Sathya Theesar, S. Jeeva; Chandran, R.; Balasubramaniam, P.
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Available in: http://www.redalyc.org/articulo.oa?id=46423465006
Delay-Dependent Exponential Synchronization Criteria for Chaotic Neural Networks with Time-Varying Delays

S. Jeeva Sathya Theesar · R. Chandran · P. Balasubramaniam

Abstract The problem of exponentially synchronizing class of delayed neural networks is studied. Both constant and time-varying delays are considered, to obtain the delay-dependent state feedback synchronization gain matrix. By means of the method of Lyapunov–Krasovskii functional, combined with linear matrix inequalities, exponential synchronization of the master–slave structure of neural networks is achieved. The delay interval is decomposed into multiple nonequidistant subintervals, on which Lyapunov–Krasovskii functionals are constructed. On the basis of these functionals, a new exponential synchronization condition, one that is time-delay dependent, is proposed in terms of linear matrix inequalities. A numerical example showing the effectiveness of the proposed method is presented.

Keywords Exponential synchronization · Delayed neural networks · Time-varying delay · Delay decomposition · Maximum admissible upper bound (MAUB)

1 Introduction

The problem of synchronization arises in numerous practical problems in physics, ecology, and physiology. In 1990, the pioneering work of Pecora and Carroll [1, 2] brought attention to the importance of control and synchronization of chaotic systems. Since then, chaos synchronization has been widely investigated with a view to its applications in secure communication systems [3, 4]. Neural networks have likewise become a field of active research in the past two decades in view of their potential applications in the modeling of complex dynamics; for more details, see [5–20] and references therein. Hopfield neural networks [5] and cellular neural networks [6] have been widely applied in a number of engineering and scientific fields, including image processing, computing technology, and solving linear and nonlinear algebraic equations. Thus, the stability of neural networks [7–11] and the state-estimation problem [12, 13] have become thrust areas of research.

In various dynamical systems, chaotic oscillations are marked by instability. Chaos synchronization phenomena in dynamical systems have offered insight into the functionality of neurobiological networks, in which the natural behavior of neurons displays irregular (chaotic) dynamics. The synchronization of chaotic neural networks has attracted additional attention from researchers since artificial neural network models were shown to exhibit chaotic behavior [14–19]. Reference [16] has discussed sufficient conditions for exponential synchronization of neural networks with time-varying delays in terms of feasible solutions in the form of linear matrix inequalities (LMIs). To obtain sufficient delay-dependent exponential synchronization conditions in terms of LMIs for neural networks with time-varying...
Delays, ref. [19] relied on nonlinear feedback control based on the Lyapunov method.

Delay-dependent criteria have been derived in ref. [20], with the technique of equidistant decomposition of the delay time interval for the asymptotic synchronization problem. Here, the exponential synchronization problem is considered for fast convergence of the error system to the trivial solution. Different error feedback controls are computed on the basis of exponential convergence rate-dependent LMI conditions.

To the best of the authors’ knowledge, the non-equidistant delay decomposition approach to delay-dependent exponential synchronization analysis for neural networks with time-varying delay has never been tackled in the literature. In an illustration of the delay decomposition approach to delay-derived LMIs, the time delay implies less conservatism of delay-dependent exponential synchronization analysis for equidistant delay decomposition approach to delay-derived LMIs.

Implemented in the MATLAB LMI toolbox to solve the solution of the master neural networks (1), and $a_{ij}$ and $b_{ij}$ indicate the interconnection strength among the neurons without and with time-varying delays, respectively. The neuron activation function $f_j$ describes the neuron response to each other, $I_i$ denotes the constant external input, and $u_i(t)$ is a unidirectional-coupled term, a control input designed with a given control objective in mind. We assume the time-varying delay $\tau(t)$ to be a bounded function satisfying

$$0 \leq \tau(t) \leq \bar{\tau} \quad \text{and} \quad \dot{\tau}(t) \leq \mu \quad \text{for} \quad t \geq 0.$$  

The system of (1) and (2) possesses initial conditions $x_i(t) = \phi_i(t) \in \mathbb{C}([-\tau, 0], \mathbb{R})$ and $y_i(t) = \phi_i(t) \in \mathbb{C}([-\tau, 0], \mathbb{R})$, known as the delay history functions for the master and slave systems (1) and (2), respectively, where $\mathbb{C}([-\tau, 0], \mathbb{R})$ denotes the set of all continuous functions from $[-\tau, 0]$ to $\mathbb{R}$.

We further assume $f_j(\cdot)$ to satisfy the following condition:

(A1) Each function $f_j : \mathbb{R} \rightarrow \mathbb{R}$ is bounded and satisfies the Lipschitz condition with Lipschitz constant $l_j > 0$ that is

$$0 \leq \frac{f_j(u) - f_j(v)}{u - v} \leq l_j \quad (j = 1, 2, \ldots, n)$$

for any $u, v \in \mathbb{R}$, with $u \neq v$.

Define the synchronization error $z_i(t) = x_i(t) - y_i(t)$. From (1) and (2), we can then derive the following error dynamic system:

$$\dot{z}_i(t) = -c_i z_i(t) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij} f_j(x_j(t - \tau(t))) + I_i, \quad i = 1, 2, \ldots, n$$

while the corresponding slave system is described by the equality

$$\dot{y}_i(t) = -c_i y_i(t) + \sum_{j=1}^{n} a_{ij} f_j(y_j(t)) + \sum_{j=1}^{n} b_{ij} f_j(y_j(t - \tau(t))) + I_i + u_i(t), \quad i = 1, 2, \ldots, n$$

To the best of the authors’ knowledge, the non-equidistant delay decomposition approach to delay-dependent exponential synchronization analysis for neural networks with time-varying delay has never been tackled in the literature. In an illustration of the delay decomposition approach to delay-derived LMIs, the time delay implies less conservatism of delay-dependent exponential synchronization analysis for equidistant delay decomposition approach to delay-derived LMIs. The gain matrix of the controller for a slave system is determined on the basis of LMIs. An interior point algorithm is implemented in the MATLAB LMI toolbox to solve the derived LMIs.

