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The Noncommutative Harmonic Oscillator Based on Symplectic Representation of Galilei Group

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Abstract We study symplectic unitary representations for the Galilei group and derive the Schrödinger equation in phase space. Our formalism is based on the noncommutative structure of the star product. Guided by group theoretical concepts, we construct a physically consistent phase-space theory in which each state is described by a quasi-probability amplitude associated with the Wigner function. As applications, we derive the Wigner functions for the 3D harmonic oscillator and the noncommutative oscillator in phase space.

Keywords Moyal product · Phase space · Quantum fields

1 Introduction

The concept of noncommutativity in Physics dates back to the birth of Quantum Mechanics. Heisenberg's uncertainty principle gave physical substance to that notion. From a more mathematical viewpoint, the assumption that spatial and momentum coordinates do not commute has lead to noncommutative geometry, which has offered new insight. As a matter of fact, Heisenberg himself suggested that an uncertainty relation among the spatial coordinates might avoid the singularities due to particle self-energies [1–3]. The first elaborate analysis in this line of research is, however, due to Snyder [4, 5], a former student of Oppenheimer, who proposed a new vision of space–time. According to Snyder, the space–time should be understood as a collection of minimum-size cells, forming a lattice structure rather than a continuum. In this picture, it is inadequate to define space–time points, because noncommutativity bars accurate measurements of particle positions.

Spatial noncommutativity can be introduced by means of Hermitian operators standing for the space–time coordinates, $\widehat{x}^{\mu}$, and satisfying the algebra $[\widehat{x}^{\mu}, \widehat{x}^{\nu}] = i\theta^{\mu\nu}$, where the $\theta^{\mu\nu}$ is the component of a constant antisymmetric tensor. The commutation relations imply $\Delta\widehat{x}^{\mu}\Delta\widehat{x}^{\nu} \geq \frac{1}{2}|\theta^{\mu\nu}|$, so that the noncommutativity becomes relevant for distances of the order of $\sqrt{|\theta^{\mu\nu}|}$. These are the basics of noncommutative geometry.

Over the last decades, interest in this type of noncommutative geometry progressively grew as applications were made to non-Abelian theories [6], gravitation [7–9], the standard model [10–12], and the quantum Hall effect [13]. The formalism is nonetheless still incomplete. In particular, no Wigner function analysis of noncommutativity in phase space has been presented.

In 1932, in a development that was contemporary to the initial studies of noncommutative geometry, Wigner introduced a quantum formalism in the phase space Γ, with a view to applications in quantum kinetic theory [14]. His approach associates each operator A in the Hilbert space $\mathcal{H}$ with a function $a_W(q, p)$ defined in Γ [14–17]. The application $\Omega_W : A \to a_W(q, p)$ is such that the

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associative algebra of operators in $\mathcal{H}$ defines an associative noncommutative algebra in Γ.

The noncommutativity stems from the nature of the product between two operators in $\mathcal{H}$. Given two operators A and B, we have the mapping $\Omega : AB \rightarrow a_W(q,p)\star b_W(q,p)$. Here, the (noncommutative) star product $\star$ is defined by the identity $a_W(q,p) \star b_W(q,p) = a_W(q,p) \exp\left[\frac{i}{2}\left(\frac{\overleftarrow{\partial}}{\partial q}\frac{\overrightarrow{\partial}}{\partial p} - \frac{\overleftarrow{\partial}}{\partial p}\frac{\overrightarrow{\partial}}{\partial q}\right)\right] b_W(q,p)$, the so-called Moyal product.

Although the phase space and Moyal product have been explored in different ways [15–38], only recently has a physically consistent representation theory been developed. First, irreducible unitary representations of kinematical groups in Γ have been studied with operators of type $\widehat{A} = a_W\star$ acting on the function b_W, i.e., $\widehat{A}b_W(q,p) = a_W(q,p) \star b_W(q,p)$. For the Galilei group, this symplectic star representation yields a phase-space Schrödinger equation, the role of wave functions (the quasi-amplitudes of probability) being played by the Wigner functions (the quasi-distributions of probability) [39]. This method affords a derivation of the Wigner functions without the intricacies of the Liouville–von Neumann equation, which provided the original starting point in Wigner's approach, and leads to a prescription to derive symplectic star representations for the Poincaré symmetry. This in turn leads to phase-space representations of the Klein–Gordon and Dirac fields [40, 41]. This algebraic formalism is a natural candidate to derive Wigner functions for nonclassical radiation states and for noncommutative space–time systems, such as the quantum Hall effect.

