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GENERAL AND APPLIED PHYSICS

Energy Content of Colliding Plane Waves Using Approximate Noether Symmetries

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Abstract We study the energy content of colliding plane waves using approximate Noether symmetries. For this purpose, we use the approximate Lie symmetry method for Lagrangians for differential equations. We formulate the first-order perturbed Lagrangian for colliding plane electromagnetic and gravitational waves. In both cases, we show that no nontrivial first-order approximate symmetry generator exists.

Keywords Colliding plane waves · Approximate symmetries · Conservation laws

1 Introduction

Conserved quantities, such as the energy and the linear and angular momenta, play important roles in the description of physical systems. Because the energy is not conserved, this quantity and its localization are of paramount importance in general relativity. This question becomes particularly interesting in the context of gravitational waves, the ripples in the fabric of spacetime that are induced by accelerating masses and travel at the speed of light [1]. The existence of such waves was once questioned because by definition they have zero stress-energy tensors. Nonetheless, gravitational waves are found in the solution of Einstein's field equations [2]. This problem, which arises because energy is not well-defined in general relativity, was solved by Ehlers and Kundt [3], Pirani [4], and Weber and Wheeler [5], who considered a sphere of test particles in the path of the waves and showed that the particles acquire constant momentum from the waves. Qadir and Sharif [6] presented an operational procedure, embodying the same principle, to show that gravitational waves impart momentum.

Enormous efforts have been made in search of a satisfactory general solution to this challenging problem. The pseudo-tensors, originally introduced by Einstein, provided a means to enforce global energy-momentum conservation and led many authors to develop alternative prescriptions (see, e.g., [7–13]). None of these nontensorial complexes—all of which are combinations of T_a^b and a pseudo-tensor t_a^b explicating the energy and momentum density of the gravitational field—proved unambiguous.

In a more recent, alternative approach, the concept of approximate symmetry has been used to define the energy of gravitational waves. A number of approximate symmetries have been defined, different methods [14, 15] being available to find them. Within this concept, a measure of the extent of the time-translational symmetry breakdown has been presented [16–18], which yields the so-called almost symmetric space and almost Killing vector, i.e., the vector field corresponding to the almost symmetric space [19]. Unfortunately, each of these attempts has its own drawbacks.

Noether symmetries, also known as symmetries of the Lagrangian, provide a systematic treatment of the differential equations (DEs) arising in many practical problems [20]. Noether and Lie symmetries have many important applications, such as the linearization of nonlinear equations, the reduction of the order of ordinary

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differential equations (ODEs), as well as of the number of independent variables of partial differential equations (PDEs), etc. The double reduction of DEs and the existence of conservation laws or first integrals (Noether invariants) are other interesting features of Noether symmetries [21]. The more symmetries an equation possesses, the easier it will be to integrate it.

The field equations at the basis of the general theory of relativity (GR) are highly nonlinear coupled PDEs. To solve them exactly is one of the major problems in GR. The Noether symmetries approach has been proved to be fruitful in this regard, due to their wide range of applications to DEs [22, 23]. The majority of the physically interesting solutions to the field equations in the literature display some kind of symmetry [24].

The Killing vectors (KVs) form a subalgebra of Noether symmetries, which in turn form a subalgebra of Lie point symmetries. The investigation of Noether symmetries and the corresponding conservation laws for certain static models showed that Noether symmetries yield nontrivial conservation laws that differ from the KVs [22]. The perturbed Lagrangian yields the approximate symmetries and the approximate conserved quantities as constants of motion. By contracting the first-order nontrivial approximate symmetry with the momentum 4-vector, one can obtain energy non-conservation due to the variation of time [25]. Noether's theorem showing that each continuous symmetry generator of the Lagrangian corresponds to a conserved quantity—the conservation of energy coming from the translational invariance of time—is very convenient to define the energy on the basis of the concept of time symmetry.

In the transition from the Minkowski (flat) spacetime to non-flat spacetimes, some of the conservation laws are lost. The connection between spacetime isometries and its related differential equations (geodesics) [26, 27] for different spacetimes recovers the lost conservation laws and yields the corresponding energy rescaling factors [28–31]. We have resorted to this procedure to discuss energy in stringy charged black-hole solutions [32] and the Bardeen model [33]. A recent paper [25] has used the same procedure to discuss the energy contents of pp and cylindrical gravitational waves. Here we extend the procedure to colliding plane electromagnetic and gravitational waves to check the existence of conserved quantities.