2 Synchronization Problem Formulation and Preliminaries

Consider unidirectional-coupled neural networks described by the following delay differential equation. The master system is defined by the equality

$$\dot{x}_i(t) = -c_i x_i(t) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij} f_j(x_j(t - \tau(t))) + I_i, \quad i = 1, 2, \ldots, n$$

where $n \geq 2$ denotes the number of neurons in the networks, $x_i$ and $y_i$ are the state variables associated with $i^{th}$ neuron of master and slave systems, $c_i x_i(t)$ is an appropriately behaved function imposing bounds on the solution of the master neural networks (1), and $a_{ij}$ and $b_{ij}$ indicate the interconnection strength among the neurons without and with time-varying delays, respectively. The neuron activation function $f_j$ describes the neuron response to each other, $I_i$ denotes the constant external input, and $u_i(t)$ is a unidirectional-coupled term, a control input designed with a given control objective in mind. We assume the time-varying delay $\tau(t)$ to be a bounded function satisfying

$$0 \leq \tau(t) \leq \bar{\tau} \quad \text{and} \quad \dot{\tau}(t) \leq \mu \quad \text{for} \quad t \geq 0.$$  

The system of (1) and (2) possesses initial conditions $x_i(t) = \phi_i(t) \in \mathbb{C}([-\tau, 0], \mathbb{R})$ and $y_i(t) = \phi_i(t) \in \mathbb{C}([-\tau, 0], \mathbb{R})$, known as the delay history functions for the master and slave systems (1) and (2), respectively, where $\mathbb{C}([-\tau, 0], \mathbb{R})$ denotes the set of all continuous functions from $[-\tau, 0]$ to $\mathbb{R}$.

We further assume $f_j(\cdot)$ to satisfy the following condition:

(A1) Each function $f_j : \mathbb{R} \rightarrow \mathbb{R}$ is bounded and satisfies the Lipschitz condition with Lipschitz constant $l_j > 0$ that is

$$0 \leq \frac{f_j(u) - f_j(v)}{u - v} \leq l_j \quad (j = 1, 2, \ldots, n)$$

for any $u, v \in \mathbb{R}$, with $u \neq v$.

Define the synchronization error $z_i(t) = x_i(t) - y_i(t)$. From (1) and (2), we can then derive the following error dynamic system:

$$\dot{z}_i(t) = -c_i z_i(t) + \sum_{j=1}^{n} a_{ij} g_j(z_j(t)) + \sum_{j=1}^{n} b_{ij} g_j(z_j(t - \tau(t))) - u_i(t), \quad i = 1, 2, \ldots, n.$$ 

where $g_j(z_j(\cdot)) = f_j(z_j(\cdot) + y_j(\cdot)) - f_j(y_j(\cdot)).$
If the state variables of the master system are used to derive the slave system, then the control input vector with state feedback is designed as follows:

\[
\begin{bmatrix}
  u_1(t) \\
u_2(t) \\
\vdots \\
u_n(t)
\end{bmatrix}
= -\begin{bmatrix}
  k_{11} x_1(t) - y_1(t) \\
k_{21} x_2(t) - y_2(t) \\
\vdots \\
k_{n1} x_n(t) - y_n(t)
\end{bmatrix} + \begin{bmatrix}
  k_{12} x_1(t) - y_1(t) \\
k_{22} x_2(t) - y_2(t) \\
\vdots \\
k_{n2} x_n(t) - y_n(t)
\end{bmatrix}
\]

where \( z(t) = [z_1(t), z_2(t), \ldots, z_n(t)]^T \), and \( K = (k_{ij})_{n \times n} \in \mathbb{R}^{n \times n} \) is the state feedback control gain matrix to be determined for exponentially synchronizing both the master and slave systems.

Following ref. [16], we now define a new error \( \hat{e}(t) = e^{\tau(t)} z_i(t) \). From (3) and (4), we can then rewrite the error dynamic system in the following form:

\[
\dot{\hat{e}}_i(t) = -(c_i - \alpha) \hat{e}_i(t) + \sum_{j=1}^{n} a_{ij} \hat{g}_j(\hat{e}_j(t)) \\
+ \sum_{j=1}^{n} b_{ij} \hat{g}_j(\hat{e}_j(t) \tau(t))) \\
- \sum_{j=1}^{n} k_{ij} \hat{e}_j(t), \ i = 1, 2, \ldots, n
\]

(5)

where \( \hat{g}_j(\hat{e}(t)) = e^{\tau(t)} g_j(z(t)) \) and \( \hat{g}_j(\hat{e}(t) \tau(t))) = e^{\tau(t)} g_j(z(t) \tau(t)) \).

Let \( \hat{e} = [\hat{e}_1(\cdots), \hat{e}_2(\cdot), \ldots, \hat{e}_n(\cdot)]^T, \hat{g}(\hat{e}(\cdot)) = [\hat{g}_1(\hat{e}_1(\cdot)), \hat{g}_2(\hat{e}_2(\cdot)), \ldots, \hat{g}_n(\hat{e}_n(\cdot))]^T, \hat{C} = \hat{C} + \hat{K}, \hat{C} = diag([c_1 - \alpha), (c_2 - \alpha), \ldots, (c_n - \alpha)], A = (a_{ij})_{n \times n}, \) and \( B = (b_{ij})_{n \times n} \). Equation (5) can then be transformed to the following compact form:

\[
\dot{\hat{e}} = -\hat{C} \hat{e} + A \hat{g}(\hat{e}(t)) + B \hat{g}(\hat{e}(t) \tau(t)))
\]

(6)

Preliminary to deriving the main results, we state the following definition and lemmas:

**Definition 1** [24] The system (1) and uncontrolled system (2) (that is with \( u = 0 \) in (2)) are said to be exponentially synchronized if there exists a constant \( \gamma(\alpha) \geq 1 \) and \( \alpha > 0 \) such that \( |x_i(t) - y_i(t)| \leq \gamma(\alpha) \exp^{-\alpha t} \) for any \( t \geq 0 \). The constant \( \alpha \) is said to be the degree of exponential synchronization.

**Lemma 2.1** [24] From the definition of \( \hat{e}(t) \) and solution \( z(t) \) of the system (3), if the origin of \( \hat{e}(t) \) is asymptotically convergent, then \( z(t) \) is exponentially convergent with synchronization degree \( \alpha \).