Here, we explore the elements of this approach. Considering the Galilei group, we study the eigenvalue problem of the phase-space Schrödinger equation to first treat the 3D harmonic oscillator and derive the quasi-amplitude of probability and the corresponding Wigner function. These results provide a starting point for our analysis of nonclassical electromagnetic radiation states and of phase-space Bose–Einstein condensation. As a second application, we consider a 2D noncommutative harmonic oscillator, a prototype of the Hall effect in phase space.

The presentation is organized as follows: In Section 2, we define a Hilbert space $\mathcal{H}(\Gamma)$ over a phase space, including a natural symplectic structure. We then take the space $\mathcal{H}(\Gamma)$ as the carrier space for unitary representations of the Galilei group. In Section 3, we construct the generators $A_w(q,p)\star$ for the Galilei group, hence deriving a representation for the phase-space Schrödinger equation. In Section 4, we present solutions of the Schrödinger equation in phase space for the 3D harmonic oscillator. In Section 5, we consider the noncommutative oscillator in phase space. In Section 6, we present concluding remarks.

2 Hilbert Space and Symplectic Structure

Consider an analytical manifold $\mathbb{M}$, where each point is specified by coordinates q. The coordinates of each point in the cotangent-bundle $\Gamma = T^*\mathbb{M}$ are denoted (q,p). The $2N$-dimensional manifold Γ is equipped with a 2-form, defined by $\omega = dq \wedge dp$, called the symplectic form. With the symplectic form, the operator

$$\Lambda = \frac{\overleftarrow{\partial}}{\partial q}\frac{\overrightarrow{\partial}}{\partial p} - \frac{\overleftarrow{\partial}}{\partial p}\frac{\overrightarrow{\partial}}{\partial q} \tag{1}$$

leads to the Poisson bracket,

$$\omega(f\Lambda, g\Lambda) = \{f,g\} = f\Lambda g,$$

where

$$\{f,g\} = \frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial q}.$$

Here, $f = f(q,p)$ and $g = g(q,p)$.

The manifold $\Gamma = T^*\mathbb{M}$, endowed with this symplectic structure, is then called the phase space, and the algebraic set of the analytical functions $f(q,p)$ is denoted by $C^\infty(\Gamma)$. The vector fields over Γ are given by

$$X_f = f\Lambda = \frac{\partial f}{\partial q}\frac{\partial}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial}{\partial q}.$$

A Hilbert space associated with Γ is introduced by a set of complex functions, $\psi(q,p)$, which are square integrable in $C^\infty(\Gamma)$, i.e.,

$$\int dpdq\psi^\dagger(q,p)\psi(q,p) < \infty.$$

These functions may be then defined as $\psi(q,p) = \langle q,p|\psi\rangle$, with

$$\int dpdq\, |q,p\rangle\langle q,p| = 1,$$

such that

$$\langle\psi|\phi\rangle = \int dpdq\, \psi^\dagger(q,p)\phi(q,p),$$

where $\langle\psi|$ is a dual vector of $|\psi\rangle$. This Hilbert space, denoted $\mathcal{H}(\Gamma)$, is here the carrier space for representations of Lie algebras.

Consider $\ell = \{a_i, i = 1,2,3,...\}$ a Lie algebra over the (real) field $\mathbb{R}$, of a Lie group $\mathcal{G}$, characterized by the algebraic relations $(a_i, a_j) = C_{ijk}a_k$, where $C_{ijk} \in \mathbb{R}$ is the structure constant and $(,)$ is the Lie product. We construct unitary symplectic representations for ℓ, denoted by ℓ_{Sp}, using the star product.The associative product in $\mathcal{H}(\Gamma)$

which is obtained from the operator Λ in Eq. 1, as a mapping $e^{ia\Lambda} = \star : \ \Gamma \times \Gamma \to \Gamma$, defined the equality

$$(f \star g)(q, p) = f(q, p)e^{ia\Lambda}g(q, p) = \exp\left[ia\left(\partial_q \partial_{p'} - \partial_p \partial_{q'}\right)\right] \times f(q, p)g(q', p')|_{q', p'=q, p}, \tag{2}$$

where f and g are functions in $C^{\infty}(\Gamma)$ and $\partial_x = \partial/\partial x$ $(x = p, q)$. The constant a, which fixes units, has no special meaning. The usual associative product is obtained by letting $a = 0$. To each function $f(q, p)$ an operator $\widehat{f} = f(q, p)\star$ is associated, which will be used as the generator of unitary transformations.