The plan of the paper is as follows: In the next section, we review the mathematical framework for exact and approximate symmetry methods for the Lagrangian of DEs. Sections 2.1 and 2.2 are dedicated to the approximate Noether symmetries of colliding plane electromagnetic and gravitational waves, respectively. The last section provides a summary and results.

2 Mathematical Formulation

Noether's theorem was proved in 1915 and published in 1918 by Emmy Noether. It connects the continuous symmetries of a physical system with conserved quantities and takes advantage of various transformations to provide insight into general theories and information on conservation laws [34]. If the results of an experiment are independent of position (space homogeneity) and time, then the formulated Lagrangian is symmetric under continuous spacetime translations, and the theorem leads to the laws of energy and linear momentum conservation (http://mathworld.wolfram.com/NoethersSymmetryTheorem.html) [35]. Similarly, if the experiment proves independent of a rotation angle, then the physical system is rotationally symmetric, and the theorem implies conservation of angular momentum.

Noether's theorem has a wide range of applications in theoretical physics and variational calculus. Dissipative systems constitute an example of systems that cannot be modeled with a Lagrangian. Noether's theorem is therefore inapplicable, and no conservation law follows from their continuous symmetries.

The original form of the theorem covers Lagrangians containing only first-order derivatives. In many practical problems, the DEs include small terms, known as the perturbation parameters, which are associated with small errors or corrections and can probe the DEs in certain limits. The Lie point symmetries of such perturbed DEs are very important. The application of Noether symmetry analysis to the Lagrangian of a system identifies those symmetries that directly yield the desired conserved quantities, only the first-order prolongation of the symmetry generator being needed.

We now present the procedure that identifies the symmetries of the Lagrangian [36, 37]. To this end, we consider the vector field $\mathbf{X}$ defined by the equality

$$\mathbf{X} = \xi(s, x^a)\frac{\partial}{\partial s} + \eta^b(s, x^a)\frac{\partial}{\partial x^b}$$

and its first-order prolongation

$$\mathbf{X}^{[1]} = \mathbf{X} + \left(\eta^b_{,s} + \eta^b_{,a}\dot{x}^a - \xi_{,s}\dot{x}^b - \xi_{,a}\dot{x}^a\dot{x}^b\right)\frac{\partial}{\partial \dot{x}^b},$$
$$(a, b = 0, 1, 2, 3). \qquad (1)$$

Suppose now that a second-order ODE, i.e., an Euler–Lagrange equation, is given by

$$\ddot{x}^a = g\left(s, x^a, \dot{x}^a\right). \tag{2}$$

The vector field $\mathbf{X}$ is said to be the Noether point symmetry of the Lagrangian $L(s, x^a, \dot{x}^a)$ associated with (2) if there exists a function $A(s, x^a)$ such that the following condition is satisfied:

$$\mathbf{X}^{[1]}L + (D_s\xi)L = D_s A. \tag{3}$$

Here, D_s is the total derivative operator, given by

$$D_s = \frac{\partial}{\partial s} + \dot{x}^a \frac{\partial}{\partial x^a} \tag{4}$$

and A is a gauge function.

The significance of Noether symmetries is given by the following theorem:

Theorem 1 *Let $L(s, x^a, \dot{x}^a)$ be the Lagrangian corresponding to second-order ODE given by (2) and $\mathbf{X}$ be its corresponding Noether point symmetry. Then the first integral of motion associated with $\mathbf{X}$ is defined by the equality [38]*

$$I = \xi L + \left(\eta^a - \dot{x}^a \xi\right) \frac{\partial L}{\partial \dot{x}^a} - A. \tag{5}$$

We can now define approximate Noether symmetries for DEs. The first-order perturbed Lagrangian corresponding to the first-order perturbed DE

$$\mathbf{E} = \mathbf{E}_0 + \epsilon \mathbf{E}_1 + O\left(\epsilon^2\right)$$

is given by the expression

$$L(s,\ x^a, \dot{x}^a, \epsilon) = L_0(s, x^a, \dot{x}^a) + \epsilon L_1(s, x^a, \dot{x}^a) + O\left(\epsilon^2\right) \tag{6}$$

so that the function $\int_V L\,\mathrm{d}s$ is invariant under the one-parameter group of transformations with the approximate Lie symmetry generator

$$\mathbf{X} = \mathbf{X}_0 + \epsilon \mathbf{X}_1 + O\left(\epsilon^2\right)$$

and

$$A = A_0 + \epsilon A_1$$

is the first-order perturbed gauge function. Here

$$\mathrm{X}_b = \xi_b \frac{\partial}{\partial s} + \eta_b^a \frac{\partial}{\partial x^a}; \quad a = 0, 1, 2, 3,\ b = 0, 1.$$

