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ATOMIC PHYSICS

Operator Entanglement of Two-Qubit Joint Unitary Operations Revisited: Schmidt Number Approach

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Abstract The operator entanglement of two-qubit joint unitary operations is revisited. The Schmidt number, an important attribute of a two-qubit unitary operation, may have connection with the entanglement measure of the unitary operator. We find that the entanglement measure of a two-qubit unitary operators is classified by the Schmidt number of the unitary operators. We also discuss the exact relation between the operator entanglement and the parameters of the unitary operator.

Keywords Unitary operator · Operator entanglement · Schmidt number · Entanglement power

1 Introduction

Unitary operations have become very important in quantum communication and in such entanglement manipulations as quantum cryptography [1], teleportation [2], entanglement swapping [3], quantum states purification [4], and entanglement production [5]. In quantum teleportation, to transfer the unknown quantum state to the remote user, the sender must apply a joint unitary operator on the unknown state particle and one of the entangled particles. In quantum entanglement swapping, a joint unitary transformation on two particles belonging to two different entangled pairs will result in the entanglement of two remote particles without direct interaction. In entanglement purification processes, joint unitary operations and measurements can transfer the entanglement from many partially entangled pairs to few nearly perfectly entangled pairs. In entanglement generation, the joint unitary operations and single qubit operations can entangle the initial product particles.

As these applications indicate, the nonlocal attribute of the bipartite joint unitary transformation plays the chiefly important role. Different aspects of the nonlocal attribute of a bipartite joint unitary operator have been studied, such as entangling power [6], operator entanglement [7, 8], and entanglement-changing power [9]. Entangling power is the mean entanglement (linear entropy) resulting from the action of U on a given distribution of pure product states [6]. Given that a quantum operator belongs to a Hilbert–Schmidt space, one can consider the entanglement of the operator itself, which is called *operator entanglement* [7]. This is a natural extension of the entanglement measures of quantum states [10–13] to the level of general quantum evolutions. Up to now, several methods have been proposed to quantify the entanglement of an unitary bipartite operator. Examples are the linear entropy [7], von Neumann entropy [8], concurrence [14], and Schmidt strength [15]. The relation between the entangling power and operator entanglement measures have also been recently discussed [7, 8, 16].

In general, the entangling power of an unitary operator is related to the operator entanglement measures

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in elaborate or indirect ways. The relation between the different operator entanglement measures is likewise complex. Clarifying the exact relation between the entangling power and different operator entanglement measures and the relation among different operator entanglement measures will help us understand the nonlocal attributes and entanglement capacity of a joint unitary operator. After specifying the entire set of nonlocal features of joint unitary operators, we will be able to choose the optimal unitary operator to produce the desired specifically entangled state and to realize the quantum communication protocols (such as teleportation, entanglement swapping, etc.) in an optimal way, by introducing the appropriate joint operations.[1] For two-qubit unitary operators, there are two methods measuring operator entanglements, i.e. the Schmidt strength and linear entropy, which are shown to have a one-to-one relation for Schmidt number 2. By contrast, no such relation exists for Schmidt number 4 [16]. This result also shows the Schmidt number to be a very important parameter of a unitary operator when its entangling power and operator entanglement are concerned.

Here, we study the operator entanglement of joint two-qubit unitary operators with different Schmidt numbers. We use the linear entropy to measure the operator entanglement of joint two-qubit unitary operators [7, 8] and study the Schmidt number and the entanglement measure of any unitary operator in the four-dimensional Hilbert–Schmidt space. The Schmidt number of two-qubit unitary operators can be 1, 2, or 4 [15]. We will show that the entanglement measure of two-qubit unitary operators is classified by the Schmidt number of the unitary operator. In the light of numerical analysis, we will be able to maximize the operator entanglement for the two-qubit unitary operators. In addition, we will clarify the relation between operator entanglement and the parameters of the unitary operator.

