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PARTICLES AND FIELDS

A Note on the Extrinsic Phase Space Path Integral Method for Quantization on Riemannian Manifold Particle Motions — An Application of the Nash Embedding Theorem

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Abstract We use Nash embedding for Riemann smooth manifolds to propose a constrained phase space path integral for quantization of one particle motion in a Riemannian manifold.

Keywords Curved space time path integral · Nash embedding · Constrained path integral

1 Introduction

The definition of path integral representations to describe the quantum propagation of spinorial or scalar particles in Riemannian manifolds plays an important role in the search for consistent frameworks to quantize gravitation. In the last decades, several important studies have surfaced, all of which exploit the intrinsic geometrical properties of the Riemann manifold is which the propagation is assumed to take place. Examples are found in Refs. [1–4].

In this note, we propose a somewhat different path integral quantization geometrical framework. Our reasoning is based on a profound theorem due to Nash, roughly asserting that every Riemannian metric in a given d-dimensional C^∞-manifold $\{M, g_{\mu\nu}(x)\}$ can be always obtained from an immersion $f^A : M \to R^{S(d)}$ ($f^A \in C^1(M)$ and rank $D_x\ f^A = d$) in a suitable Euclidean space with dimensionality strictly greater than d ($S(d) \geq 2d-1$) [5, 6]. As a consequence, one can write the metric field $g_{\mu\nu}(x)$ in the form

$$g_{\mu\nu}(x) = \sum_{A=1}^{S(d)} \frac{\partial f^A}{\partial x^\mu}\frac{\partial f^A}{\partial x^\nu} \overset{\text{def}}{\equiv} \frac{\partial f^A}{\partial x^\mu}\frac{\partial f_A}{\partial x^\nu}. \tag{1}$$

Here, $\{x\} \in Dom(f^A)$, a point of $R^{S(d)}$ (which in turn contains the manifold chart of M to which x belongs). Note that (1) is a nonlinear first-order set of partial differential equations for $f^A(x^\gamma)_{\gamma=1,\ldots,d}$, with an explicit source term $g_{\mu\nu}(x)$ as input.

2 The Phase Space Path Integral Representation

We start our representation by rewriting the Lagrangian for free motion on the manifold in new extrinsic coordinates:

$$\mathcal{L} = \overbrace{\frac{1}{2}Mg_{ij}(X^\mu(\sigma))\left(\frac{\partial X^i}{\partial\sigma}\frac{\partial X^j}{\partial\sigma}\right)(\sigma)}^{\mathcal{L}(X^i,\dot{X}^i)}$$

$$= \overbrace{\frac{1}{2}M\delta_{AB}\left(\frac{dQ^A}{d\sigma}\frac{dQ^B}{d\sigma}\right)(\sigma)}^{L(Q^A,\dot{Q}^A)} \tag{2}$$

where the new particle “extrinsic” coordinates are given explicitly by

$$Q^A(\sigma) = f^A\left(X^1(\sigma),\ldots,X^d(\sigma)\right). \tag{3}$$

with $A = 1,\ldots,S(d)$. In addition, we assume the inverse functional relations to hold (we assume the usual

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inverse theorem of advanced calculus to apply [2], at least locally). Namely, we have that

$$X^i(\sigma) = G^i\left(Q^{(A)}(\sigma)\right), \quad i = 1, \ldots, d. \tag{4}$$

In addition, when the motion is viewed in the non-curved (absolute inertial) referential system $R^{S(d)}$, we have in addition $S(d) - d$ smooth constraints[1]

$$\Phi^\ell\left(Q^B\right) = 0 \qquad \left(\ell = d+1, \ldots, S(d) - d\right). \tag{5}$$

At the classical level, the intrinsic free motion in the Riemannian manifold $[M, g_{\mu\nu}(x)]$ as given by (2) is entirely equivalent to the classical motion in the extrinsic space $R^{S(d)}$ under the constraints in (5) [7].

