Brazilian Journal of Physics
ISSN: 0103-9733
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Sociedade Brasileira de Física
Brasil

Braz J Phys (2012) 42:120–124
DOI 10.1007/s13538-012-0065-2

SOCIEDADE BRASILEIRA DE FÍSICA

PARTICLES AND FIELDS

An Alternative Approach to the Electric Charge Quantization with Non-global Potentials

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Received: 27 October 2011 / Published online: 21 January 2012

Abstract We present an alternative approach to justify the electric charge quantization by means of non-global electromagnetic potentials. We adopt a non-global potential whose singularity does not goes over to infinity and can be entirely embedded into an arbitrarily small closed ball.

Keywords Gauge theory · Dirac string · Singular potentials

1 Introduction

Since Dirac's seminal paper from 1931 [1], the problem posed by the quantization of the electric charge has been revisited from time to time in many different contexts. Outstanding examples are arguments based on quantum theory with monopoles and/or dyons [2–13], the weak-gauge principle [14], analysis of translation symmetry [3, 15, 16], geometrical considerations [17–21], specially those concerning fiber bundles [22–25], the use of non-divergent potentials [26, 27], classical theory of dyons [28, 29], charge quantization without magnetic monopoles [30, 31], and charge quantization in the context of the standard model [32–36].

In the celebrated paper by Wu and Yang [22], the quantization of electric charge was justified by non-global gauge potentials that describe magnetic monopoles with Dirac strings [2] placed in arbitrary regions of space. These gauge potentials generate non-integrable phases for the wave functions of charged particles and display an unwieldy property: They are divergent along an infinite Dirac string with ends at infinity [3]. The simplest version of the Wu–Yang procedure considers two patches covering space: The first one, $\mathcal{P}_{\rm I}$, is $\Re^3$ with the negative-z semi-axis excluded, while the second patch, $\mathcal{P}_{\rm II}$, is $\Re^3$ with the positive-z semi-axis excluded. In the patches $\mathcal{P}_{\rm I}$ and $\mathcal{P}_{\rm II}$, the field strength of a point-like magnetic monopole g at the origin is given by the vector potentials $\mathbf{A}^{\rm I}$ and $\mathbf{A}^{\rm II}$ given by

$$\mathbf{A}^{\rm I} = g\frac{1-\cos\theta}{r\sin\theta}\hat{\phi}, \tag{1a}$$

and

$$\mathbf{A}^{\rm II} = -g\frac{1+\cos\theta}{r\sin\theta}\hat{\phi}, \tag{1b}$$

respectively. Here, θ is the polar angle of our spherical coordinate system.

The overlap $\mathcal{P}_\cap = \mathcal{P}_{\rm I} \cap \mathcal{P}_{\rm II}$ between the patches is the manifold of the $\Re^3$ space with exception of the

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entire z-axis. In this overlap manifold, the two potentials (1) must be linked by the gauge transformation

$$\mathbf{A}^T = \mathbf{A}^{\mathrm{I}} - \mathbf{A}^{\mathrm{II}} = 2g\frac{1}{r\sin\theta}\hat{\phi} = \nabla\big(2g\phi\big)\,, \tag{2}$$

which has the structure of a Dirac–Aharonov–Bohm potential.

$\mathbf{A}^T$ is the potential of an infinite Dirac string occupying the entire z-axis. By requiring the non-integrable phases introduced for the wave functions of charged particles by the gauge (2) to produce no physical phenomena, we attain the so-called Dirac quantization.

The use of Dirac strings to describe a theory with magnetic monopoles has the advantage of not requiring two vector potentials to ensure gauge invariance [7, 9–11, 29, 37]. The potential is nonetheless cumbersome, for it diverges along an infinite Dirac string with ends at infinity [3].

In previous papers, we have proposed an alternative approach, a variant of the Wu–Yang treatment, to justify the electric charge quantization by means of non-global potentials [30, 31]. Although relying on the Wu–Yang potential (2), we have interpreted it as a pure non-global gauge field, in the sense that it was not defined all over a given region of space, an infinite line, that is. In this interpretation, the factor g on the right-hand side of (2) is a simple gauge parameter, not a monopole intensity. Introduced in connection with the Aharonov–Bohm effect [30], this procedure was subsequently used as a pure gauge transformation [31].