3 Galilei Lie Algebra and Schödinger Equation in Phase Space

We now study the representation of the Galilei group $\mathcal{H}(\Gamma)$, which leads us to the Schrödinger equation in phase space, and connect this representation with the Wigner formalism.

With the star operator $\widehat{A} = a\star$, where $a = a(q, p)$, we define a momentum- and a position-like operator by the equalities

$$\widehat{Q} = q\star = q + \frac{i\hbar}{2}\partial_p \tag{3}$$

and

$$\widehat{P} = p\star = p - \frac{i\hbar}{2}\partial_q, \tag{4}$$

respectively.

We can then define a boost, an angular momentum, and a Hamiltonian-like operator by the equalities

$$\widehat{K} = m\widehat{Q}_i - t\widehat{P}_i, \tag{5}$$

$$\widehat{L}_i = \epsilon_{ijk}\widehat{Q}_j\widehat{P}_k = \epsilon_{ijk}q_j p_k - \frac{i\hbar}{2}\epsilon_{ijk}q_j\frac{\partial}{\partial p_k} + \frac{i\hbar}{2}\epsilon_{ijk}p_k\frac{\partial}{\partial q_j} + \frac{\hbar^2}{4}\frac{\partial^2}{\partial q_j \partial p_k},$$

and

$$\widehat{H} = \frac{\widehat{P}^2}{2m} = \frac{1}{2m}\left(\widehat{P}_1^2 + \widehat{P}_2^2 + \widehat{P}_3^2\right) = \frac{1}{2m}\left[\left(p_1 - \frac{i\hbar}{2}\frac{\partial}{\partial q_1}\right)^2 = + \left(p_2 - \frac{i\hbar}{2}\frac{\partial}{\partial q_2}\right)^2 + \left(p_3 - \frac{i\hbar}{2}\frac{\partial}{\partial q_3}\right)^2\right],$$

respectively.

Tedious manipulations lead to the following set of commutation relations for this set of unitary operators:

$$\left[\widehat{L}_i, \widehat{L}_j\right] = i\hbar\epsilon_{ijk}\widehat{L}_k,$$

$$\left[\widehat{L}_i, \widehat{K}_j\right] = i\hbar\epsilon_{ijk}\widehat{K}_k,$$

$$\left[\widehat{L}_i, \widehat{P}_j\right] = i\hbar\epsilon_{ijk}\widehat{P}_k,$$

$$\left[\widehat{K}_i, \widehat{P}_j\right] = i\hbar m\delta_{ij}\mathbf{1},$$

$$\left[\widehat{K}_i, \widehat{H}\right] = i\hbar\widehat{P}_i,$$

all other commutation relations vanishing.

This is the Galilei Lie algebra, with a central extension given by m. The operators defining the Galilei symmetry $\widehat{P}$, $\widehat{K}$, $\widehat{L}$, and $\widehat{H}$ are the generators of translations, boost, rotations, and time translations, respectively. To obtain this physical content, we first notice that $\widehat{Q}$ and $\widehat{P}$ transform under the boost as the physical position and momentum, respectively, i.e.,

$$\exp\left(-i\mathbf{v}.\frac{\widehat{K}}{\hbar}\right)\widehat{Q}_j\exp\left(i\mathbf{v}.\frac{\widehat{K}}{\hbar}\right) = \widehat{Q}_j + v_j t\mathbf{1}, \tag{6}$$

$$\exp\left(-i\mathbf{v}.\frac{\widehat{K}}{\hbar}\right)\widehat{P}_j\exp\left(i\mathbf{v}.\frac{\widehat{K}}{\hbar}\right) = \widehat{P}_j + mv_j\mathbf{1}. \tag{7}$$

Furthermore, the operators $\widehat{Q}$ and $\widehat{P}$ do not commute with each other, that is,

$$\left[\widehat{Q}_i, \widehat{P}_j\right] = i\hbar\delta_{ij}\mathbf{1}.$$

$\widehat{Q}$ and $\widehat{P}$ can therefore be taken to be the position and momentum physical observables, respectively. To be consistent, the generators $\widehat{L}$ are interpreted as the angular momentum observable, and $\widehat{H}$ is taken to be the Hamiltonian operator. The Casimir invariants of the Lie algebra are given by

$$I_1 = \widehat{H} - \frac{\widehat{P}^2}{2m} \quad \text{and} \quad I_2 = \widehat{L} - \frac{1}{m}\widehat{K} \times \widehat{P},$$

where I_1 describes the Hamiltonian of a free particle and I_2 is associated with the spin degrees of freedom. Here, we are concerned with the scalar representation, i.e., spin zero, such that $I_2 = 0$.