The conditions for calculating the exact and first-order symmetries are then the following:

$$\mathbf{X}_0^{[1]} L_0 + (D_s\xi_0) L_0 = D_s A_0, \tag{7}$$

$$\mathbf{X}_1^{[1]} L_0 + \mathbf{X}_0^{[1]} L_1 + (D_s\xi_1) L_0 + (D_s\xi_0) L_1 = D_s A_1, \tag{8}$$

where $\mathbf{X}_0$ and $\mathbf{X}_1$ are the exact and first-order parts, respectively, of the symmetry generator $\mathbf{X}$. The perturbed Lagrangian always admits a symmetry $\epsilon\mathbf{X}_0$, known as the trivial *symmetry. If a symmetry generator $\mathbf{X}$ can be found with $\mathbf{X}_0 \neq 0$ and $\mathbf{X}_1 \neq k\mathbf{X}_0$, where k is an arbitrary constant, then $\mathbf{X}$ is called the* nontrivial *symmetry. The corresponding exact and the first-order approximate parts of the first integral are given by*

$$I_0 = \xi_0 L_0 + \left(\eta_0^a - \dot{x}^a \xi_0\right) \frac{\partial L_0}{\partial \dot{x}^a} - A_0, \tag{9}$$

$$I_1 = \xi_0 L_1 + \xi_1 L_0 + \left(\eta_0^a - \dot{x}^a \xi_0\right) \frac{\partial L_1}{\partial \dot{x}^a} + \left(\eta_1^a - \dot{x}^a \xi_1\right) \frac{\partial L_0}{\partial \dot{x}^a} - A_1. \tag{10}$$

The first integral of motion $I = I_0 + \epsilon I_1$ is said to be unstable (stable) if $I_0 = 0$ ($I_0 \neq 0$).

More detailed discussions can be found in [38–40].

2.1 Exact and Approximate Symmetries of the Colliding Plane Electromagnetic Waves

In 1974, Bell and Szekeres presented an exact solution of the field equations describing the collision and the subsequent interaction of two electromagnetic waves [41]. They considered two-step electromagnetic waves colliding on a flat background region. The spacetime is defined in terms of four regions [42]. Under the coordinate transformations $u = (t - x)/2$, $v = (t + x)/2$ [43], the spacetimes in those regions can be written as [44]

Region I $(t < z,\ t < -z)$:

$$\mathrm{d}s^2 = \frac{\mathrm{d}t^2}{2} - \mathrm{d}x^2 - \mathrm{d}y^2 - \frac{\mathrm{d}z^2}{2}, \tag{11}$$

Region II $(t > z,\ t < -z)$:

$$\mathrm{d}s^2 = \frac{\mathrm{d}t^2}{2} - \cos^2 a\left(\frac{t - z}{2}\right)\left(\mathrm{d}x^2 + \mathrm{d}y^2\right) - \frac{\mathrm{d}z^2}{2}, \tag{12}$$

Region III $(t < z,\ t > -z)$:

$$\mathrm{d}s^2 = \frac{\mathrm{d}t^2}{2} - \cos^2 b\left(\frac{t + z}{2}\right)\left(\mathrm{d}x^2 + \mathrm{d}y^2\right) - \frac{\mathrm{d}z^2}{2}, \tag{13}$$

Region IV $(t > z,\ t > -z)$:

$$\mathrm{d}s^2 = \frac{\mathrm{d}t^2}{2} - \cos^2(At - Bz)\mathrm{d}x^2 - \cos^2(Az - Bt)\mathrm{d}y^2 - \frac{\mathrm{d}z^2}{2}, \tag{14}$$

where $A=(a-b)/2$ and $B=(a+b)/2$. A special property of the solution is conformal flatness: All components of the Weyl tensor vanish.