**Lemma 2.2** [21] For any constant matrix \( R \in \mathbb{R}^{n \times n}, R = R^T > 0 \), scalar \( h \) with \( 0 \leq \tau(t) \leq h < \infty \), and a vector-valued function \( \dot{x} : [t - h, t] \rightarrow \mathbb{R}^n \), the following integration is well defined:

\[
-\tau(t) \int_{t-\tau(t)}^{t} \dot{x}(s) R \dot{x}(s) ds \leq \begin{bmatrix}
  x(t) \\
  x(t) - \tau(t)
\end{bmatrix}^T
\times \begin{bmatrix}
  -R & R \\
  * & -R
\end{bmatrix} \begin{bmatrix}
  x(t) \\
  x(t - \tau(t))
\end{bmatrix}
\]

**Lemma 2.3** [23] (Schur Complement) Let \( P, Q, \) and \( R \) be the given matrices of appropriate dimensions such that \( R > 0 \). Then,

\[
\begin{bmatrix}
  P & Q \\
  Q^T & -R
\end{bmatrix} < 0 \quad \Leftrightarrow \quad P + Q^T R^{-1} Q < 0
\]

In the next section, the synchronization criteria will be separately derived in the following two cases:

**Case 1** \( \tau(t) \) is a continuous function satisfying

\[
0 \leq \tau(t) \leq \tau < \infty, \quad \forall \ t \geq 0
\]

(7)

**Case 2** \( \tau(t) \) is a differentiable function satisfying

\[
0 \leq \tau(t) \leq \tau < \infty, \quad \dot{\tau}(t) \leq \mu < \infty, \quad \forall \ t \geq 0
\]

(8)

where \( \tau \) and \( \mu \) are scalars.

**3 Main Results**

In order to derive new delay-dependent exponential synchronization criteria for neural networks with the time-varying delay system described by (1) and (2), we introduce Lyapunov–Krasovskii functionals.

**Theorem 3.1** Under Case 1 and hypothesis (A1), for a given scalar \( \tau > 0 \), the master and slave neural networks (1) and (2) are exponentially synchronized with control gain matrix \( K = \tilde{Y} \tilde{P}^{-1} \) if there exist positive definite symmetric matrices \( \tilde{P} = \tilde{P}^T > 0, \tilde{R}_i = \tilde{R}_i^T > 0 \), any matrices \( \tilde{Y}, \) and symmetric matrices \( \tilde{Q}_i, \tilde{U}_i, \) and \( \tilde{V}_i \) such that

\[
\tilde{Q}_i = \begin{bmatrix}
  \tilde{Q}_i & \tilde{U}_i \\
  \tilde{U}_i^T & \tilde{V}_i
\end{bmatrix} > 0 \quad (i = 1, 2, \ldots, N)
\]
and diagonal matrices $\tilde{S}_j > 0$, $(j = 0, 1, 2, \ldots, N + 1)$ such that the following LMI holds for all $k \in \{1, 2, \ldots, N\}$:

$$
\begin{align*}
\Omega = & \begin{bmatrix}
\Omega_{11} & \Omega_{12} \\
\Omega'_{12} & \Omega_{22}
\end{bmatrix} \\
& = \begin{bmatrix}
\tilde{\alpha}_1 \tilde{R}_1 & 0 & \cdots & 0 & 0 \\
\vdots & \ddots & \ddots & \vdots & \vdots \\
0 & \cdots & \tilde{\alpha}_k & 0 & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & 0 & \cdots & \tilde{\alpha}_N \tilde{R}_N \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 0 & \cdots & 0 \\
\end{bmatrix} < 0
\end{align*}
$$

where

$$
\begin{align*}
\Omega_{11} &= \begin{bmatrix}
\tilde{\alpha}_1 \tilde{R}_1 & 0 & \cdots & 0 & 0 \\
\vdots & \ddots & \ddots & \vdots & \vdots \\
0 & \cdots & \tilde{\alpha}_k & 0 & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & 0 & \cdots & \tilde{\alpha}_N \tilde{R}_N \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 0 & \cdots & 0 \\
\end{bmatrix}^{(N+2)\times(N+2)} \\
\Omega_{12} &= \begin{bmatrix}
\tilde{\rho}_1 & 0 & \cdots & 0 & 0 & B \tilde{P} \\
0 & \tilde{\rho}_2 & \cdots & 0 & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & \tilde{\rho}_k & 0 & 0 \\
0 & \cdots & 0 & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & 0 & 0 & 0 & \tilde{\rho}_N \\
0 & \cdots & 0 & 0 & 0 & \tilde{\rho}_{N+1} \\
0 & \cdots & 0 & 0 & 0 & 0 \\
0 & \cdots & 0 & 0 & 0 & 0 \\
0 & \cdots & 0 & 0 & 0 & 0 \\
\end{bmatrix}^{(N+2)\times(N+2)} \\
\Omega_{22} &= \begin{bmatrix}
\tilde{\gamma}_1 & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & \cdots & \tilde{\gamma}_k & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & 0 & \cdots & \tilde{\gamma}_N & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & 0 & \cdots & 0 & \tilde{\gamma}_{N+1} \\
0 & \cdots & 0 & \cdots & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & 0 & \cdots & 0 & 0 \\
0 & \cdots & 0 & \cdots & 0 & 0 \\
0 & \cdots & 0 & \cdots & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & 0 & \cdots & 0 & 0 \\
0 & \cdots & 0 & \cdots & 0 & 0 \\
0 & \cdots & 0 & \cdots & 0 & 0 \\
0 & \cdots & 0 & \cdots & 0 & 0 \\
\end{bmatrix}^{(N+2)\times(N+2)}
\end{align*}
$$

Proof Let $N > 0$ be an integer and $\tau_j$ $(j = 0, 1, 2, \ldots, N)$ be scalars satisfying

$$
0 = \tau_0 < \tau_1 < \tau_2 < \cdots < \tau_N = \bar{\tau}.
$$

Then the delay interval $[-\tau, 0]$ is nonuniformly decomposed into $N$ segments as

$$
[-\tau, 0] = \bigcup_{j=1}^{N} [-\tau_j, -\tau_{j-1}].
$$

For convenience, we denote $\delta_j$ to be the length of the intervals $[-\tau_j, -\tau_{j-1}]$ that is $\delta_j = \tau_j - \tau_{j-1}$ $(j = 1, 2, \ldots, N)$. Then, choosing different matrix pairs $(\tilde{Q}_j, R_j)$ on $[-\tau_j, -\tau_{j-1}]$, $(j = 1, 2, \ldots, N)$, we construct the following new Lyapunov–Krasovskii functional:

$$
V(\tilde{e}(t)) = V_1(\tilde{e}(t)) + V_2(\tilde{e}(t)) + V_3(\tilde{e}(t)),
$$

where

$$
\begin{align*}
V_1(\tilde{e}(t)) &= \tilde{e}^T(t) P \tilde{e}(t), \\
V_2(\tilde{e}(t)) &= \sum_{j=1}^{N} \int_{-\tau_j}^{-\tau_{j-1}} \tilde{e}^T(t + s) \tilde{Q}_j \tilde{e}(t + s) ds, \\
V_3(\tilde{e}(t)) &= \sum_{j=1}^{N} \delta_j \int_{-\tau_j}^{-\tau_{j-1}} \int_{t+\theta}^{t} \tilde{e}^T(s) R_j \tilde{e}(s) ds d\theta
\end{align*}
$$