Here we work within the framework put forth in Dirac's classical 1931 paper [1], which sets out to probe the elementary nature of the electron by analyzing singularities of the electromagnetic field and comes to the monopole solutions. We propose to show how a class of non-global gauge-potential configurations may reveal the quantization of elementary charged particles and present an alternative approach to the quantization of the electric charge, on the basis of non-global gauge potentials. Instead of regions that extend to infinity, we now consider a class of potentials whose singularity can be entirely accommodated inside a closed ball with arbitrarily small radius. This choice proves to be advantageous as long as physical objects, such as test charged particles and closed strings, are prevented from staying in the region of undefined potential.

2 A New Class of Potentials

The class of potentials we consider is given by the following expression:

$$\begin{aligned}\mathbf{A}(\mathbf{r}) = {}& g\frac{4R^2}{(\rho-R)^2+z^2}\\ &\times\frac{1}{\big[(\rho+R)^2+z^2\big]^{1/2}}E\left(\sqrt{\frac{4\rho R}{(\rho+R)^2+z^2}}\right)\hat{z}\\ &+g\frac{2}{\rho}\sqrt{\frac{\rho^2+z^2}{(\rho+R)^2+z^2}}\\ &\times\left[\frac{\rho^2+z^2+R^2}{(\rho-R)^2+z^2}E\left(\sqrt{\frac{4\rho R}{(\rho+R)^2+z^2}}\right)\right.\\ &\left.-K\left(\sqrt{\frac{4\rho R}{(\rho+R)^2+z^2}}\right)\right]\hat{\theta}\,,\end{aligned} \tag{3}$$

where g is a constant gauge parameter, E and K are elliptic functions [38], $\rho=\sqrt{x^2+y^2}$ is the cylindrical radial coordinate, and $\theta=\arctan\big(\rho/z\big)$ is the spherical polar angle (see the "Appendix").

Equation (3) is obtained from the usual Dirac expression for the potential of a Dirac string. We therefore have the gauge condition $\nabla\cdot\mathbf{A}$ in the domain of the potential $\mathbf{A}$.

The vector potential (3) is undefined along a circle of radius R centered at the origin on the $z=0$ plane. [1] To see this, note that $\hat{\theta}$ coincides with $\hat{z}$ on the $z=0$ plane and expand the right-hand side of (3) with $z=0$ in a Laurent series around $\rho=R$, which yields

$$\mathbf{A}(z=0,\rho\to R) = 2gR(\rho-R)^{-2}\hat{z}\,. \tag{4}$$

The potential (3) is a pure gauge potential in the sense that $\nabla\times\mathbf{A}=0$ over its entire domain.

While non-global, the gauge (3) offers an advantage in comparison with the ones considered by Wu and Yang [22] and those in [30, 31]. By contrast with the Wu–Yang potentials, whose singularities extend to infinity, it is undefined in a finite region which can be entirely embedded inside a closed ball with arbitrary radius, since R can be arbitrarily small.

[1] Equation (3) is the vector potential due to a closed Dirac string on the circle where (3) is undefined.

In analogy with the Wu–Yang potential, (3) generates a non-integrable phase for the wave functions of charged particles, given by the Wilson loop

$$\Phi = q \oint_\Gamma \mathbf{A}(\mathbf{r}) \cdot \mathrm{d}\mathbf{r}\ , \tag{5}$$

where q is the charge and Γ is an arbitrary closed path that encircles once the region where the potential is undefined, that is, the circle of radius R with center at the origin on the $z = 0$ plane.

Stokes's theorem then shows that the integral on the right-hand side of (5) is independent of Γ, as long as the integration path encircles the region where the potential (3) is ill-defined only once. We can therefore choose any closed path to the integral in (5). For convenience, let the path comprise the z-axis (from $-\infty$ to ∞) and a semi-circle on the $y = 0$ plane, with center at the origin, infinite radius, starting at the point ($x = 0$, $y = 0$, $z = \infty$) and ending at ($x = 0$, $y = 0$, $z = -\infty$). Along the semi-circle, one can show that the integrand on the right-hand side of (5) vanishes and so does the integral.

To compute the integral along the z-axis, we notice that $\mathrm{d}\mathbf{r} = \mathrm{d}z\ \hat{z}$, and that

$$\mathbf{A}(\rho = 0) = \frac{2g\pi R^2}{(R^2 + z^2)^{3/2}}\hat{z}. \tag{6}$$

The following phase then results from (5):

$$\Phi = 4\pi g q. \tag{7}$$

Non-integrable phases lead to Aharonov–Bohm-like effects. In order to prevent the phase in (5) from yielding measurable physical effects, we must have the quantization condition

$$\Phi = 2n_{gq}\pi, \tag{8}$$

or equivalently the condition

$$gq = n_{gq}/2, \tag{9}$$

where n_{gq} is an integer that can depend on the particle charge q and the gauge parameter g.