We thus find that the free-manifold motion equation

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial(\dot{X}^i)}\right) = \frac{\partial\mathcal{L}}{\partial X^i} \tag{6a}$$

$$X^i(\sigma) = A^i = \mathbf{A} \tag{6b}$$

$$\dot{X}^i(\sigma) = B^i = \mathbf{B} \tag{6c}$$

is mathematically equivalent to the constrained motion in the extrinsic Euclidean space $R^{S(d)}$ (see pages 45–51 of [4]). We therefore have a new set of Euler–Lagrange equations in the ambient space $R^{S(d)}$:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{Q}^A}\right) - \frac{\partial L}{\partial Q^A} = \sum_{\ell=1}^{S(d)-d} \lambda_\ell(Q^B,\sigma)\left\{\frac{\partial\Phi^\ell}{\partial Q^A}\left(Q^B,\sigma\right)\right\}, \tag{7}$$

where the undetermined Langrange multipliers $\lambda(Q^B,\sigma)$ are functions of general coordinates $\{Q^B\}$ and also of the time σ. They are determined from the following constraints, valid at least locally in suitable manifold charts:

$$\sum_{\ell=1}^{S(d)-d}\left(\frac{\partial\Phi^\ell}{\partial Q^A}\cdot\delta Q^A\right) \equiv 0, \tag{8}$$

Note that the set of (7) and (8) must be complemented with the initial conditions

$$Q^A(0) = f^A(\mathbf{A}), \tag{9a}$$

$$\dot{Q}^A(0) = \langle \nabla f(\mathbf{A}), \mathbf{B}\rangle_{R^{S(d)-d}}. \tag{9b}$$

We now want to propose a phase space path integral to quantize the constrained $R^{S(d)}$ classical system described by (2)–(5). To this end, let us first obtain a few basic results from the path integral formalism for constrained Hamiltonians [5].

First, we consider the classical action functional for such a system, given by

$$S = \int_0^T d\sigma\left(\sum_{A=1}^{S(d)} P_A\dot{Q}^A - H\left(P_A, Q^A\right) - \sum_{\ell=1}^{S(d)-d}\lambda_\ell\left(Q^B\right)\Phi^\ell\left(Q^B\right)\right). \tag{10}$$

The set of variables $\left\{P_A, Q^A\right\}$ form the phase space $R^{2S(d)}$, while the Φ^ℓ are constraints. One can easily check that they satisfy the Poisson-algebra closeness property (all Poison-algebra structure constants vanishing naturally in our model):

$$\left\{H,\Phi^A\right\}_{PB} = \sum_{A=1}^{S(\ell)}\left[\overbrace{\left(\frac{\partial}{\partial P^A}\left(\frac{1}{2}P^BP_B\right)\right)}^{=PA}\overbrace{\frac{\partial}{\partial Q^A}\left(\Phi^\ell\left(Q^A\right)\right)}^{\equiv\nabla\Phi^\ell} - \left(\frac{\partial}{\partial Q^A}\left(\frac{1}{2}P^BP_B\right)\right)\overbrace{\frac{\partial}{\partial P^A}\left(\Phi^\ell\left(Q^A\right)\right)}^{=0}\right], \tag{11}$$

where $\{,\}_{PB}$ denotes the usual Poisson bracket. It then follows that

$$\left\{H,\Phi^A\right\}_{PB} = \left[\overbrace{\left(\frac{dQ^A}{d\sigma}\right)}^{=P^A_{(\sigma)}}\cdot\nabla\Phi^\ell\left(Q^A\right)\right] = \frac{d}{d\sigma}\left\{\Phi^\ell\left(Q^A(\sigma)\right)\right\} = 0,$$

and we also have that

$$\left\{\Phi^A, \Phi^B\right\}_{PB} = 0. \tag{12}$$

In this quantization framework for constrained classical dynamics, special care should be taken to fix a quantity that one might call "gauge", in order to make all physical variables $\Omega(Q^A, P^A)$ that are evaluated along the physical trajectories in phase space independent of the choice of Lagrange multipliers. More specifically, one must have that

$$\left\{\Omega, \Phi^\ell\right\}_{PB} = \sum_\ell d^\ell_P\Phi^P, \tag{13}$$

along with

$$\frac{d\Omega}{d\sigma} = \{H,\Omega\}_{PB} + \sum_\ell \lambda_\ell\left\{\Phi^\ell,\Omega\right\}. \tag{14}$$