(19)
with positive definite matrices \( P = P^T > 0 \), \( R_j = R_j^T > 0 \), and symmetric matrices \( Q_j \), \( U_j \), and \( V_j \) such that

\[
\bar{Q}_j = \begin{bmatrix} Q_j & U_j^T \\ U_j & V_j \end{bmatrix} > 0 \quad (j = 1, 2, \ldots, N).
\]

The derivative of \( V(\hat{e}(t)) \) in (19) with respect to \( t \) along the trajectory of (6) yields

\[
\dot{V}(\hat{e}(t)) = \dot{V}_1(\hat{e}(t)) + \dot{V}_2(\hat{e}(t)) + \dot{V}_3(\hat{e}(t)),
\]

where

\[
\dot{V}_1(\hat{e}(t)) = 2\hat{e}^T(t) P \dot{\hat{e}}(t)
\]

\[
\dot{V}_2(\hat{e}(t)) = \sum_{j=1}^{N} \left[ (\hat{e}^T(t - \tau_{j-1}) Q_j \dot{\hat{e}}(t - \tau_{j-1}) \right.
\]

\[
\left. + \hat{e}^T(t - \tau_{j-1}) 2U_j \tilde{g}(\hat{e}(t - \tau_{j-1})) \right]
\]

\[
+ \hat{g}^T(t) \left( \hat{e}^T(t - \tau_{j-1}) V_j \tilde{g}(\hat{e}(t - \tau_{j-1})) - (\hat{e}^T(t - \tau_{j}) Q_j \dot{\hat{e}}(t - \tau_{j}) \right.
\]

\[
\left. + \hat{e}^T(t - \tau_{j}) 2U_j \tilde{g}(\hat{e}(t - \tau_{j})) \right)
\]

\[
+ \hat{g}^T(t) \left( \hat{e}^T(t - \tau_{j}) V_j \tilde{g}(\hat{e}(t - \tau_{j})) \right] \right],
\]

(22)

\[
\dot{V}_3(\hat{e}(t)) = \sum_{j=1}^{N} \delta_j^2 \hat{e}^T(t) R_j \dot{\hat{e}}(t)
\]

\[- \sum_{j=1}^{N} \delta_j \int_{t - \tau_j}^{t} \hat{e}^T(s) R_j \dot{\hat{e}}(s) ds.\]  

(23)

By the definitions of \( g_j(e_j(t)) \), \( \tilde{g}_j(\hat{e}_j(t)) \), \( \tilde{g}_j(e_j(t - \tau(t))) \), and \( \tilde{g}_j(\hat{e}_j(t - \tau(t))) \) and assumption (A1), we have

\[
0 \leq \frac{g_j(e_j(t))}{e_j(t)} = \frac{f_j(x_j(t)) - f_j(y_j(t))}{x_j(t) - y_j(t)} \leq l_j,
\]

\[
0 \leq \frac{e_j(t)}{e_j(t)} = \frac{\tilde{g}_j(\hat{e}_j(t))}{\hat{e}_j(t)} \leq l_j,
\]

\[
0 \leq \frac{g_j(e_j(t - \tau(t))}{e_j(t - \tau(t)} \leq l_j \quad \text{and}
\]

\[
0 \leq \frac{\tilde{g}_j(\hat{e}_j(t - \tau(t))}{\hat{e}(t - \tau(t))} \leq l_je^{\alpha \tau(t)} \leq l_je^{\alpha \tau(t)}.\]  

(24)

From the diagonal matrices \( S_j \) (\( j = 0, 1, 2, \ldots, N + 1 \)) and from (24), we obtain the result

\[
\sum_{j=1}^{N} \left[ (\hat{e}^T(t - \tau_j) L - \hat{g}^T(\hat{e}(t - \tau_j)) \right] 2S_j \hat{g}(\hat{e}(t - \tau_j)) \right] \geq 0.
\]

(25)

\[
(\hat{e}^T(t - \tau(t)) L e^{\alpha \tau} - \hat{g}^T(\hat{e}(t - \tau(t))))
\]

\[
\times 2S_{N+1} \hat{g}(\hat{e}(t - \tau(t))) \geq 0,
\]

(26)

and from the system (6), the following result:

\[
\sum_{j=1}^{N} \delta_j^2 \hat{e}^T(t) R_j \dot{\hat{e}}(t) = \xi^T(t) \left[ G^T \sum_{j=1}^{N} \delta_j^2 R_j G \right] \xi(t)
\]

(27)

where

\[
\xi(t) = \left[ \hat{e}^T(t) \quad \hat{e}^T(t - \tau_1) \quad \hat{e}^T(t - \tau_2) \ldots \hat{e}^T(t - \tau_N) \right]
\]

\[
\times \hat{e}^T(t - \tau(t)) \quad \hat{g}^T(\hat{e}(t)) \quad \hat{g}^T(\hat{e}(t - \tau_1))
\]

\[
\times \hat{g}^T(\hat{e}(t - \tau_2)) \ldots \hat{g}^T(\hat{e}(t - \tau_N))
\]

\[
\times \hat{g}^T(\hat{e}(t - \tau(t))) \right] \right]^T.
\]

(28)

and

\[
\Gamma = [\Gamma_1 \Gamma_2] \quad \text{with} \quad \Gamma_1 = [-\tilde{C} \quad 0 \quad 0 \quad \ldots \quad 0 \quad 0 \quad 0],
\]

\[
\Gamma_2 = [A \quad 0 \quad 0 \quad \ldots \quad 0 \quad 0 \quad B].
\]

(29)