2 Operator Entanglement and Schmidt Number of Two-Qubit Joint Unitary Operations

There exist local unitary operators U_A,U_B ,V_A ,V_B and a two-qubit unitary operator U_d, so that an arbitrary two-qubit unitary operator U_{AB} can be canonically decomposed as follows [17, 18]:

$$U_{AB} = (U_A \otimes U_B) \cdot U_d \cdot (V_A \otimes V_B), \tag{1}$$

[1] A future paper will discuss the construction of appropriate joint unitary operations to optimize these quantum communication protocols.

where $U_d = \exp[-i\vec{\sigma}_A^T\, d\, \vec{\sigma}_B]$ and d is a diagonal matrix. In the light of this theory, any bipartite unitary operator can be decomposed as above. Moreover, since the entanglement measure of a unitary operator must be invariant under local unitary transformations [14], the entanglement measure of any bipartite unitary operator can be simplified into the entanglement measure of the operator U_d. In the standard computational basis, we have that [19]

$$U_d = \begin{pmatrix} e^{-ic_3}c^- & 0 & 0 & -ie^{-ic_3}s^- \\ 0 & e^{ic_3}c^+ & -ie^{ic_3}s^+ & 0 \\ 0 & -ie^{ic_3}s^+ & e^{ic_3}c^+ & 0 \\ -ie^{-ic_3}s^- & 0 & 0 & e^{-ic_3}c^- \end{pmatrix}, \tag{2}$$

where $c^\pm = \cos(c_1 \pm c_2)$, $s^\pm = \sin(c_1 \pm c_2)$ and one can always restrict oneself to the region $\pi/4 \geq c_1 \geq c_2 \geq |c_3|$, the so-called Weyl chamber [18].

Any operator U acting on the systems A and B can be written as the operator Schmidt decomposition [20]

$$U = \sum_l s_l A_l \otimes B_l, \tag{3}$$

where s_l are the Schmidt coefficients with positive values and A_l, B_l are orthonormal operator bases for A and B, respectively.

To calculate the operator entanglement of the unitary operator U_{AB}, we only need to carry out the Schmidt decomposition of the unitary operator U_d. The entanglement measure of a unitary operator can be expressed as [8]

$$E(U) = 1 - \sum_l \frac{s_l^4}{d_1^2 d_2^2}, \tag{4}$$

where d_1 and d_2 are the dimensions of A and B, respectively. We can therefore obtain the entanglement measure for the unitary operator U_d as follows:

$$\begin{aligned} E(U_d) = 1 - \frac{1}{4}\big\{ & 1 - \sin^2(c_1+c_2)\cos^2(c_1+c_2) \\ & - \sin^2(c_1-c_2)\cos^2(c_1-c_2) \\ & + [1 + 2\cos^2(2c_3)]\sin^2(c_1+c_2)\sin^2(c_1-c_2) \\ & + [1 + 2\cos^2(2c_3)]\cos^2(c_1+c_2)\cos^2(c_1-c_2)\big\}. \end{aligned} \tag{5}$$

The Schmidt number [10, 15] is the number of nonzero coefficients s_l. For the unitary operator U_d, the Schmidt coefficients s_l are

$$\begin{aligned} s_1 = \big[& \cos^2(c_1+c_2) + \cos^2(c_1-c_2) \\ & + 2\cos(2c_3)\cos(c_1+c_2)\cos(c_1-c_2)\big]^{1/2}, \end{aligned} \tag{6a}$$

Table 1 The Schmidt number versus the entanglement measure of the unitary operator U_d

Schmidt number of U_d	Operator entanglement of U_d
Sch = 1	$E(U_d) = 0$
Sch = 2	$0 < E(U_d) \le \frac{1}{2}$
Sch = 4	$0 < E(U_d) \le \frac{3}{4}$

$$s_2 = \left[\sin^2(c_1 + c_2) + \sin^2(c_1 - c_2) + 2\cos(2c_3)\sin(c_1 + c_2)\sin(c_1 - c_2)\right]^{1/2}, \qquad (6b)$$

$$s_3 = \left[\sin^2(c_1 + c_2) + \sin^2(c_1 - c_2) - 2\cos(2c_3)\sin(c_1 + c_2)\sin(c_1 - c_2)\right]^{1/2}, \qquad (6c)$$

$$s_4 = \left[\cos^2(c_1 + c_2) + \cos^2(c_1 - c_2) - 2\cos(2c_3)\cos(c_1 + c_2)\cos(c_1 - c_2)\right]^{1/2}. \qquad (6d)$$

Numerical analysis yields the relation in Table 1 between the Schmidt number and entanglement measure of the unitary operator .