Defining the operators

$$\overline{Q} = q\mathbf{1} \quad \text{and} \quad \overline{P} = p\mathbf{1},$$

we observe that, under the boost, $\overline{Q}$ and $\overline{P}$ transform as

$$\exp\left(-iv\frac{\widehat{K}}{\hbar}\right)2\overline{Q}\exp\left(iv\frac{\widehat{K}}{\hbar}\right)=2\overline{Q}+vt\mathbf{1},$$

and

$$\exp\left(-iv\frac{\widehat{K}}{\hbar}\right)2\overline{P}\exp\left(iv\frac{\widehat{K}}{\hbar}\right)=2\overline{P}+mv\mathbf{1}.$$

This shows that $\overline{Q}$ and $\overline{P}$ transform as position and momentum variables, respectively.These operators commute, i.e., $\left[\overline{Q},\overline{P}\right]=0$. Then, $\overline{Q}$ and $\overline{P}$ and therefore cannot be interpreted as observables. Nevertheless, they allow construction of a Hilbert space frame with the content of phase space. To this end, we define an orthogonal basis in $\mathcal{H}(\Gamma)$ by stating that $\overline{Q}|q,p\rangle=q|q,p\rangle$ and $\overline{P}||q,p\rangle=p|q,p\rangle$, with

$\langle q,p|q',p'\rangle=\delta(q-q')\delta(p-p')$,

such that $\int dqdp|q,p\rangle\langle q,p|=1$.

Although associated with the system state, the wave function $\psi(q,p,t)=\langle q,p|\psi(t)\rangle$ does not have the usual content of a quantum mechanical state. This point deserves a brief digression.

The time evolution of $\psi(q,p,t)$ is given by the generator of time translations:

$$\psi(t)=e^{\frac{-i\widehat{H}t}{\hbar}}\psi(0), \tag{8}$$

and its Hermitian adjoint is given by the equality

$$\psi^{\dagger}(t)=\psi^{\dagger}(0)e^{\frac{i\widehat{H}t}{\hbar}}. \tag{9}$$

We therefore obtain the result

$$i\hbar\partial_t\psi(q,p;t)=\widehat{H}(q,p)\psi(q,p;t),$$

or

$$i\hbar\partial_t\psi(q,p;t)=H(q,p)\star\psi(q,p;t), \tag{10}$$

which is the Schrödinger equation in phase space [39].

The expectation value of a physical observable $\widehat{A}(q,p)=a(q,p;t)\star$ in the state $\psi(q,p)$ is given by the equalities

$$\begin{aligned}\langle A\rangle &= \int dqdp\psi^{\dagger}(q,p)\widehat{A}(q,p)\,\psi(q,p)\\ &= \int dqdp\psi^{\dagger}(q,p)[a(q,p)\star\psi(q,p)]\\ &= \int dqdp\,a(q,p)[\psi(q,p)\star\psi^{\dagger}(q,p)].\end{aligned} \tag{11}$$

To physically interpret the formalism, we associate $\psi(q,p,t)$ with the Wigner function, $f_W(q,p)$, which is given by the expression [39],

$$f_W(q,p)=\psi(q,p,t)\star\psi^{\dagger}(q,p,t). \tag{12}$$

The Wigner function satisfies the Liouville–von Neumann equation [39] and determines the probability density both in configuration space,

$$\begin{aligned}\rho(q) &= \int dp\,[\psi(q,p)\star\psi^{\dagger}(q,p)]\\ &= \int dp\,\psi(q,p)\psi^{\dagger}(q,p),\end{aligned} \tag{13}$$

and in momentum space,

$$\begin{aligned}\rho(p) &= \int dq\,[\psi(q,p)\star\psi^{\dagger}(q,p)]\\ &= \int dq\,\psi(q,p)\psi^{\dagger}(q,p).\end{aligned} \tag{14}$$

The expression for the expectation value of an observable is therefore consistent with the Wigner formalism, i.e., from Eqs. 11 and 12, we have that

$$\langle A\rangle=\int dqdp\,a(q,p)f_W(q,p;t).$$

We therefore have a complete set of physical prescriptions to interpret the symplectic star representations, which paves the road to application. The following sections discuss the 3D harmonic oscillator and the non-commutative oscillator.