We now relate \( \hat{e}(t - \tau(t)) \) to \( \hat{e}(t), \hat{e}(t - \tau(t)), \ldots, \hat{e}(t - \tau_N) \). To this end, we take advantage of the integral terms in (23). Since \( \tau(t) \) is a continuous function satisfying (7) \( \forall t \geq 0 \), there should exist a positive integer \( k \in \{1, 2, \ldots, N\} \) such that \( \tau(t) \in [\tau_{(k-1)}, \tau_k] \). It follows that

\[
- \delta_k \int_{t - \tau_k}^{t - \tau_{k-1}} \hat{e}^T(s) R_k \dot{\hat{e}}(s) ds
\]

\[
= - \delta_k \int_{t - \tau_k}^{t - \tau(t)} \hat{e}^T(s) R_k \dot{\hat{e}}(s) ds - \delta_k \int_{t - \tau(t)}^{t - \tau_{k-1}} \hat{e}^T(s) R_k \dot{\hat{e}}(s) ds
\]

\[
\leq - [\tau_k - \tau(t)] \int_{t - \tau_k}^{t - \tau(t)} \hat{e}^T(s) R_k \dot{\hat{e}}(s) ds
\]

\[
- \times [\tau(t) - \tau_{(k-1)}] \int_{t - \tau(t)}^{t - \tau_{k-1}} \hat{e}^T(s) R_k \dot{\hat{e}}(s) ds.
\]

(30)
Applying Lemma 2.2 to the last two integral terms in (30), after simple manipulations we have

\[-\delta_k \int_{t-	au_k}^{t-	au_k-1} \ddot{e}^T(s) R_k \dot{e}(s) ds \leq \eta^T(t) \begin{bmatrix} -R_k & 0 \\ * & -R_k & R_k \\ * & * & -2R_k \end{bmatrix} \eta(t), \quad (31)\]

where

\[\eta(t) = [\ddot{e}^T(t-t_{k-1}) \quad \ddot{e}^T(t-t_k) \quad \ddot{e}^T(t-t) ]^T.\]

For \(j \neq k\), we obtain the following inequality from Lemma 2.2:

\[-\delta_j \int_{t-	au_j}^{t-	au_j-1} \ddot{e}^T(s) R_j \dot{e}(s) ds \leq \left[ \dot{e}(t-t_{j-1}) \right]^T \begin{bmatrix} -R_j & R_j \\ * & -R_j \end{bmatrix} \dot{e}(t-t_j). \quad (32)\]

Combining (31) and (32), we have

\[-\sum_{j=1}^{N} \delta_j \int_{t-t_j}^{t-t_{j-1}} \ddot{e}^T(s) R_j \dot{e}(s) ds \leq \xi^T(t)(\Psi)\xi(t), \quad (33)\]

where

\[\Psi = \begin{bmatrix} -R_1 & R_1 & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 \\ * & -R_2 & R_2 & \ldots & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 \\ * & * & -R_3 & R_3 & \ldots & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ * & * & * & \ldots & -R_N-R_{N-1} & R_N & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 \\ * & * & * & \ldots & * & -R_N & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \end{bmatrix} \] \((2N+4) \times (2N+4)\)

Substituting (21), (22), (23), and (33) in (20) and adding (25) and (26) to (20), we therefore have

\[\dot{V}(\dot{e}(t)) \leq \xi^T(t)[\Phi + \Gamma^T \sum_{j=1}^{N} \delta_j R_j \Gamma] \xi(t), \quad (34)\]

where

\[\Phi = \begin{bmatrix} \alpha_1 & R_1 & 0 & \ldots & 0 & 0 & 0 & 0 & \beta_1 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & 0 \\ \beta_1 & R_2 & \ldots & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & 0 \\ \beta_2 & \ldots & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \beta_{N-1} & \ldots & \ldots & \beta_N & R_N & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & 0 \\ \gamma_1 & \ldots & \ldots & \gamma_N & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & 0 \\ \gamma_2 & \ldots & \ldots & \gamma_{N+1} & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & 0 \\ \gamma_{N+1} & \ldots & \ldots & \gamma_{2N+1} & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \] \((2N+4) \times (2N+4)\)
there exist real diagonal matrices that master–slave systems described by (1) and (2) that:

\[ V = Q_i - Q_{i-1} - R_i - R_{i-1} \quad i = 2, \ldots, N, \]

\[ Q_N - R_N \quad i = N + 1, \]

\[ P A + U_i + L S_0 \quad i = 1, \]

\[ U_i - U_{i-1} + L S_{i-1} \quad i = 2, \ldots, N, \]

\[ -U_N + L S_N \quad i = N + 1, \]

\[ V_i - V_{i-1} - 2 S_{i-1} \quad i = 2, \ldots, N, \]

\[ -V_N - 2 S_N \quad i = N + 1. \]

A sufficient condition for synchronization of the master–slave systems described by (1) and (2) is that there exist real diagonal matrices \( S_j, (j = 1, 2, \ldots, N, N + 1) \), positive-definite matrices \( P = P^T > 0, R_i = R_i^T > 0 \) and symmetric matrices \( Q_i, U_i, \) and \( V_i \) such that

\[ \tilde{Q}_i = \begin{bmatrix} Q_i & U_i \\ \ast & V_i \end{bmatrix} > 0 \quad (i = 1, 2, \ldots, N) \]

with

\[ \dot{V}(\hat{\xi}(t)) \leq \xi^T(t) \left[ \Phi + \Gamma^T \sum_{j=1}^{N} \delta_j^2 R_j \Gamma \right] \xi(t) \]

\[ \leq -\lambda \xi^T(t) \hat{\xi}(t) \]

\[ < 0 \quad \forall \hat{\xi}(t) \neq 0 \quad \text{where} \quad \lambda > 0. \]

In order to guarantee (39), we require the following condition:

\[ \left[ \Phi + \Gamma^T \sum_{j=1}^{N} \delta_j^2 R_j \Gamma \right] < 0 \]

which with the help of Lemma 2.3 can be written as

\[ \left[ \Phi \left[ \Gamma^T \delta_1 R_1 \Gamma^T \delta_2 R_2 \cdots \Gamma^T \delta_N R_N \right] * -R_1 0 \cdots 0 \\ * * -R_2 \cdots 0 \\ \vdots \vdots \ddots \vdots \\ * * * \cdots -R_N \right] < 0, \]

\[ (3N+4) \times (3N+4) \]