In order to discuss Noether symmetries for colliding plane electromagnetic waves, we define the Lagrangian for the spacetime in (14) as follows:

$$L=\frac{\dot{t}^2}{2}-\cos^2(At-Bz)\dot{x}^2-\cos^2(Az-Bt)\dot{y}^2-\frac{\dot{z}^2}{2}. \tag{15}$$

To evaluate the exact Noether symmetries of this Lagrangian, we first substitute in (7) the Lagrangian, the first-order prolongation of the symmetry generator (1), and the total derivative operator (4) and then compare the coefficients of the coordinate derivatives and their products on both sides. This leads to the following set of DEs:

$$\xi_{,t}=0,\quad \xi_{,x}=0,\quad \xi_{,y}=0,\quad \xi_{,z}=0,\quad A_{,s}=0, \tag{16}$$

$$\eta^0_{,s}=A_{,t},\quad -2\cos^2(At-Bz)\eta^1_{,s}=A_{,x}, \tag{17}$$

$$-2\cos^2(Az-Bt)\eta^2_{,s}=A_{,y},\quad -\eta^3_{,s}=A_{,z}, \tag{18}$$

$$\eta^0_{,x}-2\cos^2(At-Bz)\eta^1_{,t}=0, \tag{19}$$

$$\eta^0_{,y}-2\cos^2(Az-Bt)\eta^2_{,t}=0,\quad \eta^0_{,z}-\eta^3_{,t}=0, \tag{20}$$

$$\cos^2(At-Bz)\eta^1_{,y}+\cos^2(Az-Bt)\eta^2_{,x}=0, \tag{21}$$

$$\eta^3_{,x}+2\cos^2(At-Bz)\eta^1_{,z}=0,\quad 2\eta^0_{,t}-\xi_{,s}=0, \tag{22}$$

$$\eta^3_{,y}+2\cos^2(Az-Bt)\eta^2_{,z}=0,\quad 2\eta^3_{,z}-\xi_{,s}=0, \tag{23}$$

$$2\eta^1_{,x}-\xi_{,s}-2(A-B)\tan(At-Bz)(\eta^0+\eta^3)=0, \tag{24}$$

$$2\eta^2_{,y}-\xi_{,s}-2(A-B)\tan(Az-Bt)(\eta^0+\eta^3)=0. \tag{25}$$

We have to solve this system of 19 DEs to determine the five unknowns ξ, η_0, η_1, η_2, and η_3, each of which is a function of the five variables s, t, x, y, and z. The solution to the above set of DEs, i.e., the exact symmetry generators, can be written as

$$\mathbf{Y}_0=s\frac{\partial}{\partial t}-s\frac{\partial}{\partial z},\quad \mathbf{Y}_1=\frac{\partial}{\partial x},\quad \mathbf{Y}_2=\frac{\partial}{\partial y}. \tag{26}$$

As already mentioned, the electromagnetic waves collide with each other on the flat background region (i.e., on a Minkowski spacetime in Cartesian coordinates) and are represented by a non-static spacetime, given by (14). We therefore define them as a perturbation over some static spacetime, i.e., the spacetime representing the interaction region (region IV) as a perturbation over the flat Minkowski spacetime. The exact static spacetime is given by

$$\mathrm{d}s^2=\frac{\mathrm{d}t^2}{2}-\mathrm{d}x^2-\mathrm{d}y^2-\frac{\mathrm{d}z^2}{2} \tag{27}$$

and the corresponding Lagrangian can be written in the form

$$L=\frac{\dot{t}^2}{2}-\dot{x}^2-\dot{y}^2-\frac{\dot{z}^2}{2}. \tag{28}$$

To evaluate the Noether symmetries of the Lagrangian, we now substitute these results in (7) and compare the coefficients of the coordinate derivatives and their products to obtain the following set of equations:

$$\xi_{,t}=0,\quad \xi_{,x}=0,\quad \xi_{,y}=0,\quad \xi_{,z}=0,\quad A_{,s}=0, \tag{29}$$

$$\eta^0_{,s}=A_{,t},\quad -2\eta^1_{,s}=A_{,x},\quad -\eta^3_{,s}=A_{,z}, \tag{30}$$

$$-2\eta^2_{,s}=A_{,y},\quad \eta^0_{,x}-2\eta^1_{,t}=0, \tag{31}$$

$$\eta^0_{,y}-2\eta^2_{,t}=0,\quad \eta^0_{,z}-\eta^3_{,t}=0,\quad \eta^1_{,y}+\eta^2_{,x}=0, \tag{32}$$

$$\eta^3_{,x}+2\eta^1_{,z}=0,\quad 2\eta^0_{,t}-\xi_{,s}=0, \tag{33}$$

$$\eta^3_{,y}+2\eta^2_{,z}=0,\quad 2\eta^3_{,z}-\xi_{,s}=0, \tag{34}$$

$$2\eta^1_{,x}-\xi_{,s}=0,\quad 2\eta^2_{,y}-\xi_{,s}=0. \tag{35}$$

The solution of this set of DEs is given by the equalities

$$\xi=\frac{c_0s^2}{2}+c_1s+c_2, \tag{36}$$

$$\eta^0 = \frac{t(c_0 s + c_1)}{2} + sc_3 + 2xc_5 + 2c_7 y + c_{12} z + c_{13}, \quad (37)$$