4 3D Harmonic Oscillator in Phase Space

In this section, we construct the solutions of the harmonic oscillator in phase space. Consider the following 3D Hamiltonian

$$H=\frac{p^2}{2m}+\frac{1}{2}m\omega^2q^2, \tag{15}$$

where $p^2=p_x^2+p_y^2+p_z^2$ and $q^2=x^2+y^2+z^2$.

In phase space, we replace the coordinates and momenta by

$$q\star=q_i+\frac{\imath}{2}\frac{\partial}{\partial p^i},\qquad p^i\star=p^i+\frac{\imath}{2}\frac{\partial}{\partial q_i},$$

respectively, where we have set $m=\omega=\hbar=1$.

Then, we have that

$$\begin{aligned}H\star=\frac{1}{2}\Bigg[&p_x^2+p_y^2+p_z^2+x^2+y^2+z^2\\ &+i\left(x\frac{\partial}{\partial p_x}-p_x\frac{\partial}{\partial x}\right)+i\left(y\frac{\partial}{\partial p_y}-p_y\frac{\partial}{\partial y}\right)\\ &+i\left(z\frac{\partial}{\partial p_z}-p_z\frac{\partial}{\partial z}\right)\\ &-\frac{1}{4}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}+\frac{\partial^2}{\partial p_x^2}+\frac{\partial^2}{\partial p_y^2}+\frac{\partial^2}{\partial p_z^2}\right)\Bigg].\end{aligned}$$

To solve the equation $H \star \Psi = E\Psi$, we change variables with the definition

$$\zeta = \frac{1}{2}\left(p_x^2 + p_y^2 + p_z^2 + x^2 + y^2 + z^2\right),$$

such that

$$\frac{\partial \Psi}{\partial x} = x\frac{\partial \Psi}{\partial \zeta}, \qquad \frac{\partial \Psi}{\partial p_x} = p_x\frac{\partial \Psi}{\partial \zeta},$$
$$\frac{\partial \Psi}{\partial y} = y\frac{\partial \Psi}{\partial \zeta}, \qquad \frac{\partial \Psi}{\partial p_y} = p_y\frac{\partial \Psi}{\partial \zeta},$$
$$\frac{\partial \Psi}{\partial z} = z\frac{\partial \Psi}{\partial \zeta}, \qquad \frac{\partial \Psi}{\partial p_z} = p_z\frac{\partial \Psi}{\partial \zeta},$$

and

$$\frac{\partial^2 \Psi}{\partial x^2} = \frac{\partial \Psi}{\partial \zeta} + x^2\frac{\partial^2 \Psi}{\partial \zeta^2}, \qquad \frac{\partial^2 \Psi}{\partial p_x^2} = \frac{\partial \Psi}{\partial \zeta} + p_x^2\frac{\partial \Psi}{\partial \zeta},$$
$$\frac{\partial^2 \Psi}{\partial y^2} = \frac{\partial \Psi}{\partial \zeta} + y^2\frac{\partial^2 \Psi}{\partial \zeta^2}, \qquad \frac{\partial^2 \Psi}{\partial p_y^2} = \frac{\partial \Psi}{\partial \zeta} + p_y^2\frac{\partial^2 \Psi}{\partial \zeta^2},$$
$$\frac{\partial^2 \Psi}{\partial z^2} = \frac{\partial \Psi}{\partial \zeta} + z^2\frac{\partial^2 \Psi}{\partial \zeta^2}, \qquad \frac{\partial^2 \Psi}{\partial p_z^2} = \frac{\partial \Psi}{\partial \zeta} + p_z^2\frac{\partial^2 \Psi}{\partial \zeta^2}.$$

Consequently, the imaginary part of $H \star \Psi = E\Psi$ vanishes. This leads to

$$\zeta\frac{\partial^2 \Psi}{\partial \zeta^2} + 3\frac{\partial \Psi}{\partial \zeta} - 4(\zeta - E)\Psi = 0. \tag{16}$$

To change variables again, we define $r = 4\zeta$, write $\Psi = \exp(-r/2)\,\chi(r)$, and obtain the following equation:

$$r\chi'' + (3 - r)\chi' + \left(E - \frac{3}{2}\right)\chi = 0, \tag{17}$$

where the prime indicates differentiation with respect to r.