(42)

with \( \Phi \) and \( \Gamma \) defined in (35) and (29), respectively. Equation (42) contains bilinear matrix inequalities, the direct solution of which may not be efficient. Thus, we introduce a new matrix transformation for LMIs. Premultiply and postmultiply (42) with

\[ \text{diag}\{ P^{-1}, P^{-1}, \ldots, P^{-1}, P^{-1}, R_i^{-1}, R_2^{-1}, \ldots, R_N^{-1} \}, \]

and change the variables as follows:

\[ P^{-1} Q_i P^{-1} = \hat{Q}_i, \quad P^{-1} U_i P^{-1} = \hat{U}_i, \quad P^{-1} V_i P^{-1} = \hat{V}_i, \]

\[ P^{-1} R_i P^{-1} = \hat{R}_i, \quad (i = 1, 2, \ldots, N), \]

\[ P^{-1} S_j P^{-1} = \hat{S}_j, \quad (j = 0, 1, 2, \ldots, N, N + 1), \]

\[ \hat{K} P^{-1} = \hat{Y}, \quad P^{-1} = \hat{P} \quad \text{and} \quad \hat{C} = C + K. \]

We then obtain the result

\[ \Omega = \begin{bmatrix} \Omega_{11} & \Omega_{12} & \Gamma^T \delta_1 \Gamma \delta_2 \cdots \Gamma^T \delta_N \\ \ast & \Omega_{22} & \Gamma^T \delta_1 \Gamma^T \delta_2 \cdots \Gamma^T \delta_N \\ \ast * -R_1^{-1} & 0 & \cdots & 0 \\ \ast * * -R_2^{-1} \cdots & 0 \\ \vdots \vdots \vdots \ddots \vdots \\ \ast * * * \cdots -R_N^{-1} \end{bmatrix} < 0, \]

\[ (43) \]

where \( \Omega_{11}, \Omega_{12}, \Omega_{22}, \Gamma_1, \) and \( \Gamma_2 \) are defined in (10), (11), (12), (16), and (17).

Equation (43) is not an LMI condition because of the terms \( R_i^{-1} \), which are equal to \( \hat{P} \hat{R}_i^{-1} \hat{P} \). As explained in ref. [13], the inequality \(-\hat{P} \hat{R}_i^{-1} \hat{P} \leq \hat{R} - 2 \hat{P} \) is obtained from \((\hat{P} - \hat{R})^T \hat{R}^{-1} (\hat{P} - \hat{R}) = \hat{P} R^{-1} \hat{P} - 2 \hat{P} + \hat{R} \geq 0.\) Equation (43) can be rewritten as \( \Omega < 0 \), where \( \Omega \) is defined by (9). Considering all possible k’s in the set \( \{1, 2, \ldots, N\} \), from (9) we conclude that that \( \Omega < 0 \) holds for any \( k \in \{1, 2, \ldots, N\} \). This completes the proof.

\[ \square \]

Remark 3.2 Relative to the work in refs. [14] and [19], we have presented a less conservative relaxed sufficient synchronization condition by removing the constraint \( \hat{\tau}(t) = \mu < 1 \).

Remark 3.3 Case 1 includes the constant-time delay as a special case. One can apply Theorem 3.1 to the special case to derive a result.

Case 2 \( \tau(t) \) is a differentiable function satisfying (8). In this case, the derivative of the time-varying delay is known. We will use this additional information to
provide a less conservative result. To this end, we will rewrite \( V(\dot{e}(t)) \) as

\[
\dot{V}(\dot{e}(t)) = V(\dot{e}(t)) + \int_{t-\tau(t)}^{t} \dot{\tilde{g}}(\dot{e}(s))^T W \tilde{g}(\dot{e}(s)) \, ds,
\]

(44)

where \( W = W^T > 0 \) and \( \tilde{W} = P^{-1} W P^{-1} \).

**Theorem 3.4** Under Case 2 and hypothesis (A1), for a given scalar \( \overline{\tau} > 0 \) and \( \mu > 0 \), the master–slave neural networks (1) and (2) are exponentially synchronized with control gain matrix \( K = \tilde{Y} \tilde{P}^{-1} \) if there exist positive definite symmetric matrices \( \tilde{P} = \tilde{P}^T = P^{-1} > 0 \), \( \tilde{W} = \tilde{W}^T > 0 \), \( \tilde{R}_i = \tilde{R}^T_i > 0 \), any matrix \( \tilde{Y} \), and symmetric matrices \( \tilde{Q}_i, \tilde{U}_i, \) and \( \tilde{V}_i \) such that

\[
\dot{\tilde{Q}}_i = \begin{bmatrix} \tilde{Q}_i & \tilde{U}_i & \tilde{V}_i \end{bmatrix} > 0 \quad (i = 1, 2, \ldots, N)
\]

and diagonal matrices \( \tilde{S}_j > 0 \), \( (j = 0, 1, 2, \ldots, N, N + 1) \) such that the following LMI holds for all \( k \in \{1, 2, \ldots, N\} \):

\[
\dot{\tilde{\Omega}} = \begin{bmatrix}
\Omega_{11} & \Omega_{12} & \tilde{G}_i \delta_1 & \tilde{G}_i \delta_2 & \cdots & \tilde{G}_i \delta_N \\
* & \Omega_{22} & \tilde{G}_2 \delta_1 & \tilde{G}_2 \delta_2 & \cdots & \tilde{G}_2 \delta_N \\
* & * & \tilde{R}_1 - 2\tilde{P} & 0 & \cdots & 0 \\
* & * & * & \tilde{R}_2 - 2\tilde{P} & \cdots & 0 \\
* & * & * & * & \cdots & \tilde{R}_N - 2\tilde{P} \\
\end{bmatrix} < 0
\]

(45)

where

\[
\tilde{Q}_{11} = \Omega_{11} + \text{diag}[\tilde{W}, \ldots, 0, -(1-\mu)\tilde{W}],
\]

\[
\tilde{Q}_{22} = \Omega_{22} + \text{diag}[\tilde{W}, \ldots, 0, -(1-\mu)\tilde{W}],
\]

with \( \Omega_{11} \) and \( \Omega_{22} \) are defined in (10) and (12).