$$\eta^1 = \frac{x(c_0 s + c_1)}{2} + sc_4 + tc_5 + c_8 y + c_{14} z + c_{15}, \quad (38)$$

$$\eta^2 = \frac{y(c_0 s + c_1)}{2} + sc_6 + tc_7 + c_{16} z - xc_8 + c_{17}, \quad (39)$$

$$\eta^3 = \frac{z(c_0 s + c_1)}{2} - sc_9 + tc_{12} - 2xc_{14} - 2yc_{16} + c_{11}, \quad (40)$$

$$A(t,x,y,z) = c_0\left(\frac{t^2}{4} - \frac{x^2}{2} - \frac{y^2}{2} - \frac{z^2}{4}\right) + tc_3 - 2xc_4 - 2yc_6 + c_9 z + c_{10}, \quad (41)$$

where all c's are arbitrary integration constants. Clearly, this is a 17-dimensional algebra in which many symmetries provide no conservation laws. In this algebra, ten symmetries correspond to the generators forming the Poincaré group and seven provide the other significant generators, discussed in detail in [30]. These symmetries can be rearranged to the following form:

$$\mathbf{Y}_0 = \frac{st}{2}\frac{\partial}{\partial t} + \frac{sx}{2}\frac{\partial}{\partial x} + \frac{sy}{2}\frac{\partial}{\partial y} + \frac{sz}{2}\frac{\partial}{\partial z},$$

$$\mathbf{Y}_1 = \frac{Y_0}{s},\ \mathbf{Y}_2 = s\frac{\partial}{\partial t},\ \mathbf{Y}_3 = 2x\frac{\partial}{\partial t} + t\frac{\partial}{\partial x},$$

$$\mathbf{Y}_4 = s\frac{\partial}{\partial y},\ \mathbf{Y}_5 = \frac{\partial}{\partial x},\ \mathbf{Y}_6 = 2y\frac{\partial}{\partial t} + t\frac{\partial}{\partial y},$$

$$\mathbf{Y}_7 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y},\ \mathbf{Y}_8 = -s\frac{\partial}{\partial z},\ \mathbf{Y}_9 = \frac{\partial}{\partial z},$$

$$\mathbf{Y}_{10} = z\frac{\partial}{\partial t} + t\frac{\partial}{\partial z},\ \mathbf{Y}_{11} = \frac{\partial}{\partial t},$$

$$\mathbf{Y}_{12} = z\frac{\partial}{\partial x} - 2x\frac{\partial}{\partial z},\ \mathbf{Y}_{13} = \frac{\partial}{\partial x},$$

$$\mathbf{Y}_{14} = z\frac{\partial}{\partial y} - 2y\frac{\partial}{\partial z},\ \mathbf{Y}_{15} = \frac{\partial}{\partial y},\ \mathbf{Y}_{16} = s\frac{\partial}{\partial x}.$$

In order to determine the conserved quantities from the nontrivial first-order Noether symmetry (in case it exists), we define the perturbed spacetime as follows:

$$\mathrm{d}s^2 = \frac{\mathrm{d}t^2}{2} - \left[1 + \epsilon\cos^2(At - Bz)\right]\mathrm{d}x^2 - \left[1 + \epsilon\cos^2(Az - Bt)\right]\mathrm{d}y^2 - \frac{\mathrm{d}z^2}{2}. \quad (42)$$

Briefly stated, this amounts to taking the spacetime of region IV as a perturbation over the spacetime of region I. The corresponding first-order perturbed Lagrangian is

$$L = \frac{\dot{t}^2}{2} - \left[1 + \epsilon\cos^2(At - Bz)\right]\dot{x}^2 - \left[1 + \epsilon\cos^2(Az - Bt)\right]\dot{y}^2 - \frac{\dot{z}^2}{2}. \quad (43)$$