Next, to discuss electric charge quantization, we proceed with standard arguments. Since the gauge parameter g is independent of the charge, q, we can write that

$$q = n_q q_0\ , \tag{10}$$

where n_q is an arbitrary integer that depends on q and q_0 is a constant with dimension of electric charge. Equation (10) states that any electric charge q is a multiple of a constant q_0 and is hence equivalent to the quantization condition for the electric charge [2, 22].

For completeness, we point out that the Dirac quantization condition (10) can be obtained in the context of fiber bundles and that the gauge potential (3) is an example of a non-trivial connection in a $U(1)$ fiber bundle. To see this, consider an electromagnetic system in which the fields are defined over the entire space $\Re^3$ except on a given point (the system can, for instance, be an electric charge), so that the field configuration is defined on a manifold homotopic to S^2. Let us then take an atlas for S^2 composed by two patches, U_A and U_B, in such a way that U_A contains a neighborhood around the equator and the two poles of the sphere and U_B contains a neighborhood around the equator, but no pole.

With no loss of generality, we for simplicity choose a coordinate system in which the S^2 sphere is crossed only on its poles by the loop singularity where the gauge (3) is undefined. We then have an arc-singularity internal to the S^2-sphere linking its two poles.

In U_A and U_B, the vector potential is described by distinct forms $\mathcal{A}_A$ and $\mathcal{A}_B$, respectively, which can be chosen in such a way that

$$\mathcal{A}_B = \mathcal{A}_A + \mathcal{A} \tag{11}$$

where $\mathcal{A}$ is the form obtained from (3),

$$\begin{aligned}\mathcal{A} = {} & ig\frac{4R^2}{(\rho - R)^2 + z^2}\frac{z}{\sqrt{(\rho + R)^2 + z^2}}\frac{1}{\sqrt{\rho^2 + z^2}} \\ & \times E\left(\sqrt{\frac{4\rho R}{(\rho + R)^2 + z^2}}\right)\mathbf{d}r \\ & + ig\frac{2}{\rho}\frac{1}{(\rho - R)^2 + z^2}\frac{1}{\sqrt{(\rho + R)^2 + z^2}}\mathbf{d}\theta \\ & \times\Bigg[E\left(\sqrt{\frac{4\rho R}{(\rho + R)^2 + z^2}}\right)\left[(\rho^2 + z^2)^2 - R^2(\rho^2 - z^2)\right] \\ & - K\left(\sqrt{\frac{4\rho R}{(\rho + R)^2 + z^2}}\right)\left[(\rho - R)^2 + z^2\right](\rho^2 + z^2)\Bigg]\end{aligned} \tag{12}$$

The overlap $\mathcal{U} = U_A \cap U_B$, from which the poles of the sphere are excluded, is a manifold containing a neighborhood around the equator. Within $\mathcal{U}$, we can find a submanifold homeotopic to S^1, which can be parametrized by an angle $0 \leq \xi < 2\pi$. This submanifold encircles once the arc-singularity internal to the S^2-sphere. We can also define a map from S^1—a sub-set of the overlap $\mathcal{U}$—to $U(1)$, the structure group of electrodynamics, which can be written as

$$t_{B\to A}(\xi) = \exp\left[iq\psi(\xi)\right], \quad (13)$$

where q is any coupling constant of the structure gauge group $U(1)$ (an arbitrary electric charge). For the forms (11) and (13), we have that

$$\mathrm{d}\psi = -i(\mathcal{A}_B - \mathcal{A}_A) = -i\mathcal{A}, \quad (14)$$

where $\mathcal{A}$ is given by (12).