**Proof** The proof is similar to that of Theorem 3.1, with the Lyapunov–Krasovskii functional described by (44) and can be omitted. \( \Box \)

**Remark 3.5** Instead of the inequality \( -\tilde{P} \tilde{R}_i^{-1} \tilde{P} \leq \tilde{R} - 2\tilde{P} \), to handle the nonlinear term \( \tilde{P} \tilde{R}_i^{-1} \tilde{P} \) \( (i = 1, 2, \ldots, N) \) in the proof Theorem 3.1, one can introduce new variables \( \tilde{W}_i \) \( (i = 1, 2, \ldots, N) \) such that

\[
-\tilde{P} \tilde{R}_i^{-1} \tilde{P} < -\tilde{W}_i \quad (i = 1, 2, \ldots, N),
\]

which is equivalent to

\[
\begin{bmatrix}
-\tilde{W}_i^{-1} & -\tilde{P}^{-1} \\
\tilde{P}^{-1} & -\tilde{R}_i^{-1} \\
\end{bmatrix} < 0.
\]

Equation (43) then becomes

\[
\Omega = \begin{bmatrix}
\Omega_{11} & \Omega_{12} & \tilde{G}_1 \delta_1 & \tilde{G}_1 \delta_2 & \cdots & \tilde{G}_1 \delta_N \\
* & \Omega_{22} & \tilde{G}_2 \delta_1 & \tilde{G}_2 \delta_2 & \cdots & \tilde{G}_2 \delta_N \\
* & * & \tilde{R}_1 - 2\tilde{P} & 0 & \cdots & 0 \\
* & * & * & \tilde{R}_2 - 2\tilde{P} & \cdots & 0 \\
* & * & * & * & \cdots & \tilde{R}_N - 2\tilde{P} \\
\end{bmatrix} < 0
\]

which is another criterion for exponential synchronization of master–slave systems.

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>MAUB</th>
<th>Controller ( K )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.9956</td>
<td>[ \begin{bmatrix} 5.4023 &amp; -2.7964 \ -0.3927 &amp; 5.0127 \end{bmatrix} ]</td>
</tr>
<tr>
<td>0.5</td>
<td>0.6340</td>
<td>[ \begin{bmatrix} 11.7124 &amp; -4.7274 \ -0.9764 &amp; 12.5219 \end{bmatrix} ]</td>
</tr>
<tr>
<td>1.0</td>
<td>0.3602</td>
<td>[ \begin{bmatrix} 35.3409 &amp; -9.6680 \ -2.2913 &amp; 37.0719 \end{bmatrix} ]</td>
</tr>
<tr>
<td>3.0</td>
<td>0.0332</td>
<td>[ \begin{bmatrix} 2.5247 &amp; 1.0434 \ 0.8346 &amp; 2.8094 \end{bmatrix} ]</td>
</tr>
<tr>
<td>5.0</td>
<td>0.0030</td>
<td>[ \begin{bmatrix} 3.0344 &amp; 1.1937 \ 1.0255 &amp; 3.3787 \end{bmatrix} ]</td>
</tr>
<tr>
<td>7.26</td>
<td>2.0000e–004</td>
<td>[ \begin{bmatrix} 7.1302 &amp; 3.2665 \ 2.1822 &amp; 7.8141 \end{bmatrix} ]</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>MAUB</th>
<th>Controller ( K )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>1.3215</td>
<td>[ \begin{bmatrix} 6.4715 &amp; -3.3576 \ -0.5184 &amp; 6.5584 \end{bmatrix} ]</td>
</tr>
<tr>
<td>0.5</td>
<td>0.7890</td>
<td>[ \begin{bmatrix} 17.2352 &amp; -5.8896 \ -1.3651 &amp; 18.2216 \end{bmatrix} ]</td>
</tr>
<tr>
<td>1.0</td>
<td>0.4437</td>
<td>[ \begin{bmatrix} 45.4054 &amp; -3.4238 \ 1.8483 &amp; 48.2772 \end{bmatrix} ]</td>
</tr>
<tr>
<td>3.0</td>
<td>0.0405</td>
<td>[ \begin{bmatrix} 3.7915 &amp; 1.3848 \ 1.3042 &amp; 4.2543 \end{bmatrix} ]</td>
</tr>
<tr>
<td>5.0</td>
<td>0.0036</td>
<td>[ \begin{bmatrix} 4.4515 &amp; 1.4494 \ 1.4363 &amp; 4.8559 \end{bmatrix} ]</td>
</tr>
<tr>
<td>7.09</td>
<td>3.0000e–004</td>
<td>[ \begin{bmatrix} 7.0034 &amp; 2.7385 \ 2.2547 &amp; 7.7772 \end{bmatrix} ]</td>
</tr>
</tbody>
</table>
Table 3 MAUBs on $\tau(t)$ for various $\mu$'s with $\alpha = 0$ for Remark 3.2

<table>
<thead>
<tr>
<th>$N$</th>
<th>$\mu=0$</th>
<th>$\mu=0.3$</th>
<th>$\mu=0.5$</th>
<th>$\mu=0.8$</th>
<th>$\mu=1.0$</th>
<th>$\mu=1.5$</th>
<th>$\mu=2.0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1.5492</td>
<td>1.5152</td>
<td>1.4796</td>
<td>1.3630</td>
<td>0.9956</td>
<td>0.9956</td>
<td>0.9956</td>
</tr>
<tr>
<td>3</td>
<td>2.3238</td>
<td>2.2728</td>
<td>2.2194</td>
<td>2.0445</td>
<td>1.3215</td>
<td>1.3215</td>
<td>1.3215</td>
</tr>
</tbody>
</table>

Remark 3.6 Reference [16] offered a criterion for exponential synchronization in terms of ARE, which is not easy to solve directly. Using a trial and error method, it can be solved numerically using the MATLAB eigenvalue solvers. To overcome these difficulties, ref. [19] analyzed sufficient conditions in terms of delay-dependent LMIs under the restrictive condition $\mu < 1$. The results of Theorems 3.1 and 3.4 make this restriction unnecessary.

Remark 3.7 Reference [11] decomposed the delay interval $[-\bar{\tau}, 0]$ uniformly into $N$ segments. Here, we have decomposed the delay interval $[-\bar{\tau}, 0]$ into $N$ nonuniform segments and used the augmented matrix $\tilde{Q}$ in $V_2(z_t)$ to obtain a less conservative result for exponential convergence rate $\alpha = 0$ in Theorem 3.1.

Next, we compare our procedure with the result in ref. [11].