[1]They are defined abstractly as the $S(d) - d$ dimensional manifold of the ambient phase space where the classical particle motion takes place.

To fix the gauge, one needs only to introduce, in the phase space of the system, a set of surfaces $\chi^\ell(Q^A, P^A) = 0$ satisfying the conditions [5]

$$\{\chi^\ell, \chi^r\}_{PB} = 0 \tag{15}$$

and

$$\det\left[\{\chi^\ell, \Phi^r\}_{PB}\right] \neq 0, \tag{16}$$

where $\left[\{\chi^\ell, \Phi^r\}_{PB}\right]$ is a $[S(d) - d] \times [S(d) - d]$ matrix.

Under these conditions, a canonical transformation can be found which turns the gauge-fixing functions $\chi^\ell(Q^A, P^A) \equiv \pi^\ell$ into new canonical moments. Accordingly, let Q^ℓ be the coordinates conjugate to π^ℓ and Q^*, P^* be the remaining set of canonical variables. Since we can always solve the system $\Phi^\ell = 0$ and find $Q^a = Q^a(Q^*, P^*)$, the constraints $\Phi^\ell = 0$ and the supplementary conditions obeyed by the χ^ℓ define the new physical phase space Γ^*. And within Γ^*, we have that

$$\pi^\ell \equiv 0 \qquad (\ell = 1, \ldots, S(d) - d)\,, \tag{17}$$

$$Q^a = Q^a(Q^*, \pi^*). \tag{18}$$

Next, we recall a result [8] that is central to our study. The matrix element of the quantum mechanical evolution operator is explicitly given by the following phase space path integral[2]

$$\begin{aligned}
&\left\langle (Q^A_{\text{out}}, T) \left| \exp\left(\frac{iH}{\hbar} T\right) \right| (Q^A_{\text{in}}, 0) \right\rangle \\
&= \int \exp\left\{ \frac{i}{\hbar} \int_0^T d\sigma \left[\sum_{A=1}^{s(d)} P_A \dot{Q}^A - \overbrace{H\left(P_A, Q^A\right)}^{\frac{1}{2}P^B P_B + \tilde{V}(Q^A)} \right] \right\} \\
&\quad \times (2\pi)^{S(d)-d} \det\left[\{\chi^\ell, \Phi^r\}_{PB}\right] \\
&\quad \times \prod_{\ell=1}^{S(d)-d} \delta\left(\chi^\ell\left(Q^A, P^A\right)\right) \delta\left(\Phi^\ell\left(Q^A\right)\right) \\
&\quad \times \prod_{\ell=1}^{S(d)} \left(D^F\left[Q^i(\sigma)\right] D^F\left[P^i(\sigma)\right]\right) \\
&\quad \times \delta^{(S(d))}\left(Q^A(0) - Q^A_{\text{in}}\right) \delta^{(S(d))} \\
&\quad \times \left(Q^A(T) - Q^A_{\text{out}}\right).
\end{aligned} \tag{19}$$

Here, we have reintroduced a potential $V(X^i) \equiv \tilde{V}(Q^A)$ in the motion Lagrangian, with no additional mathematical complication.

At least in our approach, we find no invariant path integral expression for matrix elements on the basis of the original manifold variables $\langle X^i_{\text{out}}, T \mid X^j_{\text{in}}, 0\rangle$. Only when the intrinsic geometrical setting is viewed as usual holonomic mechanical constraints in the extrinsic space $R^{S(d)}$ of the absolute embedding frame of the metric manifold does the Nash theorem allow us to study quantum mechanics in Riemann manifolds.