Guided by our discussion of the gauge (3), we now follow a complete turn along S^1. The function $\psi(\xi)$ then changes by

$$\Delta\psi|_{\xi\to\xi+2\pi} = \int_{\xi}^{\xi+2\pi} \mathrm{d}\psi = \int_{\xi}^{\xi+2\pi} -i\mathcal{A} = 4\pi g, \quad (15)$$

while the map (13) changes by

$$\begin{aligned} t_{\mathrm{I}\to\mathrm{II}}(\xi+2\pi) &= \exp\left[iq\psi(\xi+2\pi)\right] \\ &= \exp\left[iq(\psi(\xi) + \Delta\psi|_{\xi\to\xi+2\pi})\right] \\ &= t_{\mathrm{I}\to\mathrm{II}}(\xi)\exp\left(i4\pi q g\right). \end{aligned} \quad (16)$$

It follows that the map (13) is well-defined only if the condition (10) is satisfied.

3 Conclusions an Final Remarks

The construction leading to the quantization condition (10) is a variant of the Wu–Yang procedure with two main advantages: (a) We adopt a pure non-global gauge potential, unrelated to magnetic monopoles, and (b) the region over which the potential is undefined does not extend to infinity, as in Wu's and Yang's approach. It can instead be confined to a closed ball with arbitrary radius and may lie on a region of space where the potential field is naturally ill-defined—a string, for instance. Our analysis comprises both the vector formalism and fiber bundles. Although our discussion was restricted to the gauge produced by a circular Dirac string, the arguments can be extended to any closed Dirac string. As a final remark, we note that, in addition to the quantization (10) of the electric charge, the quantization condition $g = n_g g_0$ for the gauge parameter g also follows from our analysis, where n_g is an integer depending on g and g_0 is a constant with the dimension of g.

Acknowledgments The authors acknowledge financial support from the FAPEMIG and CNPq, without which this work could not have been carried out.

Appendix

This appendix presents the class of potentials (3) considered in this work. It was obtained from the standard expression [39]

$$\mathbf{A}(\mathbf{r}) = g\int_{\ell} \frac{\mathrm{d}\mathbf{l}' \times (\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} \quad (17)$$

for the singular potential due to a Dirac string lying along a path ℓ. Here, g is a constant gauge parameter, not necessarily related to magnetic monopoles.

We choose ℓ to be a circle of radius R centered at the origin on the xy plane. It is convenient to first compute the potential at a point $\mathbf{r}$ on the $y=0$ plane and then rotate the coordinate system to obtain the potential elsewhere. With polar coordinates such that $\mathrm{d}\mathbf{l}' = R d\varphi'\hat{\varphi}'$ and $\mathbf{r}-\mathbf{r}' = (x - R\cos\varphi')\hat{x} - R\sin\varphi'\hat{y} + z\hat{z}$, (17) leads to

$$\begin{aligned} &\mathbf{A}(x, y=0, z) = \\ &\quad = gR\int_0^{2\pi} \mathrm{d}\varphi' \frac{(\hat{x}z - \hat{z}x)\cos\varphi' + \hat{z}R + \hat{y}z\sin\varphi'}{(x^2+z^2+R^2-2Rx\cos\varphi')^{3/2}}. \end{aligned} \quad (18)$$

The contribution from the last term in the numerator of the integrand on the right-hand side vanishes. The remaining two terms are not quite so easy to handle, but their contributions can be expressed as elliptic functions of first and second kinds, K and E, respectively [38]:

$$\begin{aligned} \mathbf{A}(x, y=0, z) &= g\frac{4R^2}{(x-R)^2+z^2}\frac{1}{[(x+R)^2+z^2]^{1/2}} \\ &\quad\times E\left(\sqrt{\frac{4xR}{(x+R)^2+z^2}}\right)\hat{z} \\ &\quad + g\frac{2}{x[(x+R)^2+z^2]^{1/2}} \\ &\quad\times\left[\frac{x^2+z^2+R^2}{(x-R)^2+z^2}E\left(\sqrt{\frac{4xR}{(x+R)^2+z^2}}\right)\right. \\ &\quad\left. - K\left(\sqrt{\frac{4xR}{(x+R)^2+z^2}}\right)\right](\hat{x}z - \hat{z}x). \end{aligned} \quad (19)$$

To find the potential at an arbitrary position $\mathbf{r}$, we rotate the coordinate system around the z-axis by the azimuthal angle φ. The rotation being equivalent to the transformations $x \to \rho$, $\hat{x} \to \hat{\rho}$ and $(\hat{x}z - \hat{z}x) \to \sqrt{\rho^2+z^2}\hat{\theta}$, where $\rho = \sqrt{x^2+y^2}$ is the cylindrical radial coordinate, (19) yields at once the potential $\mathbf{A}(\mathbf{r})$ in (3).

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