Equation 17 is of the general form

$$zu'' + (\gamma - z)u' - \alpha u = 0,$$

where $u = u(z)$, a solution of which is the confluent hypergeometric function, is defined by the equality

$$u(z) = F(\alpha.\gamma, z) = 1 + \frac{\alpha}{\gamma}z + \frac{\alpha(\alpha+1)}{\gamma(\gamma+1)}\frac{z^2}{2} + \frac{\alpha(\alpha+1)(\alpha+2)}{\gamma(\gamma+1)(\gamma+2)}\frac{z^3}{6} + \ldots.$$

Comparison with Eq. 17 now shows that $\chi = F\left(-\left(E - \frac{3}{2}\right); 3; r\right)$. The confluent hypergeometric function is finite if the parameter α is a negative integer. This constraint yields the result

$$E = E_n = n + \frac{3}{2},$$

where n is an integer, which is the expected result for the energy, since $\hbar = \omega = 1$.

The solution of the phase-space Schrödinger equation is

$$\Psi_n(\zeta) = \exp(-2\zeta)\,F(-n, 3, 4\zeta), \tag{18}$$

where $F(-n, 3, 4\zeta)$ is the confluent hypergeometric function with the appropriate parameters.

The Wigner function is calculated from Eq. 12, which in this case reads $f_W^n(q, p) = \Psi_n(\zeta) \star \Psi_n(\zeta)$. We therefore have that

$$f_W^n(q, p) = C_n\,\exp(-2\zeta)\,F(-n, 3, 4\zeta), \tag{19}$$

where $C_n = \exp(-2E_n)\,F(-n, 3, 4E_n)$.

5 2D Noncommutative Oscillator in Phase Space

We now want to derive the Wigner function for a 2D noncommutative oscillator in phase space, which is defined by the Hamiltonian

$$H = \frac{1}{2}\left(x^2 + p_x^2\right) + \frac{1}{2}\left(y^2 + p_y^2\right), \tag{20}$$

where again $\hbar = 1$, $m = 1$, and $\omega = 1$.

The star product is now given by the expression

$$\star = \star_{\hbar\theta} = \exp\left\{\frac{i}{2}\sum_{i=1}^{2}\left(\overleftarrow{\partial}_{q_i}\overrightarrow{\partial}_{p_i} - \overleftarrow{\partial}_{p_i}\overrightarrow{\partial}_{q_i}\right) + \frac{i\theta}{2}\left(\overleftarrow{\partial}_x\overrightarrow{\partial}_y - \overleftarrow{\partial}_y\overrightarrow{\partial}_x\right)\right\}$$

where $q_i = (x, y)$ and $p_i = (p_x, p_y)$.

The position and momentum operators are given by the expressions

$$q_i\star = q_i + \frac{i}{2}\partial_{p_i} + \frac{i}{2}\theta_{ij}\partial_{q_j} \tag{21}$$

and

$$p_i\star = p_i + \frac{i}{2}\partial_{q_i} + \frac{i}{2}\theta_{ij}\partial_{p_j}, \tag{22}$$

respectively.

These operators satisfy the following nonzero commutation relations:

$$[q_i, p_j] = i\delta_{ij}, [q_i, q_j] = i\theta_{ij}, [p_i, p_j] = -i\theta_{ij}.$$

In addition to the usual phase-space noncommutativity, we therefore have momentum coordinates that do not commute and space coordinates that do not commute. From

Eqs. 21 and 22, we obtain the following Schrödinger equation, $H \star \psi(x, y, p_x, p_y) = E\psi(x, y, p_x, p_y)$:

$$E\psi(x,y,p_x,p_y) = \frac{1}{2}\Bigg[\left(x+\frac{i}{2}\partial_{p_x}+\frac{i}{2}\theta\partial_y\right)^2 + \left(p_x+\frac{i}{2}\partial_x-\frac{i}{2}\theta\partial_{p_y}\right)^2 + \left(y+\frac{i}{2}\partial_{p_y}-\frac{i}{2}\theta\partial_x\right)^2 + \left(p_y+\frac{i}{2}\partial_y+\frac{i}{2}\theta\partial_{p_x}\right)^2\Bigg]\psi(x,y,p_x,p_y),$$

where we have used that $\theta = \theta_{12} = -\theta_{21}$.