The phase space integral (19) can be proved not to depend on the specific choice of the supplementary conditions satisfyied by the $\chi^\ell(Q^A, P^A)$ [8].

In order to make connection with the original Feynman Lagrangean formalism, we note that, in the extrinsic space, the Feynman propagator can be easily rewritten in the following form, which remains invariant under geometrical coordinate transformations:

$$\begin{aligned}
&G\left[\left(Q^A_{\text{out}}, T\right); \left(Q^A_{\text{in}}, 0\right)\right] \\
&= \int_{Q^A(0)=Q^A_{\text{in}}}^{Q^A(T)=Q^A_{\text{out}}} \left(\prod_{0\le\sigma\le T} dQ^A(\sigma)\right) \left(W\left[Q^\ell(\sigma)\right]\right) \\
&\quad \times \left(\prod_{\ell=1}^{S(d)-d} \delta\left(\Phi^\ell\left(Q^B(\sigma)\right)\right)\right) \\
&\quad \times \left(\exp\left\{\frac{i}{\hbar}\int_0^T d\sigma \left[\frac{1}{2} M\left(\dot{Q}^A(\sigma)\right)^2 - \tilde{V}\left(Q^A(\sigma)\right)\right]\right\}\right)
\end{aligned} \tag{20}$$

with the path measure weight [see (4)][3]

$$W[Q^i(\sigma)] = \left\{ \prod_{0<\sigma<T} \det\left(\left[g_{\mu\nu}(G^i(Q^A(\sigma))) \right]^{\frac{1}{4}} \right) \right\}. \tag{21}$$

Straightforward analysis of the Lagrangean path integral described by (20) and (21) shows that its analytical disentanglement leads to sensible results only in a post-Newtonian perturbative framework of small metrical deviations from the usual Euclidean space:

$$g_{\mu\nu}(x) = \delta_{\mu\nu} + \frac{1}{C} h^{(1)}_{\mu\nu}(x) + \frac{1}{C^2} h^{(2)}_{\mu\nu}(x) + \cdots\,, \tag{22a}$$

where C denotes the speed of light [6].

[2]The Lagrange multipliers have been exchanged by the "gauge-fixed" surfaces (15) and (16) in the measure path integral.

[3]The path-integral measure in Eq. (21) is a Riemann weighted functional measure that, in a diagrammatic Feynman analysis, could be interpreted as the new tad-pole term in the action

$$W[Q^i(\sigma)] = \exp\left\{ -\frac{\delta^{(0)}(0)}{4} \int_0^T \operatorname{Tr}\left[lg\left(g_{\mu\nu}\left(G^i\left(Q^A(\sigma) \right) \right) \right) \right] \right\}.$$

We now take advantage of these phase space path integral results to describe diffusion on Riemann manifolds, one of the important classical problems in diffusion physics.

To this end, we consider the usual linear diffusion equation in the space R^n, here endowed with a metric $\{g_{\rm ab}(x)\}_{\substack{a=1,\ldots,n\\ b=1,\ldots,n}}$. Our problem is defined by the equation

$$\frac{\partial U(t,x)}{\partial t} = -\frac{1}{2}\left(\Delta_{g(x)}U\right)(x) - (VU)(x) \tag{22b}$$

with

$$U(0,x) = f(x) \in L^2\left(R^n, gd^n x\right). \tag{22c}$$

Here, $V(x)$ denotes a real valued function on $\{R^n, g_{\rm ab}(x)\}$, the *diffusion potential*.