The first plot in Fig. 1 shows how the entanglement measure of U_d depends on the parameters c_1 and c_2 for $c_3 = 0$ when the Schmidt number of U_d is 4. As the parameters c_1 and c_2 approach 0, which represents a unit matrix, the entanglement measure of the unitary operator approaches 0. For $c_1 = c_2 = \pi/4$, which represents the SWAP gate, the entanglement measure of the unitary operator reaches the maximum value 3/4. As the parameters c_1 and c_2 grow, the entanglement measure of unitary operator U_d rises.

With $c_3 \neq 0$, the evolution of the operator entanglement is analogous. The last three plots in Fig. 1 show that the minimum entanglement for the Schmidt 4 operator oscillates with c_3, the period and maximum being $\pi/2$ and 0.5, respectively. When $c_3 = \pi/4$, the minimum entanglement reaches its maximum, 0.5. The maximum operator entanglement is still 3/4.

For the Schmidt number 2, if $c_3 \neq 0$, then c_1 and c_2 must vanish. The operator entanglement can therefore be expressed in a very simple form: $E(U_d) = \frac{1}{2}\sin^2(2c_3)$. Figure 2 shows the entanglement measure of U_d as a function of the parameters c_1 and c_2 for $c_3 = 0$ when the Schmidt number of U_d is 2. This curve is the boundary line in the first plot in Fig. 1. As the parameters c_1 and c_2 approach 0, the entanglement measure of unitary operator U_d vanishes. The entanglement measure of the unitary operator rises with the parameters c_1 or c_2. With Schmidt number 2, the entanglement measure of the unitary operator can reach the extremum, 1/2.

Figure 1 shows that, to design an operation with Schmidt number 4, the designer aiming at a specific operator entanglement (or entanglement power) has an infinite number of schemes (i.e., choices of c_1, c_2, and c_3) at his/her disposal. By contrast, if the Schmidt number of the operation is 2, only two design schemes are possible. The Schmidt number 4 operations are therefore superior to the Schmidt number 2 operations. In addition, the maximum operator entanglement of the former operations can reach 3/4, while the maximum operator entanglement of the latter peaks at 1/2. Schmidt number 4 operations are clearly preferable.

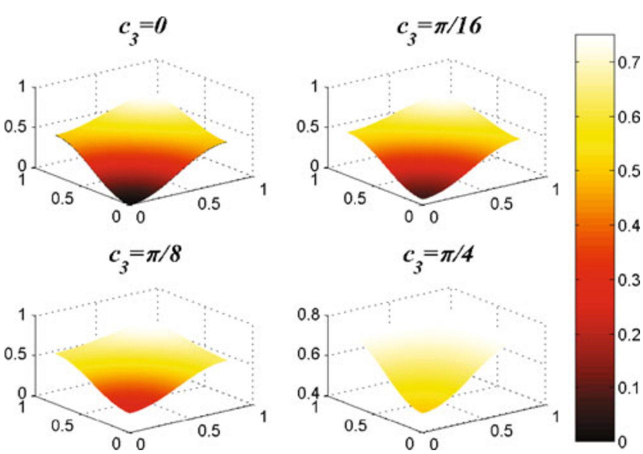

Fig. 1 Entanglement measure of the unitary operator U_d versus the parameters c_1, c_2 for parameter $c_3 = 0, \pi/16, \pi/8$, and $\pi/4$, respectively, when the Schmidt number is 4

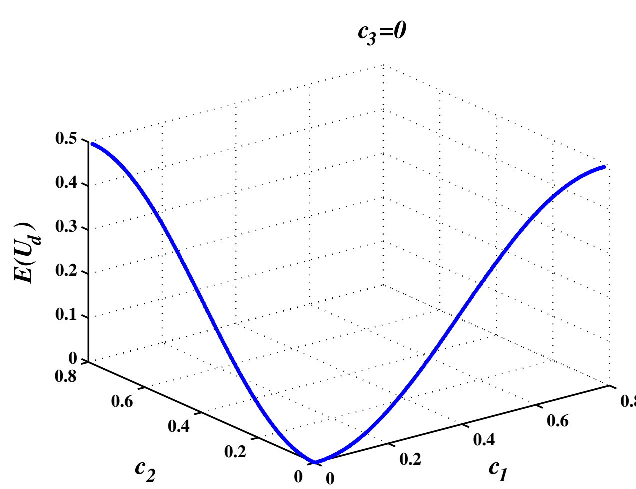

Fig. 2 Entanglement measure of unitary operator U_d as a function of the parameters c_1, c_2 for parameter $c_3 = 0$ when the Schmidt number is 2