To solve this Schrödinger equation, we define the coordinates $\widetilde{x} = x$, $\widetilde{y} = (1+\theta^2)^{-1/2}(y \star -\theta p_x)$, $\widetilde{p_x}\star = (1+\theta^2)^{-1/2}(p_x \star +\theta y\star)$, $\widetilde{p_y}\star = p_y\star$ and the star operators

$$\widetilde{x}\star = x\star,$$

$$\widetilde{y}\star = (1+\theta^2)^{-1/2}(y\star -\theta p_x\star),$$

$$\widetilde{p_x}\star = (1+\theta^2)^{-1/2}(p_x\star +\theta y\star),$$

and

$$\widetilde{p_y}\star = p_y\star .$$

The latter satisfy the following commutation relations:

$$\left[\widetilde{x}\star, \widetilde{p_x}\star\right] = (1+\theta^2)^{1/2}, \left[\widetilde{y}\star, \widetilde{p_y}\star\right] = (1+\theta^2)^{1/2}.$$

We therefore define the annihilation operators

$$\widetilde{a}_x\star = \frac{1}{\sqrt{2}}(\widetilde{x}\star +i\widetilde{p_x}\star),$$

$$\widetilde{a}_y\star = \frac{1}{\sqrt{2}}(\widetilde{y}\star +i\widetilde{p_y}\star),$$

and the creation operators

$$\widetilde{a}^\dagger_x\star = \frac{1}{\sqrt{2}}(\widetilde{x}\star -i\widetilde{p_x}\star),$$

$$\widetilde{a}^\dagger_y\star = \frac{1}{\sqrt{2}}(\widetilde{y}\star +i\widetilde{p_y}\star),$$

such that $\left[\widetilde{a}_i\star, \widetilde{a}^\dagger_j\star\right] = i(1+\theta^2)^{1/2}\delta_{ij}$, where $\widetilde{a}_1\star = \widetilde{a}_x\star$ and $\widetilde{a}_2\star = \widetilde{a}_y\star$.

The Schrödinger equation can then be written in the form

$$H\star\psi(x,y,p_x,p_y) = E\psi(\widetilde{x},\widetilde{p_x},\widetilde{y},\widetilde{p_y}) = \left[\widetilde{a}^\dagger_x\star\widetilde{a}_x\star+\widetilde{a}^\dagger_y\star\widetilde{a}_y\star +(1+\theta^2)^{1/2}\right]\psi(\widetilde{x},\widetilde{p_x},\widetilde{y},\widetilde{p_y}).$$

The energy eigenvalues are then given by the expression

$$E_{n_x n_y} = (1+\theta^2)^{1/2}(n_x+n_y+1).$$

For the ground state, $\psi_{00}(\widetilde{x},\widetilde{p_x},\widetilde{y},\widetilde{p_y}) = \phi_0(\widetilde{x},\widetilde{p_x})\chi_0\times(\widetilde{y},\widetilde{p_y})$, and we have the equations $\widetilde{a}_x\star\phi_0 = \widetilde{a}_y\star\chi_0 = 0$, which can be explicitly written as

$$\frac{1}{\sqrt{2}}\left(\widetilde{x}+\frac{i}{2}\partial_{\widetilde{p_x}}+i\widetilde{p_x}+\frac{1}{2}\partial_{\widetilde{x}}\right)\phi(\widetilde{x},\widetilde{p_x}) = 0, \quad (23)$$

and

$$\frac{1}{\sqrt{2}}\left(\widetilde{y}+\frac{i}{2}\partial_{\widetilde{p_y}}+i\widetilde{p_y}+\frac{1}{2}\partial_{\widetilde{y}}\right)\phi(\widetilde{y},\widetilde{p_y}) = 0. \quad (24)$$

To find real solutions, we have to solve the following set of equations:

$$\left(\widetilde{x}+\frac{1}{2}\partial_{\widetilde{x}}\right)\phi_0 = 0,$$

$$\left(\widetilde{y}+\frac{1}{2}\partial_{\widetilde{y}}\right)\chi_0 = 0,$$

$$\left(\widetilde{p_x}+\frac{1}{2}\partial_{\widetilde{p_x}}\right)\phi_0 = 0,$$

$$\left(\widetilde{p_y}+\frac{1}{2}\partial_{\widetilde{p_y}}\right)\chi_0 = 0.$$

The general ground state solution is given by the expression

$$\psi_{00} = C_0\exp-\left(\widetilde{x}^2+\widetilde{p_x}^2+\widetilde{y}^2+\widetilde{p_y}^2\right), \quad (25)$$

where $C_0 = \frac{1}{\pi}$ is the normalization constant.