The usual analytical complex-time continuation arguments connecting the diffusion equation to the Schrödinger equation on R^n [9] then lead to a Feynman–Kac–Wiener path integral solution of (22):

$$U(t,x) = \int_{-\infty}^{+\infty} d^n y\sqrt{g(y)}\; K(x,y,t)\, f(y), \tag{23}$$

with the evolution Kernel

$$K(x,y,t) = \overline{G}(Q_x, Q_y, t)\Big|_{\substack{Q^{x,A}=f^A(x^1,\ldots,x^n)\\ Q^{y,B}=f^B(y^1,\ldots,y^n)}}, \tag{24}$$

where $\overline{G}(Q_x, Q_y, t)$ is the analytical imaginary-time continued path integral (20). Equation (24) can therefore be written

$$K(x,y,t) = G\left[\left(f^A(x), it\right); \left(f^A(y), 0\right)\right]. \tag{25}$$

Finally, we consider a perturbative approach to the second-quantization problem in for neutral scalar fields. Here, we must assign a meaning to the covariant manifold free-field path integral under an external, covariantly coupled field source [9]:

$$Z[J(x)] = \frac{1}{Z(0)} \times \left\{\int\left(\prod_{x^\mu\in M}\sqrt{4}g\left(x^\alpha\right)d\varphi\left(x^\alpha\right)\right)\right.$$
$$\times \exp\left\{+\frac{i}{\hbar}\int_M\left[\left(\varphi\left(-\frac{1}{2}\Delta_g\right)\varphi\right)\left(x^\alpha\right)\right.\right.$$
$$\left.\left.\times\sqrt{g(x^\alpha)}d^D x\right]\right\}$$
$$\times \exp\left\{+\frac{i}{\hbar}\int_M \left(J(x^\alpha)\varphi(x^\alpha)\right)\right.$$
$$\left.\times\sqrt{g(x^\alpha)}d^D x\right\}. \tag{26}$$

We then evaluate the exactly soluble Gaussian Feynman covariant-field path integrals and employ the proper-time method to express the resulting function determinant, one obtains the "Loop Space" path integral representation for the non-normalized generating functional associated with the Feynman field path integral (26), i. e.,

$$Z[J(x)] = \det{}^{-1/2}(-\Delta_g)$$
$$\times \exp\left\{\frac{i}{\hbar}\int_M\sqrt{g(x^\alpha)}d^\nu x\int_M\sqrt{g(y^\alpha)}d^\nu y\right.$$
$$\left.\times J(x^\alpha)(-\Delta_g)^{-1}(x^\alpha,y^\alpha)J(y^\alpha)\right\}, \tag{27}$$

with the proper-time motion manifold path integrals representations [see Eqs. (23) and (24)]

$$\log\left[\det{}^{-\frac{1}{2}}\left(-\Delta_g\right)\right] = \int_0^\infty \frac{dt}{2t}\left(\overbrace{\mathrm{tr}_{L^2(M,g(x))}\left[e^{-t(-\Delta_g)}\right]}^{\equiv K(x^\alpha,x^\alpha,t)}\right) \tag{28a}$$

$$\left(-\Delta_g\right)^{-1}(x^\alpha,y^\alpha) = \int_0^\infty dt\left(\overbrace{\langle x,t|e^{-t(-\Delta_g)}|y,o\rangle}^{\equiv K(x^\alpha,y^\alpha,t)}\right). \tag{28b}$$

We then re-insert (28) into (27) to obtain the non-normalized generating functional in terms of a dynamics of covariant path integrals [9]. For a full-fledged use of Nash embedding representation theorem on quantum gravity, see [6].

3 Conclusion

As a general conclusion of our note, we stress the theoretical feasibility of using the methods of constrained gauge-fixed path integrals to give a sound justification for path-integral particle quantization on curved space times. It is also important to remark that the subject of constrained path integral is mainly of theoretical importance. And its practical use may be somewhat cumbersome, as unwieldy as the whole subject of quantization of holonomic constraint dynamics and the quantization of non-abelian gauge fields has proved to be [9]. However, our use of such techniques in the context of "ambient phase space" through the Nash embedding theorem certainly opens the possibility to study curved space time propagators by means of the well-known perturbative $\frac{1}{C}$ expansion (see (22a) in the text) as it has been done in Ref. [6].

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