To find the first-order approximate symmetries, we substitute the above results in (8) and compare the coefficients of coordinate derivatives and their products. The following set of DEs results:

$$\xi_{,t} = 0, \quad \xi_{,x} = 0, \quad \xi_{,y} = 0, \quad \xi_{,z} = 0, \quad A_{,s} = 0, \quad (44)$$

$$\eta^0_{,s} = A_{,t}, \quad -2\eta^1_{,s} - 2\cos^2(At - Bz)\left[\frac{xc_0}{2} + c_4\right] = A_{,x}, \quad (45)$$

$$-2\eta^2_{,s} - \cos^2(Az - Bt)\left[\frac{yc_0}{2} + c_6\right] = A_{,y}, \quad -\eta^3_{,s} = A_{,z}, \quad (46)$$

$$\eta^0_{,x} - 2\eta^1_{,t} - 2c_5\cos^2(At - Bz) = 0, \quad (47)$$

$$\eta^0_{,y} - 2\eta^2_{,t} - 2c_7\cos^2(Az - Bt) = 0, \quad \eta^0_{,z} - \eta^3_{,t} = 0, \quad (48)$$

$$\eta^1_{,y} + \eta^2_{,x} + c_8\left[\cos^2(At - Bz) - \cos^2(Az - Bt)\right] = 0, \quad (49)$$

$$\eta^3_{,x} + 2\eta^1_{,z} + 2c_{14}\cos^2(At - Bz) = 0, \quad 2\eta^0_{,t} - \xi_{,s} = 0, \quad (50)$$

$$\eta^3_{,y} + 2\eta^2_{,z} + 2c_{16}\cos^2(Az - Bt) = 0, \quad 2\eta^3_{,z} - \xi_{,s} = 0, \quad (51)$$

$$2\eta^1_{,x} - \xi_{,s} - 2(A - B)\cos(At - Bz)\sin(At - Bz) \times\left[\frac{t(c_0 s + c_1)}{2} + \frac{z(c_0 s + c_1)}{2} + s(c_3 - c_9) + 2x(c_5 - c_{14}) + 2y(c_7 - c_{16}) + c_{12}(t + z) + c_{13} + c_{11}\right] + (c_0 s + c_1)\cos^2(At - Bz) = 0, \quad (52)$$

$$2\eta^2_{,y} - \xi_{,s} - 2(A-B)\cos(Az-Bt)\sin(Az-Bt)$$
$$\times\left[\frac{t(c_0s+c_1)}{2}+\frac{z(c_0s+c_1)}{2}+s(c_3-c_9)+2x(c_5-c_{14})\right.$$
$$\left.+\,2y(c_7-c_{16})+c_{12}(t+z)+c_{13}+c_{11}\right]$$
$$-(c_0s+c_1)\cos^2(Az-Bt)=0. \quad (53)$$

In this system of DEs, 14 of the 17 constants correspond to exact symmetry generators. We solve the system and look for nontrivial parts of the symmetry generators. Since all the constants corresponding to the exact symmetry generators given by (36)–(40) disappear, the system (44)–(53) of DEs becomes homogeneous and yields symmetries of the static spacetime only.

2.2 Exact and Approximate Symmetries of the Colliding Plane Gravitational Waves

The spacetime representing the collision of two plane gravitational waves in the Rosen form is given by

$$ds^2 = 2e^{-M}du dv - e^{-U}\left(e^{V}dx^2 + e^{-V}dy^2\right),$$

where U, V and M are functions of the null coordinates u and v. For the collision of such waves, we divide the spacetime into the following four regions [44]:

Region I $(u<0,\ v<0)$:

$$ds^2 = 2du dv - dx^2 - dy^2, \quad (54)$$

Region II $(u>0,\ v<0)$:

$$ds^2=2(1\pm u)^{\frac{(a^2-1)}{2}}du dv-(1\pm u)^{1-a}dx^2-(1\pm u)^{1+a}dy^2, \quad (55)$$

Region III $(u<0,\ v>0)$:

$$ds^2=2(1\pm v)^{\frac{(a^2-1)}{2}}du dv-(1\pm v)^{1-a}dx^2-(1\pm v)^{1+a}dy^2, \quad (56)$$

Region IV $(u>0,\ v>0)$:

$$ds^2 = 2(1\pm u\pm v)^{\frac{(a^2-1)}{2}}du dv - (1\pm u\pm v)^{1-a}dx^2 - (1\pm u\pm v)^{1+a}dy^2, \quad (57)$$

where a is any arbitrary parameter. With $a=0$ and negative signs only, this result becomes quite similar to the solution obtained by Szekeres, Khan, and Penrose [45]. With the coordinate transformations $u=\frac{t-z}{2}$, $v=\frac{t+z}{2}$ [43] and the negative signs only, the spacetime in region IV becomes

$$ds^2 = 2(1-t)^{\frac{-1}{2}}\left(\frac{dt^2}{4}-\frac{dz^2}{4}\right)-(1-t)\left(dx^2+dy^2\right). \quad (58)$$