For $n \geq 1$, the functions ψ_n are determined by the creation operator, that is, by the relation

$$\psi_{n_x n_y} = \frac{1}{\sqrt{n!}}\left(\widetilde{a}^\dagger_x\star\widetilde{a}^\dagger_y\star\right)^n\psi_{00}. \quad (26)$$

The Wigner function associated with each $\psi_{n_x n_y}$ is

$$f_W(\widetilde{x},\widetilde{p_x},\widetilde{y},\widetilde{p_y}) = \psi_{n_x n_y}\star\psi^\dagger_{n_x n_y}.$$

In particular, for $n_x = 1, n_y = 1$, we find that

$$f^1_W(\widetilde{x},\widetilde{p_x},\widetilde{y},\widetilde{p_y}) \sim \left[1-2\left((\widetilde{x})^2+(\widetilde{p_x})^2\right)\right]e^{-\left((\widetilde{x})^2+(\widetilde{p_x})^2\right)} \times\left[1-2\left((\widetilde{y})^2+(\widetilde{p_y})^2\right)\right]e^{-\left((\widetilde{y})^2+(\widetilde{p_y})^2\right)},$$

and for $n_x = 2, n_y = 2$,

$$f^2_W(\widetilde{x},\widetilde{p_x},\widetilde{y},\widetilde{p_y})\sim\left[2-4\left((\widetilde{x})^2+(\widetilde{p_x})^2\right)+\left((\widetilde{x})^2+(\widetilde{p_x})^2\right)^2\right] \times e^{-\left((\widetilde{x})^2+(\widetilde{p_x})^2\right)}\left[2-4\left((\widetilde{y})^2+(\widetilde{p_y})^2\right) + \left((\widetilde{y})^2+(\widetilde{p_y})^2\right)^2\right]e^{-\left((\widetilde{y})^2+(\widetilde{p_y})^2\right)}.$$

For arbitrary n_x and n_y, we have the result

$$f_W^n \sim Ln\left[\left(\widetilde{p_x}\right)^2 + \left(\widetilde{x}\right)^2\right)\right] \times Ln\left[\left(\widetilde{y}\right)^2 + \left(\widetilde{p_y}\right)^2\right] e^{-\left(\left(\widetilde{x}\right)^2+\left(\widetilde{p_x}\right)^2+\left(\widetilde{y}\right)^2+\left(\widetilde{p_y}\right)^2\right)},$$

where Ln are the Laguerre polynomials.

Going back to the original variables, we have that

$$f_W^n(x, y, p_x, p_y; \theta) \sim Ln\left[\left(x^2 + (1+\theta^2)^{-1}(p_x+\theta y)^2\right)\right] \times Ln\left[\left(1+\theta^2\right)^{-1}(y-\theta p_x)^2 + p_y^2\right] \times e^{-\left((x^2+\beta(p_x+\theta y)^2)+\beta(y-\theta p_x)^2+p_y^2\right)}, \quad (27)$$

where $\beta = (1+\theta^2)^{-1}$.

Equation 27 was derived, by a different method, in Refs. [42] and [43]. It is important to notice that additional solutions can be found, associated with different combinations of the quasi-amplitudes of probability. The Wigner function for the noncommutative oscillator depends on the parameter θ, which finds physical application in such problems as the quantum Hall effect [44–46].

6 Concluding Remarks

We have set forth a symplectic representation of the Galilei group, which yields quantum theories in phase space. We have derived a Schrödinger equation and, as illustrations, studied the 3D harmonic oscillator and the noncommutative oscillator in phase space. In both cases, we obtained the Wigner functions. The symplectic representation is constructed on the basis of the Moyal or star product, an ingredient of noncommutative geometry. A Hilbert space is then defined from a manifold with the features of phase space. The states are represented by a quasi-amplitude of probability, a wave function in phase space, the definition of which makes connection with the Wigner function, i.e., the quasi-probability density. Nontrivial, yet consistent, the association with the Wigner function provides a physical interpretation of the theory. Analogous interpretations are not found in other studies of representations in phase space [25, 26].

One aspect of the procedure deserves emphasis. Our formalism explores unitary representations to calculate Wigner functions. This constitutes an important advantage over the more traditional constructions of the Wigner method, which entail several intricacies associated with the Liouville–von Neumann equation. Furthermore, the formalism we have described opens new perspectives for applications of the Wigner function method in quantum field theory. This aspect of the formalism will be discussed in a forthcoming paper.

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