3 An Example from Cavity QED

As an illustration of the abstract relation between operator entanglement and the parameters of the unitary operator, we consider two two-level atoms (1, 2) trapped in a single-mode optical cavity. We assume that the two atoms are coupled to the cavity mode with the same coupling constant g. The excited state $|e\rangle_i$ and the ground state $|g\rangle_i$, $(i = 1, 2)$ are the two levels used to encode quantum information. The two atoms have different transition frequencies, $\omega_1 \neq \omega_2$. The frequency of the cavity mode is denoted ω_0. The atom 1 is resonantly driven by an external classical field with coupling constant Ω. Suppose the cavity mode is initially prepared in a vacuum state, under the large detuning condition $\delta_1 = \omega_1 - \omega_0 \gg g$, $\delta_2 = \omega_2 - \omega_0 \gg g$ and in the strong driving regime $\Omega \gg g^2/\delta_1$. The effective Hamiltonian of the total system can then be expressed as [21]

$$H_{\text{eff}} = \frac{\lambda}{2}\sigma_1^x \sigma_2^x, \tag{7}$$

where $\lambda = g^2/\delta_1$ is the effective coupling constant between the two atoms and σ_i^x is the Pauli operator of the ith atom. The unitary transformation induced by this effective Hamiltonian can be expressed as

$$U_{\text{eff}} = \begin{pmatrix} \cos\left(\frac{\lambda t}{2}\right) & 0 & 0 & -i\sin\left(\frac{\lambda t}{2}\right) \\ 0 & \cos\left(\frac{\lambda t}{2}\right) & -i\sin\left(\frac{\lambda t}{2}\right) & 0 \\ 0 & -i\sin\left(\frac{\lambda t}{2}\right) & \cos\left(\frac{\lambda t}{2}\right) & 0 \\ -i\sin\left(\frac{\lambda t}{2}\right) & 0 & 0 & \cos\left(\frac{\lambda t}{2}\right) \end{pmatrix}. \tag{8}$$

If we set $c_1 = \lambda t/2, c_2 = 0$, and $c_3 = 0$ in (2), the right-hand side coincides with the joint unitary operator in Eq. (8). That is to say, the above-mentioned physical process is just a physical realization of the joint unitary operation (2). The Schmidt number of the operator (8) is 2, and its operator entanglement measure can be written $E(U_{\text{eff}}) = \frac{1}{2}\sin^2(\lambda t)$. The relationship between the operator entanglement measure and the effective interaction time λt between the two atoms is depicted in Fig. 3, which shows that the maximum operator entanglement is 1/2 with Sch = 2.

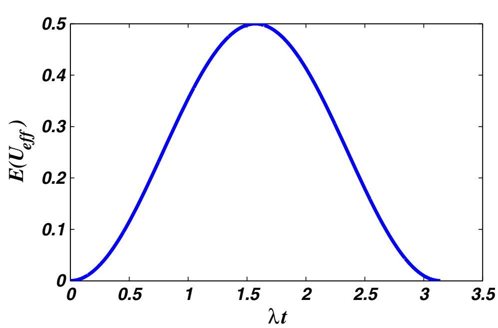

Fig. 3 Entanglement measure of unitary operator U_{eff} versus the effective interaction time λt between the two atoms. Here the Schmidt number of U_{eff} is 2

4 Conclusion

We have discussed the linear entropy and the Schmidt number of an arbitrary two-qubit unitary operator. We have shown that the Schmidt number is closely related to the entanglement measure of unitary operators. For a given operator entanglement in the range $(0, \frac{1}{2}]$, there exist infinite unitary operators with Schmidt number 4, but only 2 unitary operators with Schmidt number 2. For specified operator entanglement, unitary operators with Schmidt number 4 can hence be more easily realized than unitary operators with Schmidt number 2. While the range for the operator entanglement is $(0, \frac{3}{4}]$ for the unitary operators with Schmidt number 4, the range declines to $(0, \frac{1}{2}]$ for unitary operators with Schmidt number 2. It follows that, for certain entanglement requirements, only unitary operators with Schmidt number 4 are available.

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