To identify the exact symmetries of the Lagrangian for this spacetime,

$$L = 2(1-t)^{\frac{-1}{2}}\left(\frac{\dot{t}^2}{4}-\frac{\dot{z}^2}{4}\right)-(1-t)\left(\dot{x}^2+\dot{y}^2\right) \quad (59)$$

we substitute the first-order prolongation operator, the total derivative operator, and the above-defined Lagrangian in (7) and compare the coefficients of the derivatives and their products to obtain the following set of DEs:

$$\xi_{,t}=0,\quad \xi_{,x}=0,\quad \xi_{,y}=0,\quad \xi_{,z}=0,\quad A_{,s}=0, \quad (60)$$
$$\eta^0_{,s}\frac{1}{\sqrt{1-t}}=A_{,t},\quad -2(1-t)\eta^1_{,s}=A_{,x}, \quad (61)$$
$$-2(1-t)\eta^2_{,s}=A_{,y},\quad -\eta^3_{,s}\frac{1}{\sqrt{1-t}}=A_{,z}, \quad (62)$$
$$\eta^0_{,x}-2(1-t)^{3/2}\eta^1_{,t}=0, \quad (63)$$
$$\eta^0_{,y}-2(1-t)^{3/2}\eta^2_{,t}=0,\quad \eta^0_{,z}-\eta^3_{,t}=0, \quad (64)$$
$$\eta^1_{,y}+\eta^2_{,x}=0,\quad \eta^3_{,x}+2(1-t)^{3/2}\eta^1_{,z}=0, \quad (65)$$
$$2(1-t)\left[2\eta^0_{,t}-\xi_{,s}\right]+\eta^0=0,\quad \eta^3_{,y}+2(1-t)^{3/2}\eta^2_{,z}=0, \quad (66)$$
$$2(1-t)\left(2\eta^3_{,z}-\xi_{,s}\right)-\eta^0=0, \quad (67)$$
$$(1-t)\left[2\eta^1_{,x}-\xi_{,s}\right]-\eta^0=0,\quad (1-t)\left[2\eta^2_{,y}-\xi_{,s}\right]-\eta^0=0. \quad (68)$$

The solution of this system of DEs yields

$$\eta^0=0,\quad \eta^1=b_6y+b_7,\quad \eta^2=-b_6x+b_8,$$
$$\eta^3=b_5,\quad \xi_2=b_2,\quad A=b_4, \quad (69)$$

where all b's are constants, which shows that the gauge function is constant. These can be rearranged to the following form:

$$A=b_0,\ \xi_2=c_0,\ \mathbf{Y}_1=\frac{\partial}{\partial z},\ \mathbf{Y}_2=y\frac{\partial}{\partial x}-x\frac{\partial}{\partial y},$$
$$\mathbf{Y}_3=\frac{\partial}{\partial x},\ \mathbf{Y}_4=\frac{\partial}{\partial y}.$$

In order to discuss the corresponding conserved quantities, we take (58) as a perturbation over the background static spacetime as we have done for the colliding electromagnetic waves. The background is the Minkowski spacetime in Cartesian coordinates, given by (27), and its Noether symmetries are given by (36)–(40).

We next evaluate the approximate Noether symmetries. The first-order perturbed spacetime for the collision of gravitational waves is given by

$$\mathrm{d}s^2 = \left[1+\epsilon(1-t)^{\frac{-1}{2}}\right]\left(\frac{\mathrm{d}t^2}{2}-\frac{\mathrm{d}z^2}{2}\right) - \left[1+\epsilon(1-t)\right]\left(\mathrm{d}x^2+\mathrm{d}y^2\right).$$

The corresponding Lagrangian is

$$L = \left[1+\epsilon(1-t)^{\frac{-1}{2}}\right]\left(\frac{\dot{t}^2}{2}-\frac{\dot{z}^2}{2}\right) - \left[1+\epsilon(1-t)\right]\left(\dot{x}^2+\dot{y}^2\right). \quad (70)$$