Synchronization Algorithm

Our chief purpose is to design a linear error state feedback controller of the form (4), such that the master (1) and the slave (2) systems are exponentially synchronized. Theorems 3.1 and 3.4 provide new criteria for synchronization that depend on the delay. In consistency with the above results, the maximum admissible upper bound of $\bar{\tau}$ is formulated as an optimization problem for the symmetric, positive definite decision variables $\tilde{P}$, $\tilde{Q}_i$, $\tilde{U}_i$, and $\tilde{R}_i$ for $i = 1, 2, \ldots, N$ and $\tilde{S}_j (j = 0, 1, \ldots, N + 1)$ and for all $k \in \{1, 2, \ldots, N\}$.

For example, consider the following problem of finding the maximum admissible upper bound for Case 1 from Theorem 3.1 as

$$\max \bar{\tau} \quad \text{s.t.} \quad \text{LMI (9)}.$$
If the problem described by (46) has a feasible solution set for all \( i \) and \( k \), then there is a delay limit \( \bar{\tau} \), and the corresponding control gain \( K \) exists such that the master system (1) and the slave system (2) are synchronized exponentially. The suboptimal problem can be easily solved by the interior point algorithm in the Matlab LMI toolbox or the cone-complementary algorithm implemented in YALMIP using the SeDuMi solver or any other LMI solvers. In order to obtain the control gain \( K \) while maximizing the delay \( \bar{\tau} \), an iterative algorithm is presented as follows:

**Step 1:** Fix the number of decomposition \( N' \). Set \( j = 0, N = N' \), and \( \delta = 0 \).

**Step 2:** Solve the LMI feasibility problem given in (46) for the positive definite matrices \( \tilde{P}, \tilde{Q}_i, \tilde{R}_k, \tilde{S}_1, \tilde{S}_2 \), and any matrix \( \tilde{Y} \) for \( i = 1, 2, \ldots, N \) and for all \( k \in \{1, 2, \ldots, N\} \).

**Step 3:** If a feasible solution exists and positive value for \( \delta \) exists, then \( \bar{\tau} = N' \delta \), and the control gain is \( K = \tilde{Y} \tilde{P}^{-1} \).

**Step 4:** Set \( j = j + 1 \). If \( K \) and \( \bar{\tau} \) are desirable, end the process. Else, go to step 2 with \( N = N' + 1 \).

### 4 Numerical Example

This section discusses an example from refs. [16] and [19] to show the effectiveness of the derived results.

**Example 4.1** Consider the following two-dimensional neural networks with time-varying delay described by the master system as

\[
x_i(t) = -c_i x_i(t) + \sum_{j=1}^{2} a_{ij} g_j(x_j(t)) + \sum_{j=1}^{2} b_{ij} g_j(x_j(t - \tau(t))), \quad i = 1, 2
\]

where

\[
C = \text{diag}(1, 1) \quad A = (a_{ij})_{2 \times 2} = \begin{bmatrix} 2 & -0.1 \\ -5 & 2 \end{bmatrix},
\]

\[
B = (b_{ij})_{2 \times 2} = \begin{bmatrix} -1.5 & -0.1 \\ -0.2 & -1.5 \end{bmatrix}
\]
and \( g_i(x_i) = \tanh(x_i) \). To achieve synchronization, the slave system is designed as follows:

\[
\dot{y}_i(t) = -c_i y_i(t) + \sum_{j=1}^{2} a_{ij} g_j(y_j(t)) + \sum_{j=1}^{2} b_{ij} g_j(y_j(t - \tau(t))) + u_i(t), \quad i = 1, 2. \tag{48}
\]

As an illustration, ref. [19] exponentially synchronized the system of (47) and (48) for the given exponential degree \( \alpha = 0.6 \) with the controller gain matrix \( K = \begin{bmatrix} 11.6 & 0 \\ 0 & 11.6 \end{bmatrix} \).

The authors also computed the largest exponential synchronization degree \( \alpha = 1.003 \), larger than the result reported in ref. [16]. For comparison, we have determined the maximum admissible upper bounds \( \tau \) for the delay \( \tau(t) \) and the largest exponential synchronization degree for the systems (47) and (48) to be synchronized. The calculated maximum admissible upper bounds and corresponding controller gain matrices of the above systems for various \( N \) are listed in Tables 1 and 2. The largest exponential synchronization degree, \( \alpha = 7.26 \), is larger than the result in ref. [19]. Our results can thus be easily applied to chaotic neural networks with time-varying delay, even though the derivative of the delay is unknown. Table 3 shows the calculated maximum admissible upper bounds \( \tau \) for various \( \mu \) and \( \alpha \), obtained from Theorem 3.4 and numerical results for Remark 3.2.

In view of the results in Theorems 3.1 and 3.4, we have simulated (47) with appropriate parameters and shown their phase portraits in Fig. 1. With no controller applied, even small perturbations in the initial condition \([0.01 \cos(t), 0.01 \sec(t)]^T\) cause large variations in the system evolution, as shown in Fig. 2.

Next, with Theorem 3.1 and the numerical results in Table 2, we simulated the controlled error trajectories for exponential synchronization degrees \( \alpha = 0 \) and \( \alpha = 0.9 \), the results being plotted in Figs. 3 and 4, respectively. Comparison between the two figures shows that exponential synchronization is substantially faster than without the exponential degree (than for \( \alpha = 0 \), that is).

Our findings are therefore of practical interest, since they are more general and carry less conservatism than the results in the literature. They can also be applied to control chaos in a large class of chaotic systems modeled by neural networks.

5 Conclusion

We have derived an exponential synchronization criterion for delayed neural networks. The delay decomposition approach, a new delay-dependent synchronization method for analysis of delayed neural networks, has led us to a new synchronization condition in terms of LMIs, which depends on the size of the time delay. We determined the maximum admissible upper
bounds $\tau$ for the delay $\tau(t)$, the largest exponential synchronization degree, and the corresponding controller gain matrices. Illustrative numerical results have been presented to demonstrate the efficiency of the derived results and its superiority over previously published results.

Acknowledgements The work of one of us (RC) is supported by the University Grant Commission, Government of India, under Faculty Development Programme, XI plan grant. The authors would like to thank the reviewers for their valuable comments and suggestions to improve the quality of this paper.

References

22. Q-L. Han, A delay decomposition approach to stability and H∞ control of linear time-delay systems—part II: H∞ control, in Proceedings of the 7th World Congress on Intelligent Control and Automation (Chongqing, China, 25–27 June 2008)