-8M^2$$

$$-|q|\mu)\sqrt{(\frac{1}{2}|q|+\mu)^2-M^2}+6M^2(|q|+\mu)$$

$$\times\left(\log\left(\frac{1}{2}|q|+\mu+\sqrt{\left(\frac{1}{2}|q|+\mu\right)^2-M^2}\right)-\log M\right)\Big).$$

Although the eigenfunctions of the Dirac hamiltonian are not eigenfunctions of spin (except for the lowest Landau level), the average spin is determined by the components ϕ_1, ϕ_2. We might expect that the magnetization would be obtained multiplying the previous expression by $Q/(2M)$, but that is not really correct since it does not lead to agreement with (16).

Appendix B

Derivation of the formula

$$a\sum_{\nu=1}^{N}f(a\nu)\approx\int_{x=0}^{Na}\mathrm{d}x f(x)+\frac{1}{2}a(f(Na)-f(0))$$
$$+\frac{1}{12}a^2(f'(Na)-f'(0))+\cdots.$$

For $a(\nu-1)<x\le a\nu$, expand $f(x)=f(a\nu)+(x-a\nu)f'(a\nu)+\cdots$ and integrate in the interval $[a(\nu-1),a\nu]$,

$$\int_{x=a(\nu-1)}^{a\nu}\mathrm{d}x f(x)=[x-a\nu]_{x=a(\nu-1)}^{x=a\nu}f(a\nu)$$
$$+\frac{1}{2}[(x-a\nu)^2]_{x=a(\nu-1)}^{x=a\nu}f'(a\nu)+\cdots$$
$$=af(a\nu)-\frac{1}{2}a^2f'(a\nu)+\frac{1}{6}a^3f''(a\nu)+\cdots.$$

Thus,

$$\sum_{\nu=1}^{N}\int_{x=a(\nu-1)}^{a\nu}\mathrm{d}x f(x)=\int_{x=0}^{aN}\mathrm{d}x f(x)=a\sum_{\nu=1}^{N}f(a\nu)$$
$$-\frac{1}{2}a^2\sum_{\nu=1}^{N}f'(a\nu)+\cdots,$$

implying

$$a\sum_{\nu=1}^{N}f(a\nu)=\int_{x=0}^{aN}\mathrm{d}x f(x)+\frac{1}{2}a^2\sum_{\nu=1}^{N}f'(a\nu)$$
$$-\frac{1}{6}a^3\sum_{\nu=1}^{N}f''(a\nu)+\cdots$$
$$\approx\int_{x=0}^{aN}\mathrm{d}x f(x)+\frac{1}{2}a\int_{x=0}^{aN}\mathrm{d}x f'(x)$$
$$+\left(\frac{1}{4}-\frac{1}{6}\right)a^2\int_{x=0}^{aN}\mathrm{d}x f''(x)+\cdots$$
$$=\int_{x=0}^{aN}\mathrm{d}x f(x)+\frac{1}{2}a(f(Na)-f(0))$$
$$+\frac{1}{12}a^2(f'(Na)-f'(0))+\cdots.$$

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