Substitution in (8) yields the following system of DEs:

$$\xi_{,t}=0,\ \xi_{,x}=0,\ \xi_{,y}=0,\ \xi_{,z}=0,\ A_{,s}=0, \quad (71)$$

$$\eta^0_{,s}+\left(\frac{c_0 t}{2}+c_3\right)\frac{1}{\sqrt{1-t}}=A_{,t}, \quad (72)$$

$$-2\eta^1_{,s}-2(1-t)\left(\frac{xc_0}{2}+c_4\right)=A_{,x}, \quad (73)$$

$$-2\eta^2_{,s}-2(1-t)\left(\frac{c_0 y}{2}+c_6\right)=A_{,y}, \quad (74)$$

$$-\eta^3_{,s}-\frac{1}{\sqrt{1-t}}\left(\frac{c_0 z}{2}-c_9\right)=A_{,z}, \quad (75)$$

$$\left[\eta^0_{,x}-2\eta^1_{,t}\right]+\frac{2}{\sqrt{1-t}}c_5\left(1-(1-t)^{3/2}\right)=0, \quad (76)$$

$$\left[\eta^0_{,y}-2\eta^2_{,t}\right]+\frac{2}{(1-t)^{1/2}}c_7\left(1-(1-t)^{3/2}\right)=0, \quad (77)$$

$$\eta^0_{,z}-\eta^3_{,t}=0,\ \eta^1_{,y}+\eta^2_{,x}=0, \quad (78)$$

$$\left[\eta^3_{,x}+2\eta^1_{,z}\right]-\frac{2c_{14}}{\sqrt{1-t}}\left(1-(1-t)^{3/2}\right)=0, \quad (79)$$

$$\left[2\eta^0_{,t}-\xi_{,s}\right]+\frac{1}{2(1-t)^{3/2}}\left[\frac{t}{2}(c_0 s+c_1)+sc_3 + 2xc_5+2c_7 y+c_{12}z+c_{13}\right]=0, \quad (80)$$

$$\eta^3_{,y}+2\eta^2_{,z}-\frac{2c_{16}}{\sqrt{1-t}}\left(1-(1-t)^{3/2}\right)=0, \quad (81)$$

$$\left(2\eta^3_{,z}-\xi_{,s}\right)+\frac{1}{2(1-t)^{3/2}}\left[\frac{t}{2}(c_0 s+c_1)+sc_3 + 2xc_5+2c_7 y+c_{12}z+c_{13}\right]=0, \quad (82)$$

$$\left[2\eta^1_{,x}-\xi_{,s}\right]-\left(\frac{t}{2}(c_0 s+c_1)+sc_3+2xc_5+2c_7 y + c_{12}z+c_{13}\right)=0, \quad (83)$$

$$\left[2\eta^2_{,y}-\xi_{,s}\right]-\left(\frac{t}{2}(c_0 s+c_1)+sc_3+2xc_5+2c_7 y + c_{12}z+c_{13}\right)=0. \quad (84)$$

In this system of 19 DEs, 12 of the 17 constants correspond to the exact symmetry generators defined in (36)–(40). In the final solution of this system of simultaneous DEs, all of these constants disappear, making the system homogeneous and ruling out nontrivial generators.

3 Summary and Discussion

We have discussed the energy contents of colliding plane waves. For this purpose, we have applied the slightly broken or approximate Lie symmetry methods for Lagrangians to colliding plane electromagnetic and gravitational waves, since the contraction of the nontrivial approximate Noether symmetry (corresponding to time) with the momentum 4-vector can be used to define the energy imparted to test particles [25]. We have defined the first-order perturbed Lagrangian for both these solutions and checked for nontrivial symmetries.

For colliding plane electromagnetic (gravitational) waves, we have derived a system of 19 DEs. For the colliding electromagnetic (gravitational) waves, 14 (12) of the 17 constants correspond to the exact symmetry generators defined by (36)–(40) appear. In the solution of the systems of DEs, all of these constants disappear, in both cases. The resulting systems of DEs thus become homogeneous and reproduce those found in the exact case (i.e., Minkowski spacetime in Cartesian coordinates). Consequently, only the exact symmetry generators of static spacetime, which correspond to the 17-dimensional Lie algebra for Minkowski spacetime in Cartesian coordinates, are obtained. These symmetry generators provide the conservation laws for energy, linear momentum, angular momentum, and spin angular momentum. There is hence no nontrivial symmetry.

We conclude that for the first-order perturbed Lagrangians for colliding plane electromagnetic and gravitational waves, no conserved quantities exist. This is similar to the one found for pp and cylindrical gravitational waves [25], for which no nontrivial symmetry generator exists and the conserved quantities cannot be determined. Although our approach cannot be applied to electromagnetic waves, since there is no self-interaction, significant insight on self-damping or enhancement of gravitational waves may result from application of the procedure to the system of geodesic equations for the pertinent spacetimes.

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